Modeling the Flow Around a Fixed Barrier

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Introduction

This project seeks to compare the accuracy of the Complex Variable Boundary Element Method (CVBEM) and the Finite Element Method (FEM) in generating numerical solutions to PDEs. To provide a basis for such comparison, this project considers a test problem modeling a groundwater flow regime. Figure 1 shows the problem geometry superimposed on an analytically generated flownet. Note the large rectangular barrier, the green circle marking the location of a detected contaminant, and the candidate contamination sources (blue triangles). One point represents a Leaking Underground Storage Tank (LUST), the true source of the detected contamination, and is represented by a red triangle. This investigation will assess the performance of the CVBEM and FEM by comparing the two methods' accuracy in predicting the true source of the detected contaminant. This is accomplished by back-tracing the approximated streamline intersecting the detection location to the set of candidate sources.

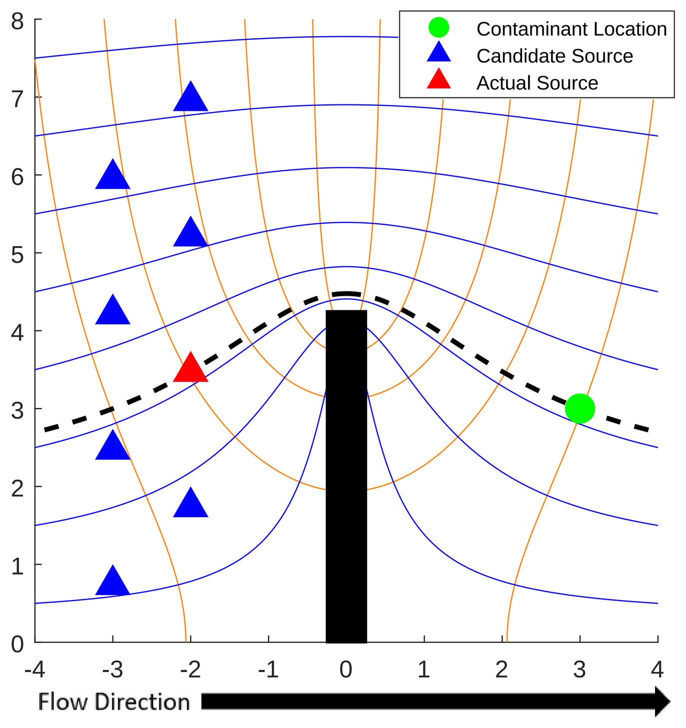

Figure 1: Analytic solution of the groundwater flow situation. Potential isocontours are shown as orange lines, and stream isocontours (streamlines) are shown as blue lines. The analytic solution is given by $\omega(z) = \sqrt{z^2 + 16}$, as stated in [1].

CVBEM Methodology

The CVBEM is a well known numerical PDE solver derived from the numerical integration of the Cauchy integral equation.

$$\omega(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta - z}$$

When applying the CVBEM, it is assumed that the area of interest is simply connected and bounded by a simple closed contour, Γ. Γ is then discretized into a set of interpolation points connected by line segments. When these conditions are satisfied, numerical integration of the Cauchy integral formula results in the CVBEM approximation function [2]:

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j (z - z_j) \ln(z - z_j),$$

where z_j denotes the branch points of the basis functions and are known as computational nodes.

Node Positioning Algorithms

Until recently, the CVBEM placed computational nodes, z_j, at regular intervals along a predetermined pattern. This method was geometrically convenient but inaccurate. Future iterations of the CVBEM have attempted to increase precision through the introduction of node positioning algorithms (NPA).

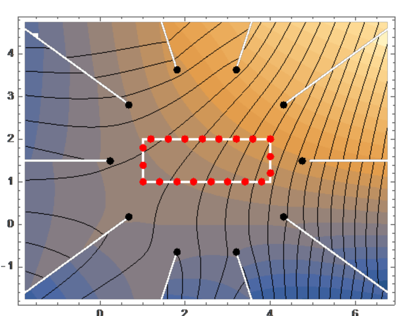

Figure 2: A CVBEM approximation generated with no NPA. Note that the computational nodes (black) are placed evenly along a circle and collocation points (red) are evenly distributed along the boundary of the problem domain. [3]

NPA1

The first improvement to the original model is the Nodal Positioning Algorithm 1 (NPA1). The algorithm begins by generating a sets of candidate nodes and collocation points. Since the CVBEM approximation and the target function are both analytic, the maximum error of the CVBEM approximation must occur on the problem boundary by the maximum modulus principle. To reduce error as much as possible with each new node, NPA1 places two collocation points on the problem boundary where the two greatest local maxima of the error function occur and then selects the next candidate node such that error is minimized.

NPA2

NPA2, the second improvement to the coupled CVBEM/NPA methodologies, introduces a refinement algorithm that is used each time a new node is added to the model. Every time the refinement alogrithm is applied, the original model is either retained or a node in the model is exchanged for a candidate node that would result in decreased error. As a result, the refinement procedure has a monotonically non-increasing effect on the maximum error of the approximation function. Research shows the NPA2 can improve accuracy by at least an order of magnitude—and up to four orders of magnitude—when the degrees of freedom are greater than ten [4].

