An Examination of the Utility of the Identity Theorem in Justifying Collocation Computational Methods for Solving Engineering Problems

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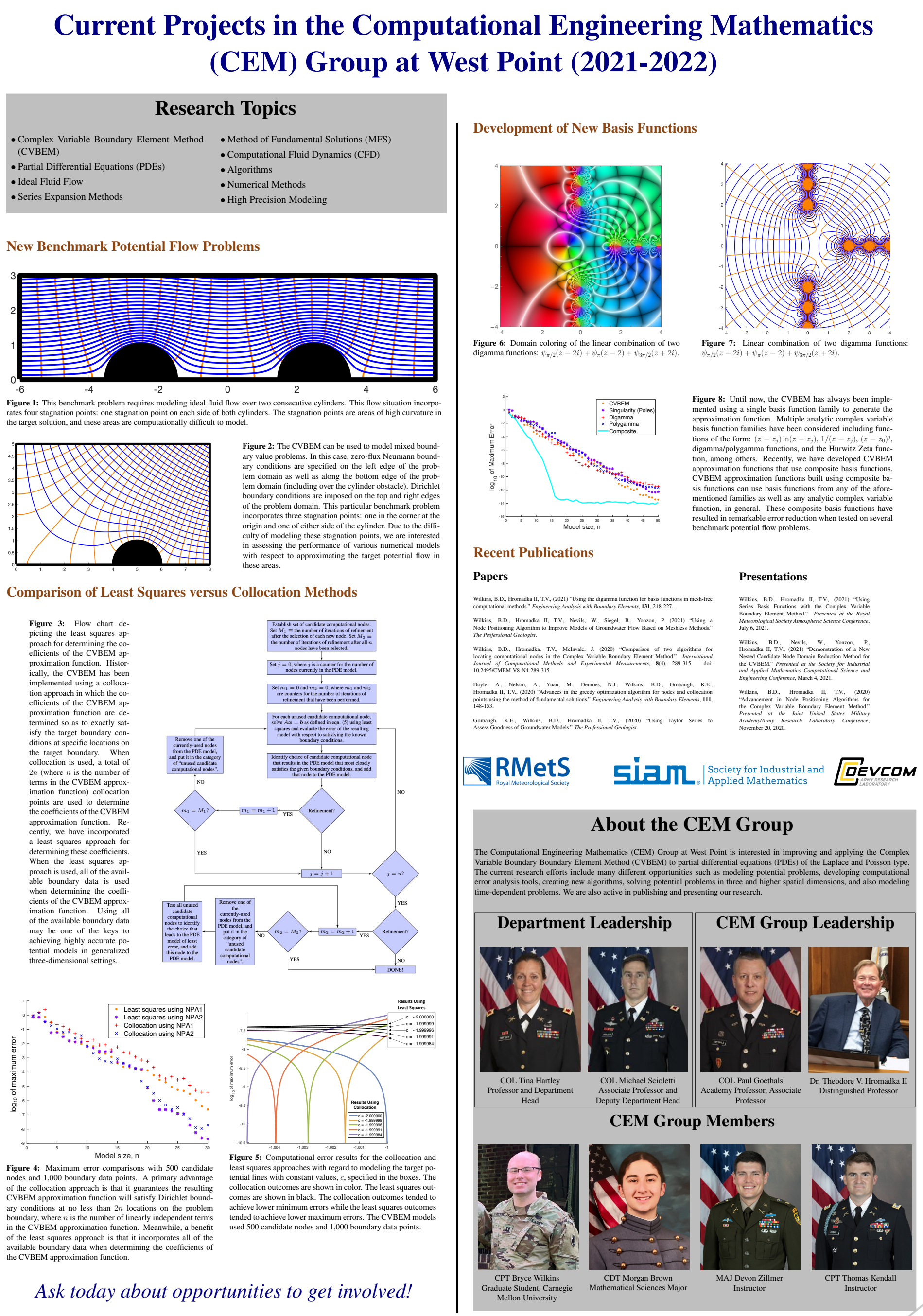

Current Projects in the Computational Engineering Mathematics (CEM) Group at West Point (2021-2022)

Research Topics

- Complex Variable Boundary Element Method (CVBEM)
- Partial Differential Equations (PDEs)
- Ideal Fluid Flow
- Series Expansion Methods
- Method of Fundamental Solutions (MFS)
- Computational Fluid Dynamics (CFD)
- Algorithms
- Numerical Methods
- High Precision Modeling

New Benchmark Potential Flow Problems

Figure 1: This benchmark problem requires modeling ideal fluid flow over two consecutive cylinders. This flow situation incorporates four stagnation points: one stagnation point on each side of both cylinders. The stagnation points are areas of high curvature in the target solution, and these areas are computationally difficult to model.

