Preface

This book introduces mathematical tools that are used in analyzing many physical problems. It has often been noted that, what at first appears to be an abstract generalization turns out to be not only useful but necessary for many applications.

To provide a focus, we establish what is needed to prove the main theorems of Chaps. 5 and 6, leaving unproved only two results from potential theory. The results of these two chapters improve on what we have previously published. These theorems provide easy methods for the numerical solution of the Dirichlet problem in two and three dimensions, which for concreteness we interpret as a problem in steady-state heat conduction. We refer to the literature for numerical applications, noting here only that such applications amount to a minimization over the parameters of the theorems.

If a reader is encouraged to learn more about Complex Variables or Linear Operators on Banach spaces, we will have accomplished our goal.

Theodore Hromadka
Robert Whitley

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Foundations of the Complex Variable Boundary Element Method

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Chapter 2
Metric Spaces

Abstract Ideas are introduced to state and understand the Dirichlet problem in two, three, and N dimensions. These include open sets, boundaries, compact sets, and (uniformly) continuous functions. The concepts of completeness, sups and infs, compact sets, and continuous functions are used in the statement and proof of The Maximum Principle for the solution to the Dirichlet Problem in R^N and the Laplace equation in N variables.

Keywords Metric Space · R^N · Convergence · Geometric series · Continuous functions on a metric space · Open and closed sets · Compact sets · Compact sets in R^N · Sup · Inf · Completeness · The Maximum Principle

The combination of Theorem 4 and the maximum principle shows that there is an approximate solution to the heat equation for the unit square, given a continuous function g on the edges of the square, with error as small as desired. Restricting the continuous function g to the side with $0 \leq x \leq 1$, $y = 0$, it can be approximated to within a given $\epsilon > 0$ by the finite sum of of Theorem 4. If you add together the sums for each of the four sides, this sum $U_N(x, t)$ is harmonic inside the square and continuous on the edges because it is the finite sum of functions which have these properties. And, assuming that an exact solution $U(x, t)$ to this problem exists, the maximum principle shows that $|U_N(x, t) - U(x, t)| < \epsilon$ holds throughout the entire square. If, for example, you want to compute some isothermal lines, i.e., curved lines in the square where each point has the same constant temperature, you can compute them using $U_N(x, t)$ with the possible error as small as desired.

In obtaining these results, we have used the fact that the continuous function g is uniformly continuous and attains its maximum on [0, 1], and that $U_N(x, t)$ is continuous and also attains its maximum on the square and hence on the edge of the square. Further, in the proof of the Maximum Principle we have used general facts about the unit square, what is inside the square and what constitutes the edges of the square.

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This ideas need to be made precise so that domains more general than the square can be considered, in R^2 as well as in R^3. As is often the case, making the discussion more precise will open up wide areas of application. To discuss the accuracy of an (numerical) approximate solution of a problem a way of measuring the distance between solutions is required. A metric is such a measure.

Definition 5 A metric space is a set S on which is defined a measure of distance between points of S, a metric d, which is a map from pairs of elements of S to the non-negative real numbers with the properties, for x, y, and z in S: (1) $d(x, y) \geq 0$, and $d(x, y) = 0$ only if $x = y$, (2) $d(x, y) = d(y, x)$, and the triangle inequality (3) $d(x, z) \leq d(x, y) + d(y, z)$.

This definition abstracts the properties of the metric $d(x, y) = |x - y|$ given by the absolute value function on the real numbers.

Example 7 On R^N, for $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ the function

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_1^N |x_j - y_j|.$$

satisfies the conditions for a metric.

Using the definition of the absolute value of a complex number $x + iy$, which is defined to be the length of the vector in the plane from $(0, 0)$ to (x, y), $|x + iy| = (x^2 + y^2)^{\frac{1}{2}}$, the formula above also gives a metric on C^N. In this case, to establish the triangle inequality requires the Pythagorean Theorem in the plane, which is a fact about a triangle.

