Demonstration of a New Nested Candidate Node Domain Reduction Method for the CVBEM

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The General CVBEM Approximation Function

- The CVBEM approximation function is a linear combination of complex variable functions that are analytic within a given problem domain, Ω:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j g_j(z), \quad z \in \Omega, \tag{1}$$

  where

  - $c_j = \alpha_j + i\beta_j$ are complex coefficients (note: 2 real coefficients),
  - $g_j(z)$ are analytic complex variable basis functions,
  - n is the number of basis functions being used in the approximation

- Each term in the approximation function corresponds to one node and two collocation points.

Problem Formulation

The Cauchy integral formula:

$$\omega(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{\omega(\zeta) d\zeta}{\zeta - z}. \qquad (2)$$

Integration of (2) results in the following sum, which is known as the CVBEM approximation function:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j (z - z_j) \ln(z - z_j). \quad (3)$$

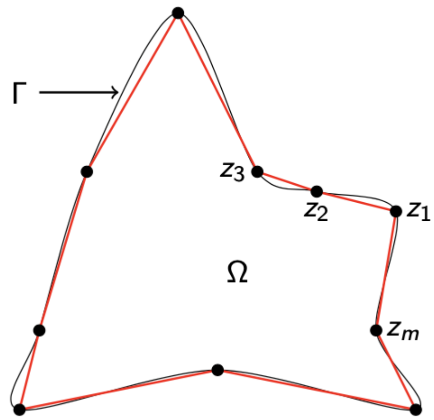


Figure: The boundary is discretized using a set of interpolation points. The interpolation points can be connected using straight line segments to create a polygonal representation.

The CVBEM Modeling Procedure

- The points z_j are the branch points of the logarithm (with branch cuts rotated) and are often referred to as computational nodes.
- The CVBEM can be viewed as a procedure for generating basis functions, such as in (4).
- The generated basis functions are used as inputs for the NPAs.

The CVBEM approximation function is as follows:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j (z - z_j) \ln(z - z_j). \qquad (4)$$

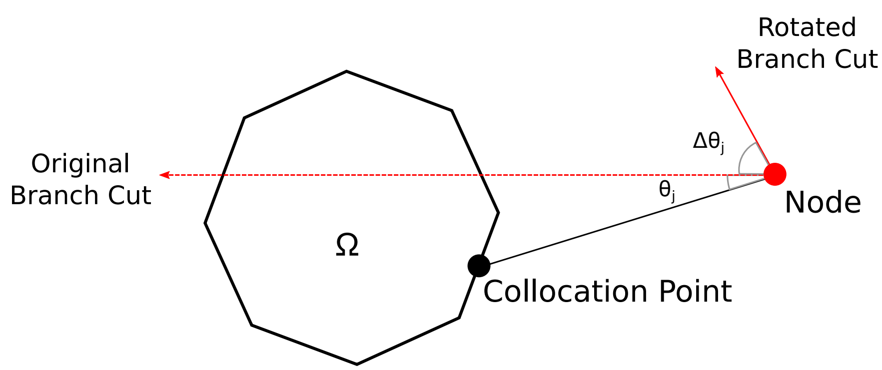


Figure: Rotation of a typical branch cut. The branch point of the basis function corresponds to a node for the NPA.

NPA0

Hromadka II, T.V. & Guymon, G.L., A Complex Variable Boundary Element Method: Development. *International Journal for Numerical Methods in Engineering*, **20**, pp. 25-37, 1984.

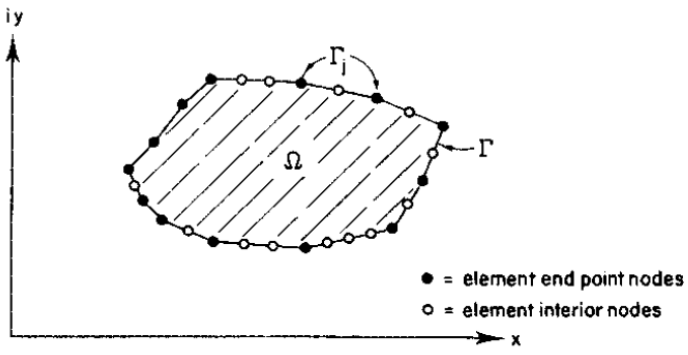


Figure: Originally, nodes were located on the problem boundary.