Figure 2: The CVBEM can be used to model mixed boundary value problems. In this case, zero-flux Neumann boundary conditions are specified on the left edge of the problem domain as well as along the bottom edge of the problem domain (including over the cylinder obstacle). Dirichlet boundary conditions are imposed on the top and right edges of the problem domain. This particular benchmark problem incorporates three stagnation points: one in the corner at the origin and one of either side of the cylinder. Due to the difficulty of modeling these stagnation points, we are interested in assessing the performance of various numerical models with respect to approximating the target potential flow in these areas.

Comparison of Least Squares versus Collocation Methods

Figure 3: Flow chart depicting the least squares approach for determining the coefficients of the CVBEM approximation function. Historically, the CVBEM has been implemented using a collocation approach in which the coefficients of the CVBEM approximation function are determined so as to exactly satisfy the target boundary conditions at specific locations on the target boundary. When collocation is used, a total of $2n$ (where n is the number of terms in the CVBEM approximation function) collocation points are used to determine the coefficients of the CVBEM approximation function. Recently, we have incorporated a least squares approach for determining these coefficients. When the least squares approach is used, all of the available boundary data is used when determining the coefficients of the CVBEM approximation function. Using all of the available boundary data may be one of the keys to achieving highly accurate potential models in generalized three-dimensional settings.

Figure 4: Maximum error comparisons with 500 candidate nodes and 1,000 boundary data points. A primary advantage of the collocation approach is that it guarantees the resulting CVBEM approximation function will satisfy Dirichlet boundary conditions at no less than $2n$ locations on the problem boundary, where n is the number of linearly independent terms in the CVBEM approximation function. Meanwhile, a benefit of the least squares approach is that it incorporates all of the available boundary data when determining the coefficients of the CVBEM approximation function.

Figure 5: Computational error results for the collocation and least squares approaches with regard to modeling the target potential lines with constant values, c, specified in the boxes. The collocation outcomes are shown in color. The least squares outcomes are shown in black. The collocation outcomes tended to achieve lower minimum errors while the least squares outcomes tended to achieve lower maximum errors. The CVBEM models used 500 candidate nodes and 1,000 boundary data points.

Ask today about opportunities to get involved!

Development of New Basis Functions

Figure 6: Domain coloring of the linear combination of two digamma functions: $\psi_{\pi/2}(z-2i)+\psi_{\pi}(z-2)+\psi_{3\pi/2}(z+2i)$.

Figure 7: Linear combination of two digamma functions: $\psi_{\pi/2}(z-2i)+\psi_{\pi}(z-2)+\psi_{3\pi/2}(z+2i)$.

Figure 8: Until now, the CVBEM has always been implemented using a single basis function family to generate the approximation function. Multiple analytic complex variable basis function families have been considered including functions of the form: $(z-z_j)\ln(z-z_j)$, $1/(z-z_j)$, $(z-z_0)^j$, digamma/polygamma functions, and the Hurwitz Zeta function, among others. Recently, we have developed CVBEM approximation functions that use composite basis functions. CVBEM approximation functions built using composite basis functions can use basis functions from any of the aforementioned families as well as any analytic complex variable function, in general. These composite basis functions have resulted in remarkable error reduction when tested on several benchmark potential flow problems.

Recent Publications

Papers

Wilkins, B.D., Hromadka II, T.V., (2021) "Using the digamma function for basis functions in mesh-free computational methods." *Engineering Analysis with Boundary Elements*, **131**, 218-227.

Wilkins, B.D., Hromadka II, T.V., Nevils, W., Siegel, B., Yonzon, P. (2021) "Using a Node Positioning Algorithm to Improve Models of Groundwater Flow Based on Meshless Methods." *The Professional Geologist*.

Wilkins, B.D., Hromadka, T.V., McInvale, J. (2020) "Comparison of two algorithms for locating computational nodes in the Complex Variable Boundary Element Method." *International Journal of Computational Methods and Experimental Measurements*, **8**(4), 289-315. doi: 10.2495/CMEM-V8-N4-289-315

Doyle, A., Nelson, A., Yuan, M., Demoes, N.J., Wilkins, B.D., Grubaugh, K.E., Hromadka II, T.V., (2020) "Advances in the greedy optimization algorithm for nodes and collocation points using the method of fundamental solutions." *Engineering Analysis with Boundary Elements*, **111**, 148-153.