Example 8 On R^N, for $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)$, a metric is given by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\left(|\mathrm{x_j} - \mathrm{y_j}| : \mathrm{j} = 1, \ldots, \mathrm{N}\right).$$

and the same formula defines a metric on C^N with x_j and y_j in C.

Example 9 On any inner product space, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ defines a metric, the triangle inequality for the metric is the triangle inequality for an inner product space. Examples include the usual Euclidean metric on R^N:

$$d_2(\mathbf{x}, \mathbf{y}) = \left[\sum_1^N |x_j - y_j|^2\right]^{1/2}.$$

and the Euclidean metric on C^N, given by the same formula.

Definition 6 A sequence of points $x_1, x_2, \ldots, x_n, \ldots$ in a metric space X converges to a point x in X if $d(x_n, x) \to 0$ as $n \to \infty$.

For real numbers with the absolute value metric this is the usual definition of convergence.

An important example of a convergent sequence is given by the partial sums of a geometric series.

Lemma 4 *A geometric series has the form* $\sum_0^\infty r^n$ *where* $-1 < r < 1$. *The partial sums converge to* $\frac{1}{1-r}$.

Proof The Nth partial sum of the series is $S_N = \sum_0^N r^n$. Since $rS_N = r + r^2 + \cdots + r^{N+1}$, $S_N - rS_N = 1 - r^{N+1}$, and $S_N = \frac{1-r^{N+1}}{1-r}$. Let N tend to infinity. Q.E.D.

It is a rare event that a sequence can be shown to converge to a specific value; in general the sequence defines the limit to which it converges. For example, suppose that $c_1, c_2, \ldots$ is a sequence where each c_n is one of the integers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Then

$$a = \frac{c_1}{10} + \frac{c_2}{10^2} + \frac{c_3}{10^3} + \cdots$$

is the decimal expansion of the number $a = 0.c_1c_2c_3\ldots$ in the interval [0, 1] How do we know that this represents a number in [0, 1]? What is meant by this infinite sum being equal to a real number is that the sequence of the partial sums $S_N = \sum_1^N \frac{c_j}{10^j}$ converges as N tends to infinity, the limiting value being indicated by $\sum_1^\infty \frac{c_n}{10^n}$. How can it be shown that this limit exists? Start by considering the series where all the $c_j = 1$, with partial sum $\sum_1^N \frac{1}{10^j}$ which by the lemma converges to $\frac{1}{9}$, since

$$S_N = \frac{1}{9}\left[1 - \frac{1}{10^N}\right].$$

What about the general number $0.c_1c_2\ldots$ in [0, 1]? Consider the difference between two partial sums $S_N - S_M$, $N > M$. Since each c_n satisfies $0 \le c_n \le 9$:

$$0 \le S_N - S_M = \sum_{M+1}^{N} \frac{c_j}{10^j} \le \frac{9}{10^M} \sum_{1}^{N-M} \frac{1}{10^j} < \frac{9}{10^M}.$$

From this, it follows that the difference $S_N - S_M$ tends to zero as N and M tend to infinity. So, intuitively, the terms of the sequence are getting close to each other for large N and M, so the sequence must be getting close to something.

Definition 7 A sequence $\{x_1, x_2, \ldots\}$ in a metric space with metric d is a Cauchy sequence if $d(x_n, x_m) \to 0$ as n, m tend to infinity. A metric space is complete if each Cauchy sequence converges; that is, for any Cauchy sequence $\{x_n\}$ there corresponds an element x in the space with $d(x_n, x) \to 0$ as $n \to \infty$.

It is a basic fact that the real numbers, with the usual absolute value metric, are complete. This is true because the real numbers are constructed to be complete, starting with the integers and with little beside mathematical induction as a tool.

