Modeling the Flow Around a Constrained Circular Obstruction

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Introduction

Recent research considers comparing the accuracy of modeling PDEs with approaches such as the Finite Element Method (FEM) and the Complex Variable Boundary Element Method (CVBEM). To assess the performance of each approach, this project uses an application of groundwater flow as a proof of concept. It is assumed that a constrained circular obstruction has formed a barrier to groundwater flow, as indicated by the black circle in Figure 1. The figure also shows examples of potential contamination source points (blue triangles). One source point is a Leaking Underground Storage Tank (LUST), as indicated by the red triangle. The green dot represents the hypothetical location where a contaminant has been detected. The purpose of this study is to determine the source tank of the detected contaminant by modeling the groundwater flow regime and back-tracing the associated streamline that goes through the location of detection.

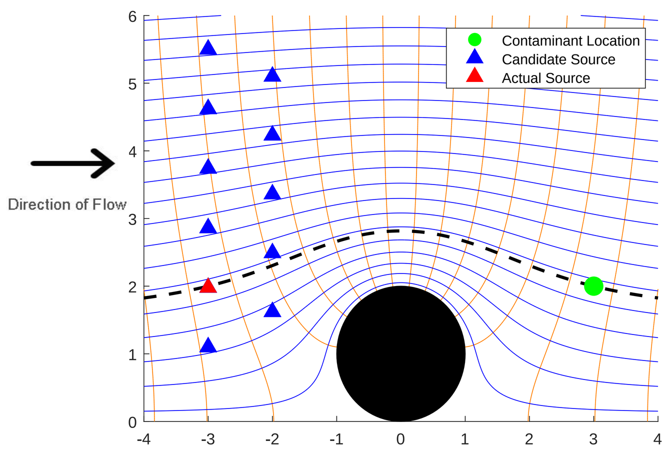

Figure 1: Analytic solution of the groundwater flow situation. Potential isocontours are shown as orange lines, and stream isocontours (stream lines) are shown as blue lines. The analytic solution is given by $\omega(z) = \pi \coth(\frac{\pi}{z})$, as stated in [1].

CVBEM Methodology

The CVBEM is a numerical solver for PDEs of the Laplace and related types that is derived from numerical integration of the Cauchy integral equation:

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

- The area of interest is assumed to be simply connected, with a simple closed boundary, denoted Γ.
- The boundary is discretized using a set of interpolation points.
- When straight line segments are used to discretize the boundary of the problem domain, the numerical integration of the Cauchy integral formula results in the following sum, which is known as the CVBEM approximation function [2]:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j (z - z_j) \ln(z - z_j)$$

The points z_j are branch points of the basis functions and are referred to as modeling or computational nodes.

Node Positioning Algorithms

Originally, the CVBEM placed modeling nodes, z_j, in a regular pattern around the problem domain without concern for finding the locations that provide minimal or reduced computational error. This method is the simplest and carries the least computational burden, but it is also the least accurate.

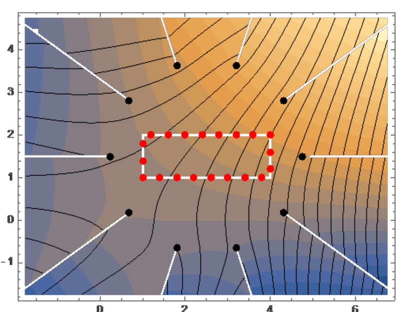

Figure 2: An early CVBEM approximation where computational nodes (black) are simply arranged in a circle around the problem domain and collocation points (red) are uniformly spaced along the problem boundary. [3]

- The improvement to the un-optimized model is the Node Positioning Algorithm 1 (NPA1). The algorithm begins by creating an initial distribution of candidate node positions and candidate collocation points. Because of the maximum modulus principle, we know the maximum error of the CVBEM approximation function occurs on the problem boundary. Consequently, as each node is added to the model, the algorithm places two collocation points on the problem boundary where the two greatest local maxima of the error function occur, and then evaluates each candidate node to find the CVBEM approximation function of least error.
- The second improvement is the Node Positioning Algorithm 2 (NPA2). NPA2 adds a refinement algorithm to the CVBEM procedure, which is utilized as each new node is added, as well as after all the nodes have been selected. The refinement procedure has a monotonically non-increasing effect on the maximum error of the approximation function because at each application of refinement, there is either no change in error and the current model is kept, or a node is exchanged to a different location, resulting in decreased error. Research shows the NPA2 can improve accuracy by at least an order of magnitude—and up to four orders of magnitude—when the degrees of freedom are greater than ten [4].