Grubaugh, K.E., Wilkins, B.D., Hromadka II, T.V., (2020) "Using Taylor Series to Assess Goodness of Groundwater Models." *The Professional Geologist*.

Presentations

Wilkins, B.D., Hromadka II, T.V., (2021) "Using Series Basis Functions with the Complex Variable Boundary Element Method." *Presented at the Royal Meteorological Society Atmospheric Science Conference*, July 6, 2021.

Wilkins, B.D., Nevils, W., Yonzon, P., Hromadka II, T.V., (2021) "Demonstration of a New Nested Candidate Node Domain Reduction Method for the CVBEM." *Presented at the Society for Industrial and Applied Mathematics Computational Science and Engineering Conference*, March 4, 2021.

Wilkins, B.D., Hromadka II, T.V., (2020) "Advancement in Node Positioning Algorithms for the Complex Variable Boundary Element Method." *Presented at the Joint United States Military Academy/Army Research Laboratory Conference*, November 20, 2020.

About the CEM Group

The Computational Engineering Mathematics (CEM) Group at West Point is interested in improving and applying the Complex Variable Boundary Boundary Element Method (CVBEM) to partial differential equations (PDEs) of the Laplace and Poisson type. The current research efforts include many different opportunities such as modeling potential problems, developing computational error analysis tools, creating new algorithms, solving potential problems in three and higher spatial dimensions, and also modeling time-dependent problems. We are also active in publishing and presenting our research.

Department Leadership

COL Tina Hartley
Professor and Department Head

COL Michael Scioletti
Associate Professor and Deputy Department Head

CEM Group Leadership

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Abstract

- Many meshless computational approaches use collocation of the model at collocation points (usually located on the boundary) in order to develop a system of linear equations which are then solved simultaneously by the usual matrix-solving methods.
- The Complex Variable Boundary Element Method (CVBEM) is a numerical technique for solving partial differential equations of the Laplace and related types. The CVBEM is often implemented using collocation, although a least squares implementation has recently been examined.
- We will examine the underpinnings of the collocation approach as used with the CVBEM. The Identity Theorem of complex variables will justify the use of collocation techniques as the number of linearly-independent terms in the CVBEM approximation function increases.

The Identity Theorem

- Why do collocation methods work? The answer may have to do with the Identity Theorem of complex variables:
  - Given functions f and g analytic on a domain D, if $f = g$ on some $S \subseteq D$, where S has an accumulation point, then $f = g$ on D.
  - Thus, an analytic function is completely determined by its values on a single open neighborhood in D.

The CVBEM develops an analytic complex variable approximation function

- To work with the Identity Theorem, one may consider that the CVBEM develops a two-dimensional complex function that is analytic over the problem domain.
  - Thus, both the real and imaginary parts of the CVBEM approximation function satisfy Laplace's equation in the problem domain.
  - Furthermore, the difference between the real part of the CVBEM approximation function and the target potential function (that is, the error function) satisfies Laplace's equation in the problem domain.

- This analytic property of the CVBEM approximation function can be used to solve other partial differential equations such as the diffusion equation and wave equation, among others, where the solution approach is to develop a computational model of the steady-state conditions using the CVBEM and then solve the remaining components of the PDE using a particular solution approach.
- For related work on the diffusion and wave equations, see:

  - Wilkins, B.D., Greenberg, J., Redmond, B., Baily, A., Flowerday, N., Kratch, A., Hromadka, T.V., Boucher, R., McInvale, H.D., Horton, S. (2017) "An Unsteady Two-Dimensional Complex Variable Boundary Element Method." *SCIRP Applied Mathematics*, **8**, 878-891. doi: 10.4236/am.2017.86069.
  - Wilkins, B.D., Hromadka, T.V., Boucher, R. (2017) "A Conceptual Numerical Model of the Wave Equation Using the Complex Variable Boundary Element Method." *SCIRP Applied Mathematics*, **8**, 724-735. doi: 10.4236/am.2017.85057.

The CVBEM error function

- The error of the CVBEM approximation function is measured as the difference between the approximation function and the target potential function:
- Let:
  - $\hat{\omega}(z) = \hat{\phi}(x, y) + i\hat{\psi}(x, y) \equiv$ the CVBEM approximation function
  - $\phi(x, y) \equiv$ the target potential function
- The error function is defined as follows:
  - $\delta(x, y) = \hat{\phi}(x, y) - \phi(x, y)$
  - Since ϕ and $\hat{\phi}$ are both potential functions, δ is also a potential function
- At collocation points, we have $\hat{\phi}(x, y) = \phi(x, y)$ and, thus, $\delta(x, y) = 0$.