The construction involves many details, and solves a mathematical problem which was open for over 2000 years, beginning when the ancient Greeks discovered that the rational numbers were not sufficient for geometry; one example being that the diagonal of a right triangle with sides each of length one is not a rational number—see exercises.

One consequence of the completeness of the real numbers is the existence of a real number for each decimal expansion. Generally whenever a numerical value is defined by a limiting process, completeness is usually required to show that this value exists.

Definition 8 Definition: Let X and Y be metric spaces and F a function mapping X to Y; $F\colon X \to Y$. The function F is continuous if whenever a sequence $\{x_n\}$ converges to x in X, the sequence $\{F(x_n)\}$ converges to $F(x)$.

One interpretation of continunity is that if a physical quantity is a continuous function F of x, then if you measure x accurately enough you will know $F(x)$ with as much accuracy as you need.

Another formulation of continuity comes from asking how closely the input value x must be known in order that the output be close to $F(x)$. In this form, if you want to know the value of $F(x)$ to within a certain error, traditionally denoted by an (arbitrary) value $\epsilon > 0$, it is required that the input value y be sufficiently close to x, this closeness is traditionally denoted by $\delta > 0$, where $|x - y| < \delta$. In spite of the fact that the appearance of two Greek letters in one definition makes it seem more complicated than the sequential definition above, the equivalent condition for continuity in the next lemma reflects an important physical property.

Lemma 5 *Let X be a metric space with metric d_x and Y another metric space with metric d_y. A function F mapping X to Y, $F\colon X \to Y$, is continuous at a point x_0 in X if and only if*

Given any $\epsilon > 0$, there is a number $\delta > 0$, depending on ϵ and x_0, with the property that if $d_x(x_0, x) < \delta$, then $d_y(F(x_0), F(x)) < \epsilon$.

Proof Suppose the $\delta - \epsilon$ condition holds. Let the sequence $x_n \to x_0$ in X, and let $\epsilon > 0$ be given. There is a $\delta > 0$ with the property that if $d_x(x, x_0) < \delta$ then $d_y(F(x), F(x_0)) < \epsilon$. Since $x_n \to x_0$, there is an integer N with $d_x(x_n, x_0) < \delta$ for $n \geq N$, and then $d_y(F(x_n), F(x_0)) < \epsilon$, showing that $F(x_n)$ converges to $F(x_0)$ and so that F is continuous at x_0.

Suppose that F is continuous at x_0. We show that the $\delta - \epsilon$ condition holds by assuming that it does not, i.e., for some $\epsilon_1 > 0$ an appropriate $\delta > 0$ cannot be found. This means that for each integer n and $\delta = 1/n$, there is a point x_n in X with $d_x(x_n, x_0) < 1/n$ but $d_y(F(x_n), F(x_0)) \geq \epsilon_1$. But then $x_n \to x_0$ yet $F(x_n)$ does not converge to $F(x_0)$, contrary to the continuity of F at x_0. Q.E.D.

A metric space has two important related classes of sets. The applied heat equation problem of Chap. 1 was stated for the unit square, and the proof of the maximum principle for the square used the notion of points being inside the square and points being on the sides of the square. These ideas need to be extended to more general domains, and that is done in terms of two classes of sets in a metric space.

Definition 9 Let X be a metric space with metric d. (1) A subset A of X is open if for each point x in A, there is a $r > 0$ so that the sphere about x with radius r,

$$S(x, r) = \{y : d(y, x) < r\}$$

is contained in A: $S(x, r) \subset A$. (2) A subset B of X is closed if whenever a sequence of point $x_1, x_2, \ldots$ of points of B converges to x, this limit point x also belongs to B.

Lemma 6 *Let X be a metric space with metric d. A subset A of X is open if and only if the set B of all points of X which are not in A, written $B = X - A = A^c$ ("c" for the set complementary to A) is closed.*

Proof Let A be open and suppose that a sequence of points $x_1, x_2, \ldots$ from B converges to a point x. This point cannot be in A, for if it were, then because A is open there would be a sphere $S(x, r) \subset A$, and, by convergence the $\{x_n\}$ would be in A for large n, and these x_n would be both in A and the complementary set B, which is not possible.

