Comparison of meshless MFS and CVBEM computational methods in analysis of groundwater flow pathways

CDTs Saleem Ali and Sebastian Neumann
Advisor: Dr. T. V. Hromadka, CPT B. D. Wilkins

Background

Solution of groundwater contamination problems continue to be of high interest to engineers and planners, among others. An important problem is identifying the source of contamination within a cluster of candidate sources. A key question is which candidate source(s) is the actual source point of the subject contamination. The approach used in modeled the flow field with a high accuracy computational model and tracked the streamlines to identify those streamlines that indicate delivery of flow to the location of the contamination detection. Then, a streamline is identified that intersects the detection point location and the actual source of the subject contamination. A study compared the finite element method to the complex variable boundary element method (CVBEM), and it was concluded that CVBEM is more accurate.

Methodology

General MFS Approximation Function

- Linear combination of real functions that are analytic on problem domain Ω:

$$\hat{\phi}(\boldsymbol{x}) = \sum_{j=1}^{n} a_j g_j(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega$$

- a_j is a real coefficient
- $g_j(\boldsymbol{x})$ are harmonic basis functions
- n is number of basis functions
- n degrees of freedom

General CVBEM Approximation Function

- Linear combination of complex functions that are analytic on problem domain Ω:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j g_j(z), \qquad z \in \Omega$$

- $c_j = \alpha_j + i\beta_j$,
- $g_j(z)$ are analytic complex basis functions
- n is number of basis functions
- $2n$ degrees of freedom

Error Function

To compute error, since we are modeling harmonic functions with harmonic functions, the absolute error function is also harmonic. Therefore, by the maximum principle of harmonic functions, the approximation function's maximum error is located on the problem boundary. For example, let ϕ be a harmonic function on domain Ω. $\hat{\phi}$ is a harmonic function that approximates ϕ. Thus, $\phi - \hat{\phi}$ is harmonic in Ω. Consequently, $max_{(x,y)\in\Omega}|\phi - \hat{\phi}| = max_{(x,y)\in\partial\Omega}|\phi - \hat{\phi}|$.

Node Position Algorithm (NPA)

1. Generate an initial set of candidate nodes outside of the problem domain and boundary and candidate collocation points on the problem boundary. The model starts with *n*=0 selected nodes and *m*=0 selected collocation points.
2. Select two collocation points to use in the model (*m*=*m*+2).
3. Loop through each node, adding it to the selected nodes to create a *n*+1 node model. Analyze error and select the *n*+1 node model with the least maximum absolute error. There are now *n*=*n*+1 selected nodes.
4. Evaluate the absolute error of the *n* node model on the boundary and locate the two greatest maxima. These two points will be the next two collocation points.
5. Repeat steps 3 and 4 until the desired number of nodes is achieved.

This study seeks to compare the accuracy of the NPA in CVBEM to the NPA in MFS and determine the feasibility of using MFS to achieve the same task. Specifically, the NPA coupled with a uniform, circular, and donut distributions of candidate nodes in MFS compared to the previously used CVBEM method with the NPA.

Example Problem

Domain: $\Omega = f\{0 \le x \le 8, 0 \le y \le 5, (x-5)^2 + y^2 \ge 1\}$
PDE: $\nabla^2\psi = 0$
BC's: $\psi(x,y) = \Re\left[z^2 + z + \frac{10}{z-5}\right], (x,y) \in \partial\Omega$