FEM Methodology

The FEM is a popular domain discretization method that is often used as a modeling procedure instead of the CVBEM. This technique begins by creating a mesh with modeling nodes and elements to discretize the problem domain. This project examined two trials with the FEM: one with a simple mesh and one with a much more refined mesh:

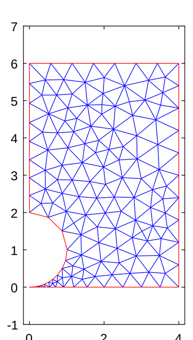

Figure 3: Trial 1 FEM mesh

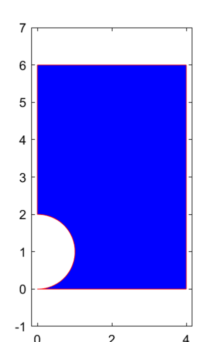

Figure 4: Trial 2 FEM mesh

Notice the FEM mesh in Figure 4 is much more refined. Since refined meshes tend to produce more accurate FEM approximations, we will use this mesh in order to graphically compare the FEM output to the analytic and CVBEM outputs. We will compare the modeling outputs of the various approximation methods at several points of interest within the problem domain.

Results

CVBEM Problem Details and Results

Parameters	Trial 1	Trial 2
Nodes in model:	20	40
Length:	4	4
Height:	6	6
Number of candidate nodes for optimization algorithm:	500	500
Number of candidate collocation points for optimization algorithm:	1000	1000

Table 1: Parameter Details

Method	Trial 1	Trial 2
NPA1:	6.26×10^{-4}	5.45×10^{-8}
NPA2:	3.58×10^{-6}	7.80×10^{-9}

Table 2: A comparison of the maximum absolute error between the NPA1 and NPA2. As expected, a higher-node model performs better than a lower-node model, and the NPA2 performs better than the NPA1.

CVBEM Graphical Results

The following figures compare the analytic solution to a 40-node CVBEM approximation in the domain. In these figures, the CVBEM isocontour outcomes overlap and are seen as dashed red and black lines.

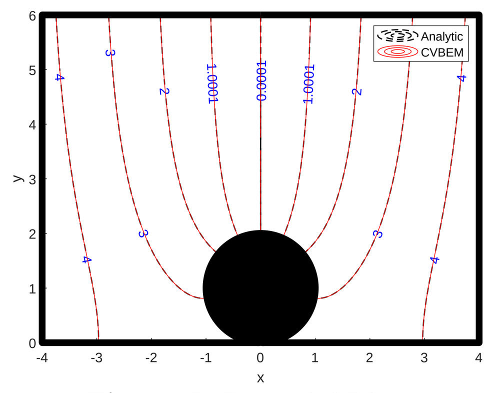

Figure 5: Potential Lines

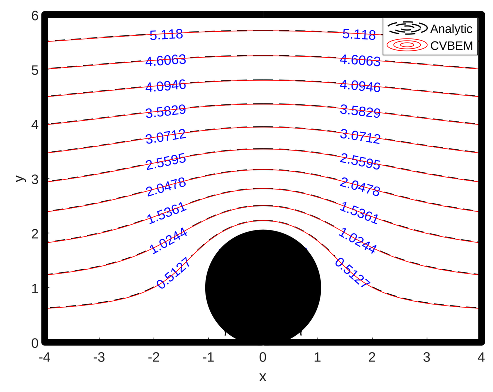

Figure 6: Streamlines

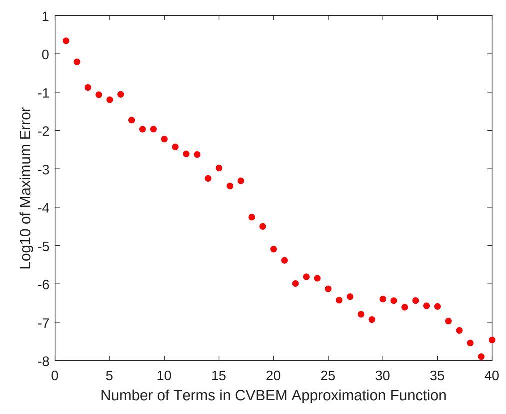

Figure 7: Maximum error of each CVBEM model of n nodes up to $n = 40$ with the NPA2.

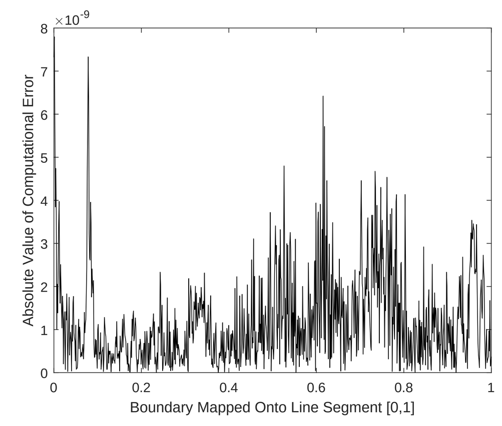

Figure 8: Absolute error on the boundary

FEM Graphical Results

The following figures compare the analytic solution to an FEM approximation in the domain. In these figures, the dashed black lines represent the analytic solution while the solid purple lines represent the FEM approximation. Notice the FEM isocontours are much less accurate when plotting streamlines.