Let B be closed, and let x be a point in A. We want to show that there is a sphere $S(x, r) \subset A$. Assume this is not true, then for each n the sphere $S(x, 1/n)$ is not contained in A and so there is a point x_n in this sphere and not in A therefore in B. But then the sequence of point $x_1, x_2, \ldots$ in B converges to x not in B, which cannot be. Q.E.D.

In the metrics space R with the absolute value metric, the interval (0, 1) is open, the interval [0, 1] is closed, and the interval (0, 1] is neither open nor closed.

For the unit square of the heat equation problem, the inside of the square is open and the edge of the square, as well as the entire square, is closed.

Definition 10 Let A be a set in a metric space X. The interior of A, written A^o is the largest open set contained in A. The closure of A, written $\overline{A}$, is the smallest closed set containing A.

Intervals in R, $(0, 1]^0 = (0, 1)$ and $\overline{(0, 1]} = [0, 1]$ The inside of the unit square is the interior of the square and the square (including its edges) is its own closure as it is itself closed.

The next idea needed in order to discuss harmonic functions on general domains corresponds to being given a continuous function g on the edge of the square. For an arbitrary set A the concept corresponding to the edge of the square is the boundary of A.

Definition 11 The boundary of a set A in a metric space X consists of those points x of X which have the property that any sphere $S(x, r)$ about x intersects both A ands its complement $X - A$.

In one dimension, if A is an interval—any of (a, b), $[a, b)$, or $[a, b]$—the boundary of A is the set consisting of the two points $\{a, b\}$.

The proof of the Maximal Principle relies on the fact that a function continuous on the closed unit square attains its maximum on the square.

Definition 12 For a set S of real numbers, the number $\sup(S)$, the supremum of S, is defined to be the least upper bound for the set S: so $M = \sup(S)$ if (1) M is an upper bound, i.e., all s in S satisfy $s \leq M$ and (2) it is the smallest upper bound, so that if $M' < M$, then there is at least one point s_0 in S for which $M' < s_0$.

In the older literature, the supremum of S was called the "least upper bound," which is much more descriptive than "sup," and was indicated by l.u.b.(S), pronounced "lub."

Example 10 $\sup((0, 1]) = 1$; $\sup((0, 1) = 1$, in the the first case the supremum of the set $(0, 1]$ is attained in the sense that it belongs to the set $(0, 1]$ while this is not true for the set $(0, 1)$ having the same supremum.

The fact that a set of real numbers has a supremum is a consequence of the completeness of the reals. For an example where completeness is not available suppose you were considering sets of rational real numbers, and wanted to find in the rational numbers a supremum for any set with an upper bound. The set $S = \{x : x^2 < 2\}$ shows that this is not possible since there is no rational number which is a least upper bound for S. For any rational number which is an upper bound for S, there is a smaller rational number which is also an upper bound—just take a better rational approximation to the real number $\sqrt{2}$ which is the least upper bound for the set in the real numbers.

Lemma 7 *A set S of real numbers which has an upper bound, $s \leq M$ for all s in S, has a least upper bound.*

Proof Let B be the set of all real numbers which are upper bounds for S which is not empty since M belongs to B, and let A be all the real numbers which are not in B.