NPA0.5

Johnson, A.N. & Hromadka II, T.V., Modeling mixed boundary conditions in a Hilbert space with the complex variable boundary element method (CVBEM). *MethodsX*, **2**, pp. 292-305, 2015.

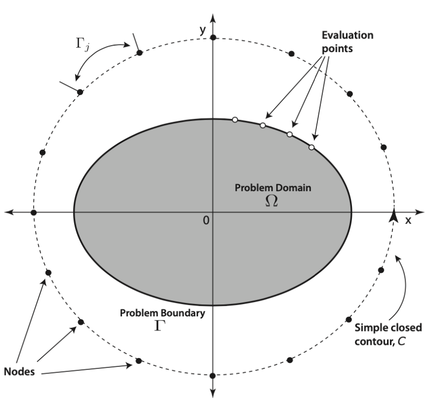


Figure: Next, nodes were located in a geometric pattern in the exterior of $\Omega \cup \partial\Omega$.

NPA1

Demoes, N.J., Bann, G.T., Wilkins, B.D., Grubaugh, K.E. & Hromadka II, T.V., Optimization Algorithm for Locating Computational Nodal Points in the Method of Fundamental Solutions to Improve Computational Accuracy in Geosciences Modeling. *The Professional Geologist*, pp. 6-12, 2019.

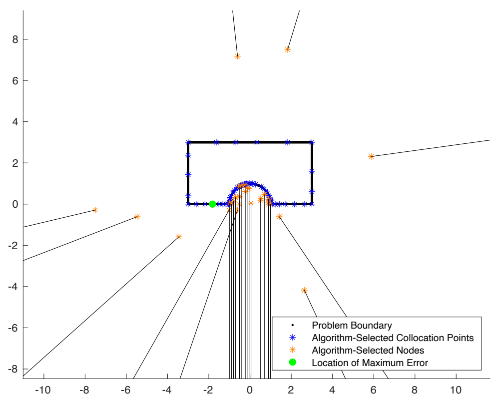


Figure: Originally, nodes were located on the problem boundary.

NPA2

Wilkins, B.D., Hromadka II, T.V. & McInvale, J., Comparison of Two Algorithms for Locating Nodes in the Complex Variable Boundary Element Method (CVBEM). *International Journal of Computational Methods and Experimental Measurements*, 2020.

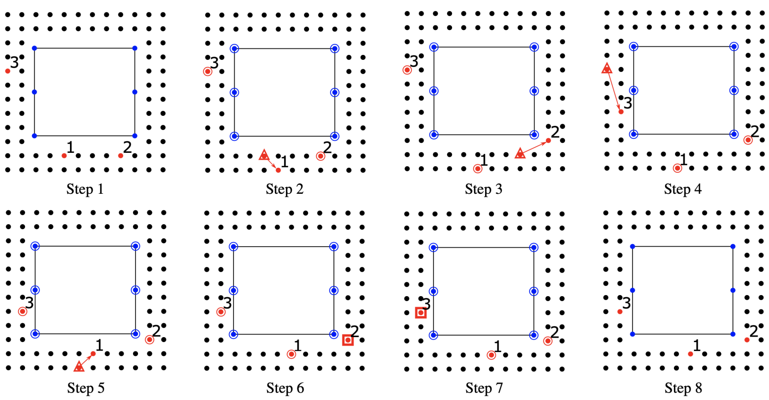


Figure: Next, nodes were located in a geometric pattern in the exterior of $\Omega \cup \partial\Omega$.