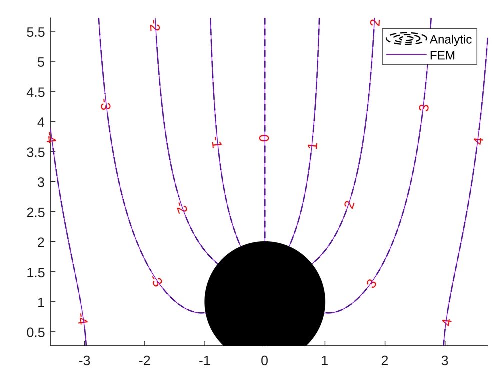

Figure 9: Potential Lines

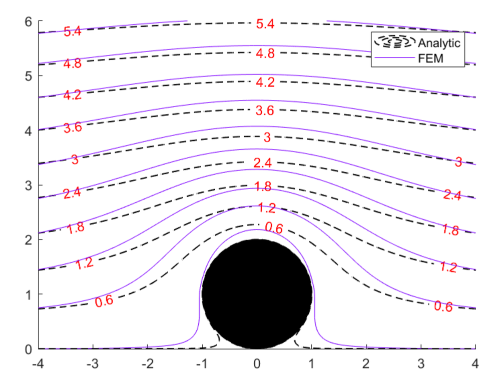

Figure 10: Streamlines

Comparison Between CVBEM and FEM Approximations

The following figures demonstrate the results of back-tracing the contamination using both the CVBEM and FEM approximations. In this situation, the dashed black lines represent the analytic solution, the red lines in Figure 11 represent the CVBEM approximation, and the purple lines in Figure 12 represent the FEM approximation. Notice the FEM model in Figure 12 would incorrectly predict the contamination source, as depicted by the magenta line.

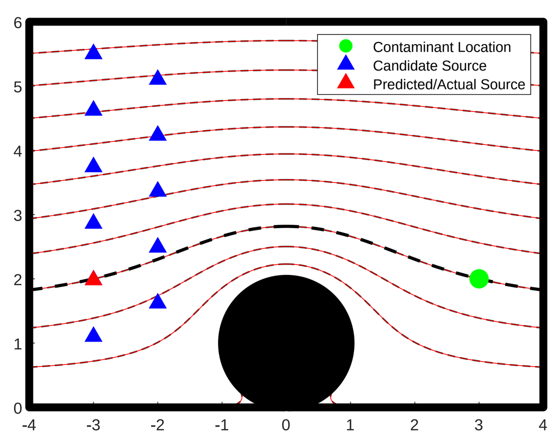

Figure 11: CVBEM

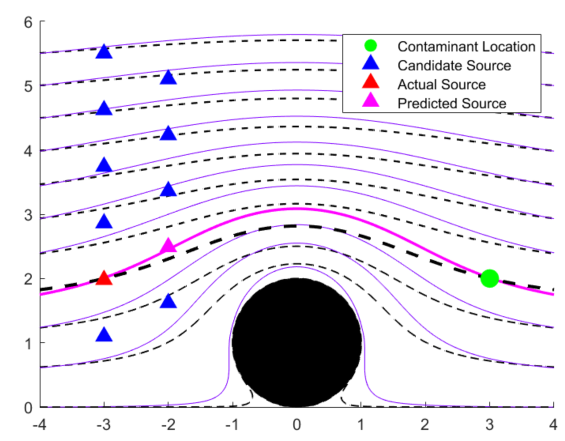

Figure 12: FEM

Conclusions

In this poster, we compared the outcomes of two numerical methods; namely, the coupled CVBEM/NPA2 methodology and the finite element method, to solve an important benchmark problem related to groundwater flow and contamination source detection. The CVBEM produced very good approximations of a typical groundwater flow problem such as potential flow over a constrained circular obstruction. In fact, the CVBEM demonstrated 7.80×10^{-9} absolute maximum error for a 40 node model with the NPA2. On the other hand, although the potential contours of the FEM model compare well to the potential contours of the analytic solution, the FEM model does not generate similarly accurate streamlines, which results in a departure of the trajectory of the approximate streamlines from the target streamlines as seen in Figure 12. In this case study, this causes the wrong source point to be identified by the FEM model. However, the CVBEM correctly identifies the LUST, which is the source of the detected contamination.

References

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Acknowledgements

This research was supported by the consulting firm Hromadka & Associates.