Begin an inductive procedure as follows. Choose a_1 in A. Since a_1 is not a bound for S there is a point s_1 in S larger than a_1. Set $b_1 = M$, and note

$$a_1 < s_1 \leq b_1.$$

and let $d = b_1 - a_1$. Consider the average value $c = \frac{a_1+b_1}{2}$. If c is a bound for S, let $b_2 = c$, and let $a_2 = a_1$; but if c is not a bound for S, take $a_2 = c$ and $b_2 = b_1$. Then since a_2 is not a bound for S but b_2 is, there is a point s_2 in S, and

$$a_2 < s_2 \leq b_2$$

with $b_2 - a_2 = \frac{d}{2}$. After n steps of we have the points

$$a_n < s_n \leq b_n$$

with a_n not a bound for S, s_n in S, b_n a bound for S, with $b_n - a_n = \frac{d}{2^{n-1}}$. All three sequences are Cauchy and so all converge (to the same limit) b. For any s in S, $s \leq b_n$, and so $s \leq b$, showing that b is an upper bound for S. Further, if c is a real number less than b, since s_n converges to b, for large enough n, $s_n > c$, showing that b is indeed the smallest upper bound. Q.E.D.

There is a related concept for lower bounds for a set of real numbers.

Definition 13 For a set S of real numbers, the number $\inf(S)$, the infimum of S, is defined to be the greatest lower bound for the set S: so $N = \inf(S)$ if (1) N is an lower bound, i.e., all s in S satisfy $s \geq N$ and (2) it is the largest lower bound, so that if $N' > N$, then there is at least one point s_0 in S with $N' > s_0$.

In the older literature, the supremum of S was called the "greatest lower bound" and was indicated by g.l.b.(S), pronounced "glub."

Lemma 8 *A set S of real numbers which has a lower bound N, $s \geq N$ for all s in S, has a greatest lower bound.*

Proof Use $\inf(S) = -\sup(-S)$. Q.E.D.

One more idea is needed in order to work with general domains for the heat equation, we need the property that a continuous function on that domain will attain its maximum. If f is a continuous function mapping a domain D into the real numbers, with $M = \sup\{f(x) : x \in X\}$, then by the definition of the finite supremum M, for each $n = 1, 2, \ldots$, since $M - \frac{1}{n} < M$, and M is the smallest upper bound for f on D, there is a point x_n in D with $M - \frac{1}{n} < f(x_n) \leq M$; the second inequality following because M is a bound for all of the function values on D. With this in mind, you can see that if the sequence $x_1, x_2, \ldots$ converged to a point x in D, then we would have $f(x) = M$ and the supremum would be attained at this point x. In order to show that the supremum is attained, it would in fact be enough to have a subsequence converge; a subsequence of a sequence $\{x_n\}$ being another sequence formed from terms of the given sequence taken in the same order as the sequence. The property of compactness, defined below, has many applications.

Definition 14 Let $x_1, x_2, x_3, \ldots$ be a sequence of points (of a metric space X). By a subsequence of this sequence is meant a sequence formed using the points of the sequence taken in the same order: $x_{n1}, x_{n2}, x_{n3}, \ldots$ with $1 \leq n1 < n2 < n3 < \cdots$.

A set D in a metric space X is compact if any sequence of points $x_1, x_2, x_3, \ldots$ in D has a subsequence which converges to a point in D.

The following result has, more or less, been built-in to the definitions.

Theorem 5 *Let D be a set in a metric space X and f a continuous function mapping D into the real numbers. If D is compact, then f attains its maximum on D, i.e., there is a point x in D, with $f(x) = \sup\{f(y) : y \in D\}$.*

Since $\inf\{f(x) : x \in D\} = -\sup\{-f(x) : x \in D\}$, a continuous function also attains its infimum on a compact set.

In order to the apply results about compactness, we need to know which sets are compact. One property is clear: if a subsequence of points of D is going to converge to a point which is required to be in D, it seems that D must be closed. An additional property is needed, and in R^N what works is that the set be bounded.

Definition 15 A set D in R^N is bounded if it is contained in some sphere: i.e., $D \subset S(x_0, r)$, i.e., there is a point x_0 in R^N (which can be taken to be zero) with $d(x_0, x) < r$ for all x in D.

