An Examination of the Utility of the Identity Theorem in Justifying Collocation Computational Methods for Solving Engineering Problems

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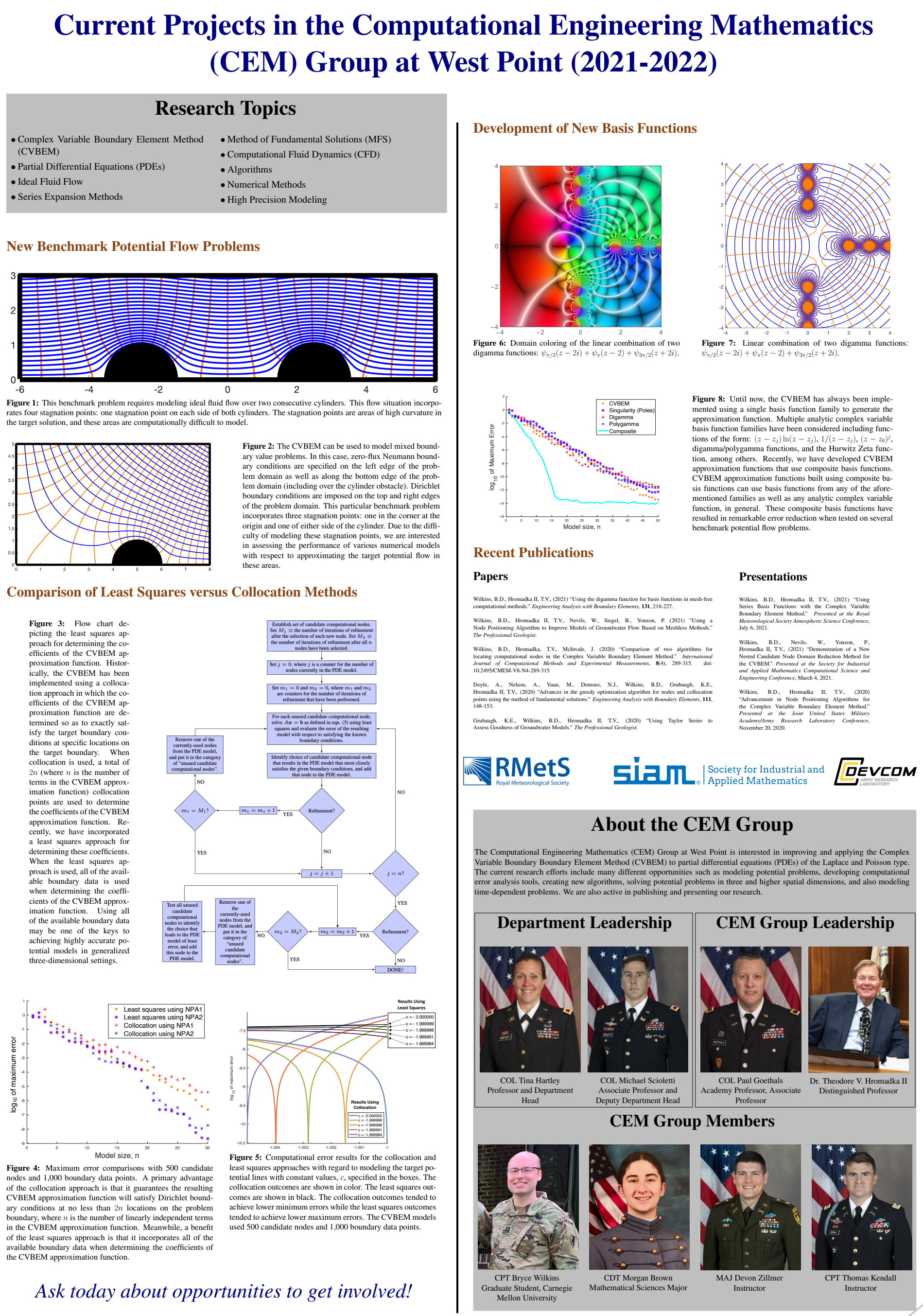


Abstract

- Many meshless computational approaches use collocation of the model at collocation points (usually located on the boundary) in order to develop a system of linear equations which are then solved simultaneously by the usual matrix-solving methods.
- The Complex Variable Boundary Element Method (CVBEM) is a numerical technique for solving partial differential equations of the Laplace and related types. The CVBEM is often implemented using collocation, although a least squares implementation has recently been examined.
- We will examine the underpinnings of the collocation approach as used with the CVBEM. The Identity Theorem of complex variables will justify the use of collocation techniques as the number of linearly-independent terms in the CVBEM approximation function increases.

The Identity Theorem

- Why do collocation methods work? The answer may have to do with the Identity Theorem of complex variables:
  - Given functions f and g analytic on a domain D, if $f = g$ on some $S \subseteq D$, where S has an accumulation point, then $f = g$ on D.
  - Thus, an analytic function is completely determined by its values on a single open neighborhood in D.

The CVBEM develops an analytic complex variable approximation function

- To work with the Identity Theorem, one may consider that the CVBEM develops a two-dimensional complex function that is analytic over the problem domain.
  - Thus, both the real and imaginary parts of the CVBEM approximation function satisfy Laplace's equation in the problem domain.
  - Furthermore, the difference between the real part of the CVBEM approximation function and the target potential function (that is, the error function) satisfies Laplace's equation in the problem domain.