The CVBEM is implemented to satisfy domain requirements for using the Identity Theorem as established by Walsh [1]

Where does the Identity Theorem apply?

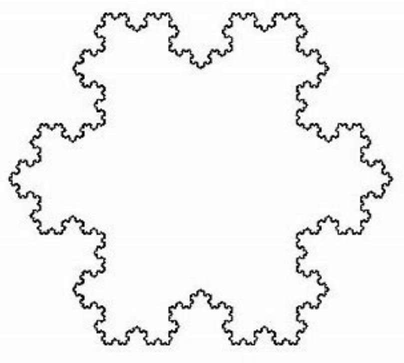
"In the study of the possibility of approximation to a given function the fundamental theorems were given by Runge in his classical paper of 1885. These theorems are of the greatest importance in the present essay; we omit the proofs, however, because they are to be found in many standard works. Moreover, the method of Hilbert, which we shall consider later in some detail, also includes a proof of Runge's theorems. We give the name Runge's first theorem to the following, although Runge's own statement was somewhat different in content: If the function $f(z)$ is analytic in a closed Jordan region G, then in that closed Jordan region $f(z)$ can be uniformly approximated as closely as desired by a polynomial in z. Runge's theorem is more readily proved for the case of a convex region than for the general case. The two concepts, possibility of uniform approximation by a polynomial with an arbitrary small error, and uniform expansion in a series or sequence of polynomials, are of course, equivalent (without reference to the present situation), in the sense that each implies the other directly. Runge's theorem specifies that the region under consideration shall be a Jordan region; it is essential that the region not be an arbitrary simply-connected region…"

--- [1] J.L. Walsh, 1935.

- For most engineering problems, a Jordan region and a simply connected region are identical.
- However, when examining specific domain issues involving cracks and similar type domain irregularities, attention needs to be paid in the candidate node positioning process so as to preserve the subtleties involved.
  - In the context of the CVBEM, this means handling branch cuts, such as by rotating them away from the problem domain.

An interesting example of a Jordan region

Background of the Jordan curve theorem



The Jordan curve theorem was "..first proposed in 1887 by French mathematician Camille Jordan, that any simple closed curve—that is, a continuous closed curve that does not cross itself (now known as a Jordan curve)—divides the plane into exactly two regions, one inside the curve and one outside, such that a path from a point in one region to a point in the other region must pass through the curve...." They continue that "... This obvious-sounding theorem proved deceptively difficult to verify. Indeed, Jordan's proof turned out to be flawed, and the first valid proof was given by American mathematician Oswald Veblen in 1905. One complication for proving the theorem involved the existence of continuous but nowhere differentiable curves. (The best-known example of such a curve is the Koch snowflake, first described by Swedish mathematician Niels Fabian Helge von Koch in 1906.)"

--- [2] Encyclopedia Britannica

- The Identity Theorem provides a backdrop for justifying the use of the collocation methods commonly used in computational modeling.
- A review of several of these proofs shows that for many of these proofs, there is significant reliance on series expansions, particularly Taylor series, that are expanded about the computational node locations.
- Because complex analytic functions necessarily have Taylor series expansions for points within the domain of function definition, the cited work of [1] J.L. Walsh (1935) again can be applied to examine such approximations as complex polynomials selected to achieve specified geometric computational error bounds.

Acknowledgements

- The Computational Engineering Mathematics Group wish to thank the United States Military Academy for their continued support and encouragement of CEM Group research efforts.
- Special thanks are paid to the leadership of the Department of Mathematical Sciences, and the department itself, for allowing this research to happen.

References

- Cited in the presentation:

  - 1. J.L. Walsh, Approximation by polynomials in the complex domain; Memorial des sciences mathematiques, fascicule 73, (1935).
  - 2. Encyclopedia Britannica: https://www.britannica.com/science/Jordan-curve-theorem

- Other interesting and relevant works:

  - 3. J.L. Walsh, Interpolation with Harmonic and Complex Polynomials to Boundary Values; Journal of Mathematics and Mechanics, (9), No. 2, (1960), pages 167-192.
  - 4. J.H. Curtiss, Interpolation by Harmonic Polynomials; Mathematical Sciences Directorate, Air Force Office of Scientific Research, Washington, D.C.; (Dec. 1, 1961), AFOSR 1737.
  - 5. T.V. Hromadka II & R.J. Whitley, Foundations of the Complex Variable Boundary Element Method; Springer Briefs in Applied Sciences and Technology, (2014).

Questions?