Theorem 6 *A subset of R^N is compact if and only if it is closed and bounded.*

Proof First consider a subset $D \subset R$, and a sequence of points $\{x_n\}$ from D. Since D is bounded it is contained in a sphere, which in R is an interval, say $D \subset [a, b]$, and then $|x_n - x_m| \leq |b - a|$. There are an infinite number of integers $1 < n_1 < n_2 < \cdots$ for which either all the x_{n_j} are in $[a, \frac{a+b}{2}]$ or in $[\frac{a+b}{2}, b]$. It could be that both of these hold, but we know that at least one does. In any case, a subsequence is obtained satisfying: $|x_{n_j} - x_{m_j}| \leq \frac{b-a}{2}$. To simplify the subscript notation, write $x_{n_j} = x_j^{(1)}$. The sequence $\{x_j^{(1)}\}$ has all its values in an interval of length $\frac{b-a}{2}$. Then there is a subsequence of this sequence which is contained in either the right-half or left-half of this new interval; let $\{x_j^{(2)}\}$ denote this subsequence and note that $|x_n^{(2)} - x_m^{(2)}| \leq \frac{b-a}{2^2}$. Continuing this process for each integer k there is a subsequence $\{x_j^{(k)}\}$ of the previous subsequence for which $|x_n^{(k)} - x_m^{(k)}| \leq \frac{b-a}{2^k}$.

Finally, the sequence with $y_n = x_n^{(n)}$ is a subsequence of the original sequence and satisfies $|y_n - y_m| \leq \frac{b-a}{2^m}$ for $n \geq m$, and this sequence $\{y_n\}$ is a Cauchy sequence, which converges, since R is complete, to a point of D, since D is closed.

The proof for D a subset of R^N follows directly from the case for R. To simplify notation, consider R^2, and a sequence $\{(x_n, y_n)\}$ taken from the closed and bounded set D. The first coordinate sequence $\{x_n\}$ is bounded and so has a convergent subsequence $\{x_{n_j}\}$ from the result for R. The related second coordinates $\{y_{n_j}\}$ being a bounded sequence has a convergent subsequence. The sequence which has this as the second coordinate, has first coordinate which is also convergent since it is a subsequence of the convergent first coordinate sequence, and so this subsequence converges in R^2 since both its coordinates are convergent sequences; it converges to a point of D since D is closed. The proof for R^N proceeds along the same lines.

The idea of the proof is easy to visualize in R^2. A closed and bounded set D in R^2 is contained in a square. Let a sequence of points from D be given. Divide the square by two lines from the midpoints of opposite sides into four squares each one-quarter the size of the first square. There is a subsequence of the original sequence in at least one of the smaller squares. Keep up this process of subdividing the square to get subsequences which whose terms are in smaller and smaller squares and therefore closer and closer together. Then, as in the case of R, a subsequence can be obtained whose terms get closer and closer together, so that this sequence is Cauchy and so converges to a point in the closed set D. This is sometimes referred to as "Lion Hunting in the Sahara Desert". To find the lion (= the limit point of the subsequence) you enclose the desert in a square, divide the desert into quarters, pick the one the lion is in, divide the new square into quarters again, pick the one the lion is in, and continue until you have a square which is just big enough to hold the lion.

To show that D, a subset of R^N must be closed and bounded if it is compact, see the exercises. Q.E.D.

The proof of Theorem 4 of Chap. 1 uses the fact below, a useful consequence of compactness.