Demonstration - Problem 1

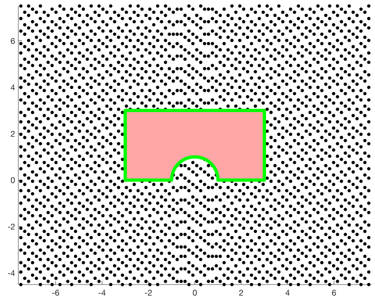


Figure: Iteration 1

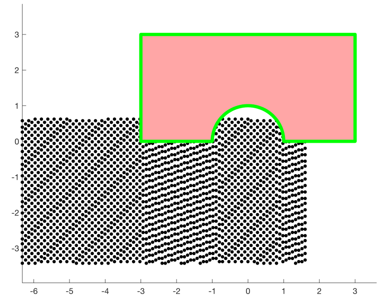


Figure: Iteration 2

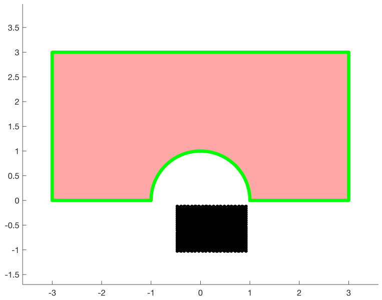


Figure: Iteration 5

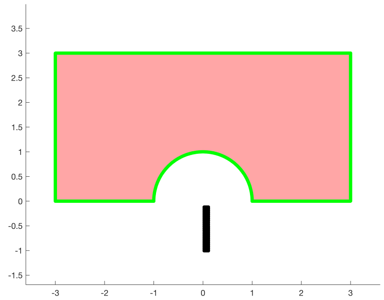


Figure: Iteration 10

Demonstration - Problem 2

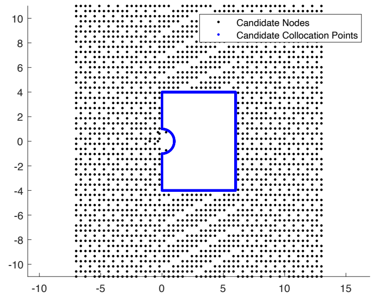


Figure: Iteration 1

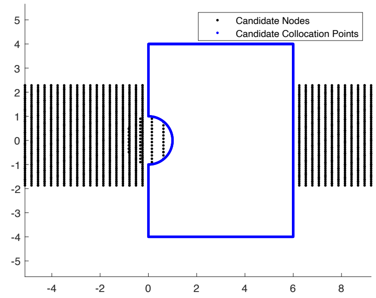


Figure: Iteration 2

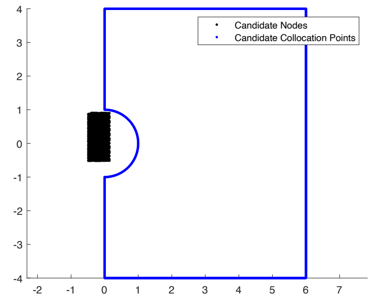


Figure: Iteration 3

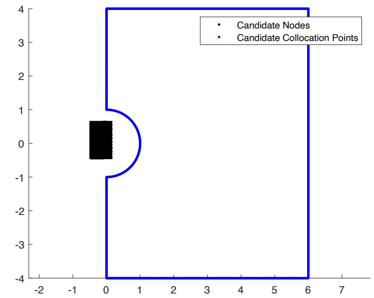


Figure: Iteration 4

Demonstration - Problem 3

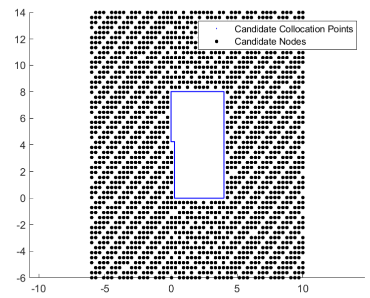


Figure: Iteration 1

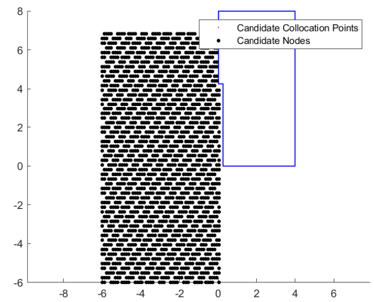


Figure: Iteration 2

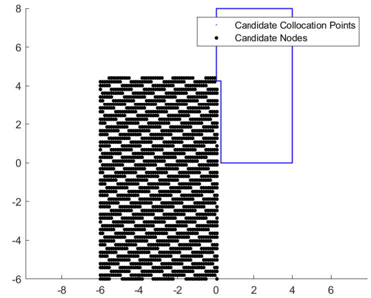


Figure: Iteration 3

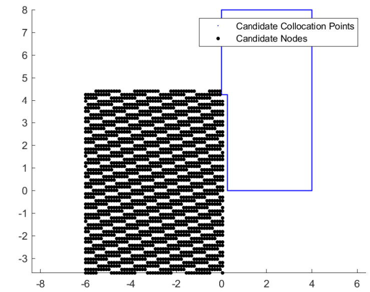


Figure: Iteration 4

Candidate Node Space Dimensions and Maximum Error Results, $n = 10$ Terms

Iteration	Problem 1		Problem 2		Problem 3	