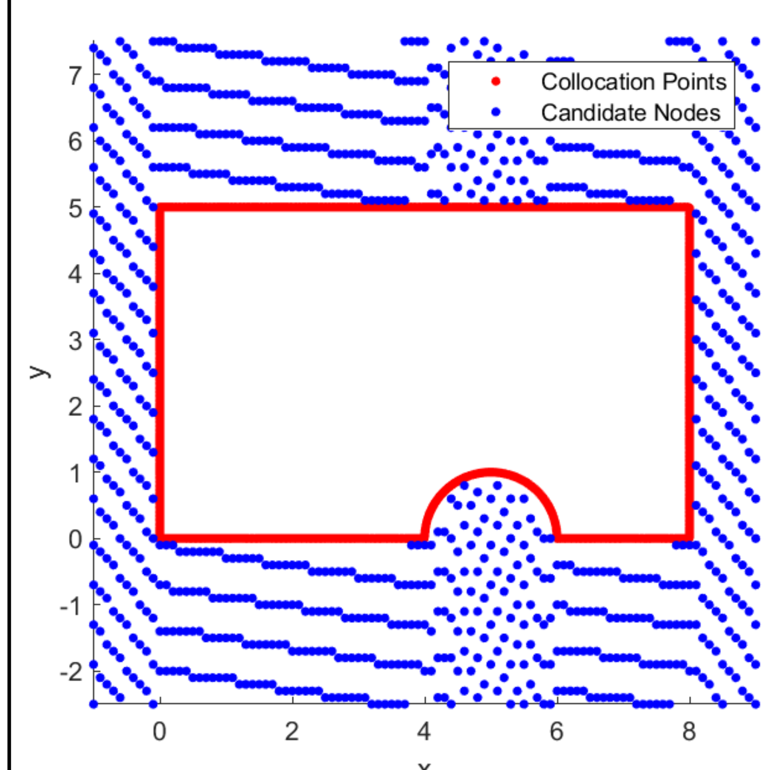


Figure 1: Candidate collocation and node locations

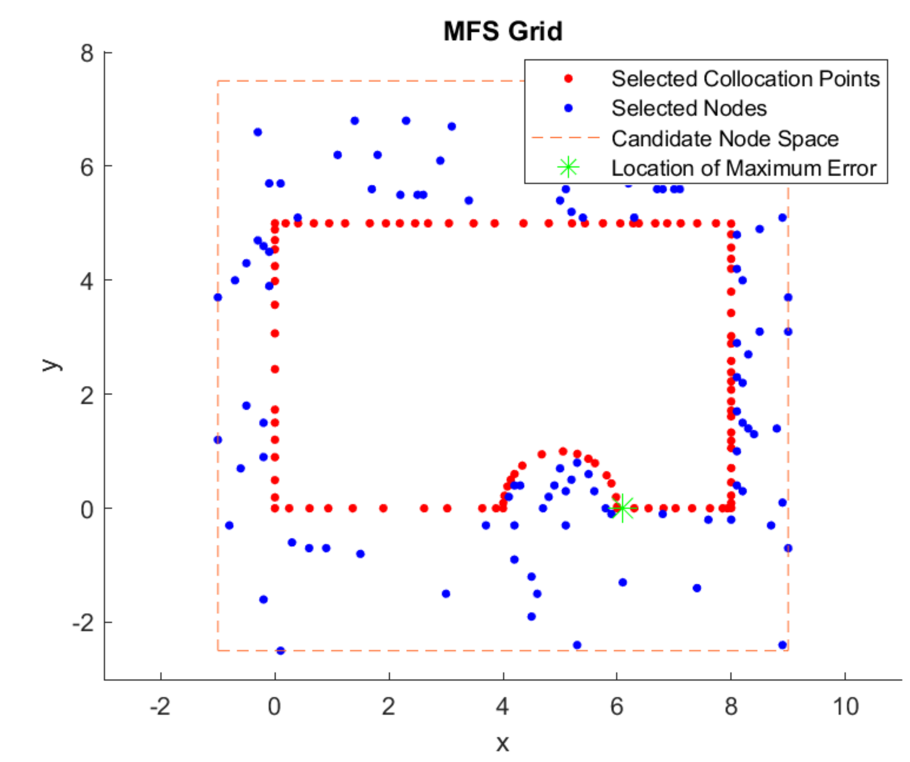
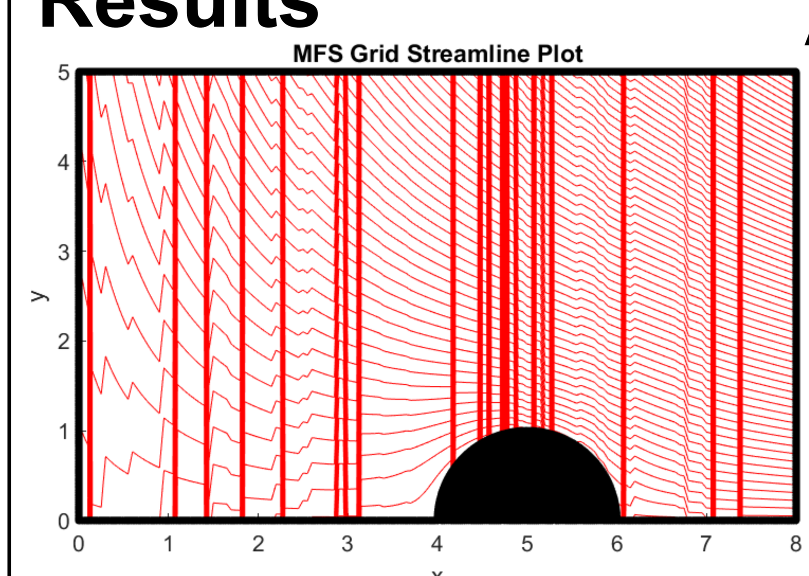