Theorem 7 *Let K be a compact metric space with metric d and f a continuous function mapping K into the real numbers. Then f is uniformly continuous, i.e., given any* $\epsilon > 0$*, there is a* $\delta > 0$ *which works for all the x in K, i.e., for any x, if* $d(x, y) < \delta$*, then* $|f(x) - f(y)| < \epsilon$*.*

Proof Let f be as in the theorem and suppose that f is not uniformly continuous. This means that for some $\epsilon_0 > 0$ you cannot find a $\delta > 0$ that will work for all x; consequently for each positive integer n and $\delta = 1/n$, there are points x_n and y_n with $d(x_n, y_n) < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. By the compactness of K, there is a subsequence $\{x'_n\}$ of $\{x_n\}$ that converges to a point x_0 in K. Since $d(y'_n, x_0) \leq d(x'_n, x_0) + d(x'_n, y'_n)$, the corresponding subsequence of the y_n converges to x_0 Passing to this subsequences and taking a limit shows that $|f(x_0) - f(x_0)| \geq \epsilon_0$, which cannot be true. Q.E.D.

2.1 Maximum Principle

The concepts are now available for a statement of the Maximum Principle in a general setting.

Theorem 8 *Let D be a bounded open set in* R^N *with boundary* Γ*. Suppose that* $U(x_1, x_2, \ldots, x_N)$ *is a function with continuous second partial derivatives which satisfies the Laplace equation*

$$\Delta U(\mathbf{x}) = \sum_1^N \frac{\partial^2 U}{\partial x_j^2} = 0 \tag{2.1}$$

in D, and which is continuous on the closure of D. Then U attains its maximum (and minimum) on the boundary Γ *of D.*

Proof The proof is a simple adaptation of the proof of the the maximum principle for the square. Q.E.D.

Note that the Maximum Principle is not an existence theorem, it indicates an important property of the solution assuming that there is a solution. The existence of a solution will be discussed later.

One consequence of the Maximum Principle is that there cannot be two different solutions U and V which satisfy the conditions of the above theorem and which are both equal to the same continuous function g on the boundary of K, for then $U - V$ satisfies the conditions of the theorem and is zero on the boundary of K, hence the maximum and the minimum of $U - V$ are both zero, which is to say that $U = V$.

2.2 Exercises

1. (Euclid) Show that $\sqrt{2}$ is not a rational number, i.e., it is not equal to the quotient of two integers n/m, $m \neq 0$. Suppose that $\sqrt{2} = n/m$. You can assume that n and m do not have a common factor. (If, say, $n = 3\text{N}$ and $m = 3\text{M}$, divide out the 3, and write $n/m = N/M$). Squaring, $2\ m^2 = n^2$. The square of an odd number is odd, so n must be even, $n = 2k$. Then $m^2 = 2k^2$, and m is also even; a contradiction.
2. Using the completeness of R^N, show that C^N is complete.
3. Show that a compact subset of R^N must be closed and bounded.
4. Show that a subset of C^N is compact if and only if it is closed and bounded.
5. Let X be a metric space with metric d. Define

$$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

   (i) Show that $d_0(x, y)$ is a metric, and that a subset of S is closed with respect to the metric d if and only if it is closed with respect to the metric d_0, and (X, d) and (X, d_0) also have the same open sets. (ii) Every subset A of X with the metric d_0 is bounded. (iii) Consequently, it is false in general that a subset of an arbitrary metric space is compact if it is closed and bounded. (iv) The subset A of the metric space (X, d) is totally bounded if every sequence of points of A has a Cauchy subsequence. Show that A is compact if and only it is closed and totally bounded.
6. Let f and g be continuous maps from a metric space X to the real numbers. Show that $f(x) + g(x)$, $f(x)g(x)$, and, if g is never zero, $\frac{1}{g(x)}$, are continuous.
7. Show that if K is a compact metric space and f a continuous map of K into another metric space, then f is uniformly continuous.
8. Give an example of a continuous real-valued function defined on (0, 1) which is continuous but not uniformly continuous.
9. Show that $\sqrt{x}$ is uniformly continuous on [0, 1].
10. Let f map the compact metric space X into the metric space Y. Show that the image of X under f, i.e., the set of all points y in Y for which there is an x in X with $y = f(x)$, is compact. For example, if K is a compact set in R^2, the projections onto the x-axis and y-axis are compact sets in R.

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