	x-dim	y-dim	x-dim	y-dim	x-dim	y-dim
1	[-7.50,7.50]	[-4.50,7.50]	[-7.00,13.00]	[-11.00,11.00]	[-6.00,10.00]	[-6.00, 14.00]
2	[-6.32,1.60]	[-3.42,0.62]	[-5.11,9.22]	[-1.86,2.28]	[-6.00, 0.03]	[-6.00, 6.83]
3	[-1.15,0.97]	[-1.51,0.48]	[-0.48,0.14]	[-0.51,0.89]	[-6.00, 0.03]	[-6.00, 4.40]
4	[-0.48,0.95]	[-1.15,-0.11]	[-0.47,0.14]	[-0.43,0.63]	[-6.00, 0.03]	[-3.64, 4.40]
5	[-0.46,0.92]	[-1.02,-0.11]	[-0.14,0.14]	[-0.41,0.49]	[-6.00, 0.03]	[-3.64, 3.95]

Table: Dimensions of the candidate node space in each iteration of the nested NPA procedure.

Iteration	Problem 1	Problem 2	Problem 3
1	1.9199e-02	1.3552e-02	7.0559e-02
2	5.2030e-03	1.2578e-04	6.8413e-03
3	2.1795e-04	6.0301e-06	2.0753e-02
4	2.4750e-04	2.3947e-06	9.6491e-04
5	1.2136e-04	1.8089e-07	6.0232e-04

Table: Maximum error results for the three demonstration problems.

Error Comparison Using Nested Procedure as Primer

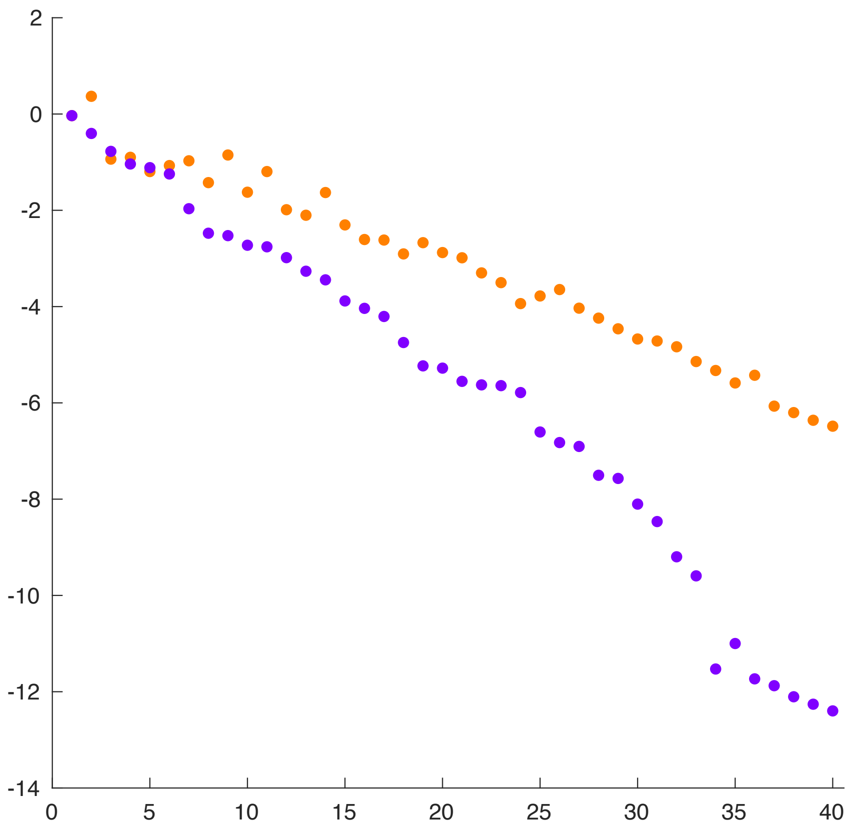
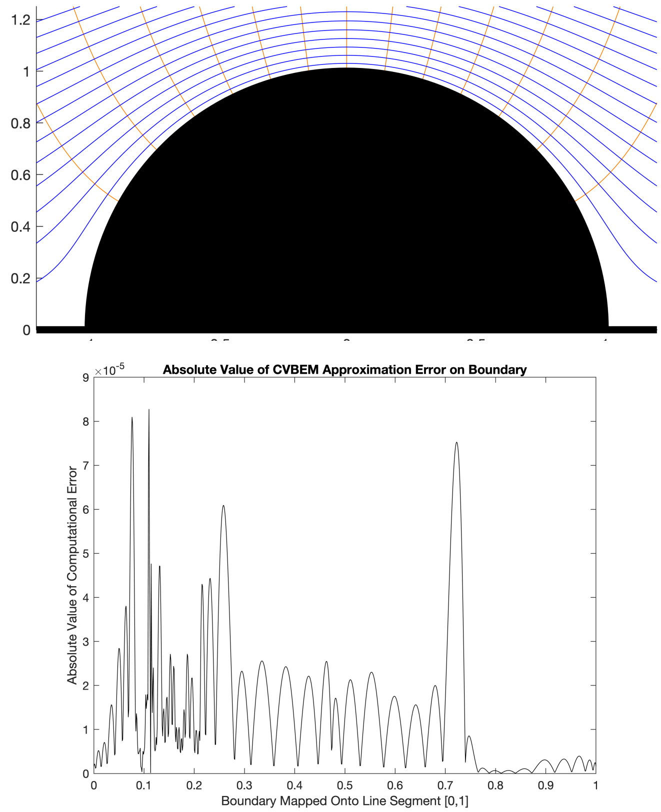