- This analytic property of the CVBEM approximation function can be used to solve other partial differential equations such as the diffusion equation and wave equation, among others, where the solution approach is to develop a computational model of the steady-state conditions using the CVBEM and then solve the remaining components of the PDE using a particular solution approach.
- For related work on the diffusion and wave equations, see:

  - Wilkins, B.D., Greenberg, J., Redmond, B., Baily, A., Flowerday, N., Kratch, A., Hromadka, T.V., Boucher, R., McInvale, H.D., Horton, S. (2017) "An Unsteady Two-Dimensional Complex Variable Boundary Element Method." *SCIRP Applied Mathematics*, **8**, 878-891. doi: 10.4236/am.2017.86069.
  - Wilkins, B.D., Hromadka, T.V., Boucher, R. (2017) "A Conceptual Numerical Model of the Wave Equation Using the Complex Variable Boundary Element Method." *SCIRP Applied Mathematics*, **8**, 724-735. doi: 10.4236/am.2017.85057.

The CVBEM error function

- The error of the CVBEM approximation function is measured as the difference between the approximation function and the target potential function:
- Let:
  - $\hat{\omega}(z) = \hat{\phi}(x, y) + i\hat{\psi}(x, y) \equiv$ the CVBEM approximation function
  - $\phi(x, y) \equiv$ the target potential function
- The error function is defined as follows:
  - $\delta(x, y) = \hat{\phi}(x, y) - \phi(x, y)$
  - Since ϕ and $\hat{\phi}$ are both potential functions, δ is also a potential function
- At collocation points, we have $\hat{\phi}(x, y) = \phi(x, y)$ and, thus, $\delta(x, y) = 0$.

The CVBEM is implemented to satisfy domain requirements for using the Identity Theorem as established by Walsh [1]

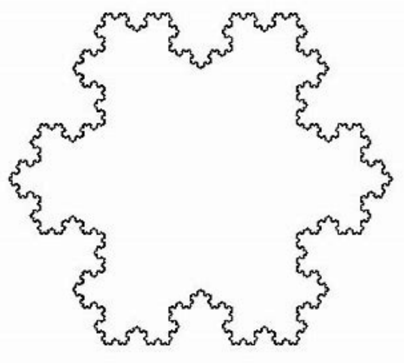
Where does the Identity Theorem apply?

"In the study of the possibility of approximation to a given function the fundamental theorems were given by Runge in his classical paper of 1885. These theorems are of the greatest importance in the present essay; we omit the proofs, however, because they are to be found in many standard works. Moreover, the method of Hilbert, which we shall consider later in some detail, also includes a proof of Runge's theorems. We give the name Runge's first theorem to the following, although Runge's own statement was somewhat different in content: If the function $f(z)$ is analytic in a closed Jordan region G, then in that closed Jordan region $f(z)$ can be uniformly approximated as closely as desired by a polynomial in z. Runge's theorem is more readily proved for the case of a convex region than for the general case. The two concepts, possibility of uniform approximation by a polynomial with an arbitrary small error, and uniform expansion in a series or sequence of polynomials, are of course, equivalent (without reference to the present situation), in the sense that each implies the other directly. Runge's theorem specifies that the region under consideration shall be a Jordan region; it is essential that the region not be an arbitrary simply-connected region…"

--- [1] J.L. Walsh, 1935.

- For most engineering problems, a Jordan region and a simply connected region are identical.
- However, when examining specific domain issues involving cracks and similar type domain irregularities, attention needs to be paid in the candidate node positioning process so as to preserve the subtleties involved.
  - In the context of the CVBEM, this means handling branch cuts, such as by rotating them away from the problem domain.

An interesting example of a Jordan region



Background of the Jordan curve theorem

The Jordan curve theorem was "..first proposed in 1887 by French mathematician Camille Jordan, that any simple closed curve—that is, a continuous closed curve that does not cross itself (now known as a Jordan curve)—divides the plane into exactly two regions, one inside the curve and one outside, such that a path from a point in one region to a point in the other region must pass through the curve...." They continue that "... This obvious-sounding theorem proved deceptively difficult to verify. Indeed, Jordan's proof turned out to be flawed, and the first valid proof was given by American mathematician Oswald Veblen in 1905. One complication for proving the theorem involved the existence of continuous but nowhere differentiable curves. (The best-known example of such a curve is the Koch snowflake, first described by Swedish mathematician Niels Fabian Helge von Koch in 1906.)"

--- [2] Encyclopedia Britannica

- The Identity Theorem provides a backdrop for justifying the use of the collocation methods commonly used in computational modeling.
- A review of several of these proofs shows that for many of these proofs, there is significant reliance on series expansions, particularly Taylor series, that are expanded about the computational node locations.
- Because complex analytic functions necessarily have Taylor series expansions for points within the domain of function definition, the cited work of [1] J.L. Walsh (1935) again can be applied to examine such approximations as complex polynomials selected to achieve specified geometric computational error bounds.

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References

- Cited in the presentation:

  - 1. J.L. Walsh, Approximation by polynomials in the complex domain; Memorial des sciences mathematiques, fascicule 73, (1935).
  - 2. Encyclopedia Britannica: https://www.britannica.com/science/Jordan-curve-theorem

- Other interesting and relevant works:

  - 3. J.L. Walsh, Interpolation with Harmonic and Complex Polynomials to Boundary Values; Journal of Mathematics and Mechanics, (9), No. 2, (1960), pages 167-192.
  - 4. J.H. Curtiss, Interpolation by Harmonic Polynomials; Mathematical Sciences Directorate, Air Force Office of Scientific Research, Washington, D.C.; (Dec. 1, 1961), AFOSR 1737.
  - 5. T.V. Hromadka II & R.J. Whitley, Foundations of the Complex Variable Boundary Element Method; Springer Briefs in Applied Sciences and Technology, (2014).

Questions?