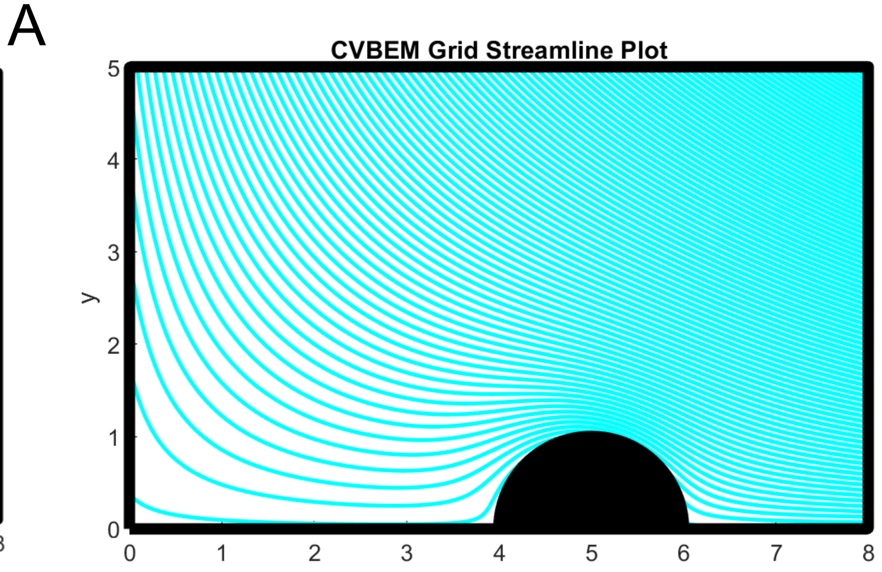
Figure 2: Problem geometry after NPA

Results

A



B



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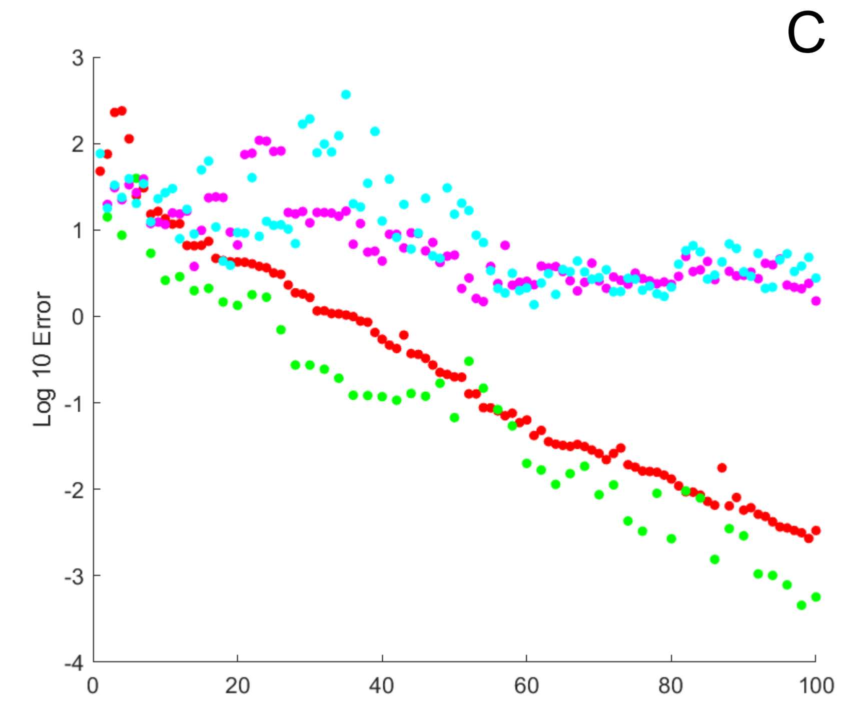

Figures 3A-C: 3A and 3B are the resulting streamline plots from the MFS and CVBEM approximations. 3C plots the log10 error over the CVBEM and MFS approximations

CVBEM produced the most accurate model beating MFS grid by approximately 1 order of magnitude. However, at 52, 54,56, and 84 degrees of freedom, MFS grid performs slightly better than CVBEM, which indicates that MFS can achieve error at least as low as the CVBEM. The error of MFS circle and donut both appear to plateau after approximately 55 degrees of freedom, which indicates that their candidate note distribution is not conducive to computational accuracy. The streamline plots for ground water flow analysis were only analyzed MFS grid and CVBEM grid because only MFS grid produced relatively similar computational error to CVBEM grid. However, the streamline plot attained from MFS grid through application of the Cauchy Riemann contained jagged streamlines compared to the CVBEM streamline plot. Thus, ground water contamination analysis cannot currently be done using MFS. Further investigation should explore and possibly resolve why this behavior occurs and compare the computational efficiency of MFS in comparison to CVBEM because it may perform faster due to only using real numbers.

Acknowledgements

Thank you to both my advisors for all your guidance!