Figure: $\log_{10}$ of maximum error. Orange points indicate use of NPA1. Blue points indicate use of nested NPA.

Flow Nets and Other Outcomes



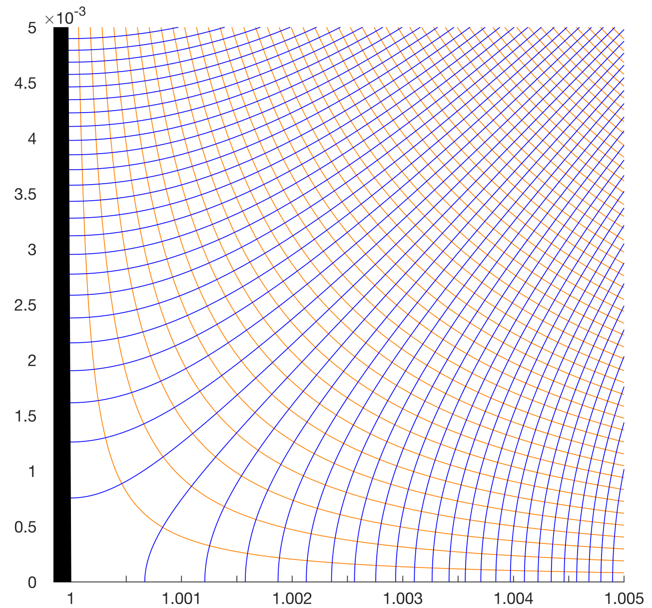


Figure: (Top Left) approximation near obstacle. (Bottom Left) absolute error evaluated on boundary. (Above) approximation near the right stagnation point.

Next Generation

Currently under development...

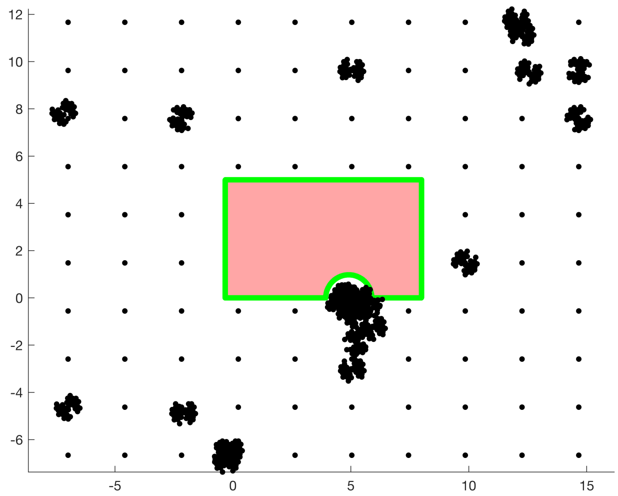


Figure: The latest NPA allows for variable candidate node density with increased node density in possible areas of interest.

Questions