FEM Methodology

The FEM is a very versatile and popular numerical methodology that can be used to solve many different PDEs, including the present Laplace equation. The FEM generates a mesh of modeling nodes and elements to discretize the problem domain. This project examined two trials with the FEM: one with a coarse mesh and one with a much finer mesh:

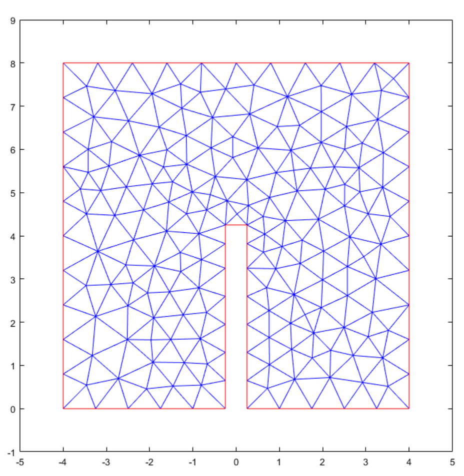

Figure 3: Trial 1 FEM Mesh

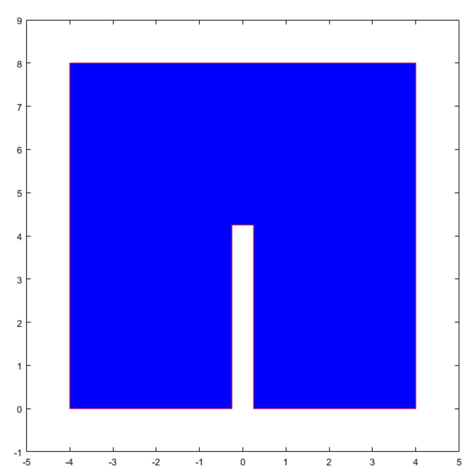

Figure 4: Trial 2 FEM Mesh

A finer mesh generally leads to a more accurate FEM approximation. As a result, we will use the mesh in Figure 4 for the purpose of drawing comparisons between the FEM and CVBEM modeling outcomes.

Results

CVBEM Problem Details and Results

Parameters	Trial 1	Trial 2
Terms:	20	40
Length:	4	4
Height:	8	8
Number of candidate nodes for optimization algorithm:	500	1000
Number of candidate collocation points for optimization algorithm:	1000	2000

Table 1: Modeling Parameters

Function	Trial 1	Trial 2
NPA1:	1.01×10^{-2}	3.82×10^{-5}
NPA2:	4.37×10^{-4}	2.38×10^{-5}

Table 2: Model Error; as expected, both incorporating more nodes and utilizing NPA2 over NPA1 result in reduced error.

CVBEM Graphical Results

The following figures compare the analytic solution to a 40-node CVBEM approximation in the problem domain. In these figures, the CVBEM potential and streamlines are shown in green and analytic isocontours are displayed as dashed black lines. Both contour plots were generated using NPA2.

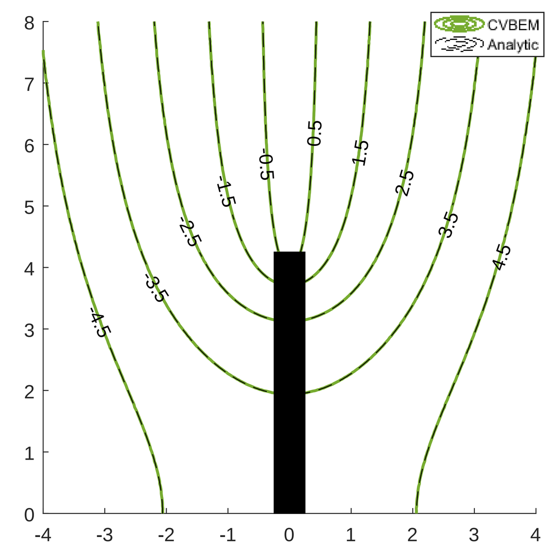

Figure 5: Potential Lines

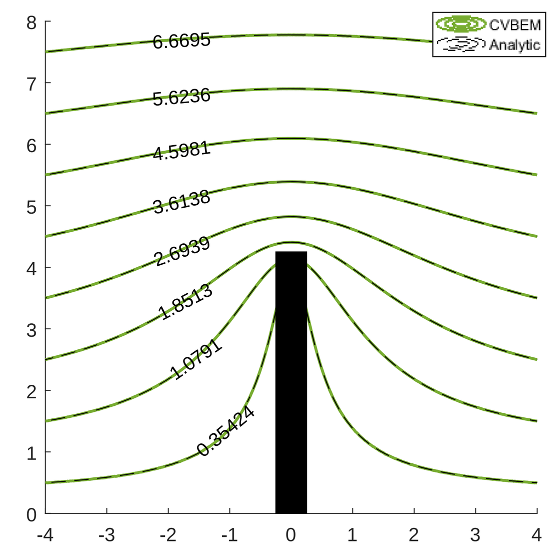

Figure 6: Streamlines

FEM Graphical Results

The following figures compare the analytic solution to our FEM approximation. In these figures, the FEM potential and streamlines are shown in blue and analytic isocontours are displayed as dashed black lines.

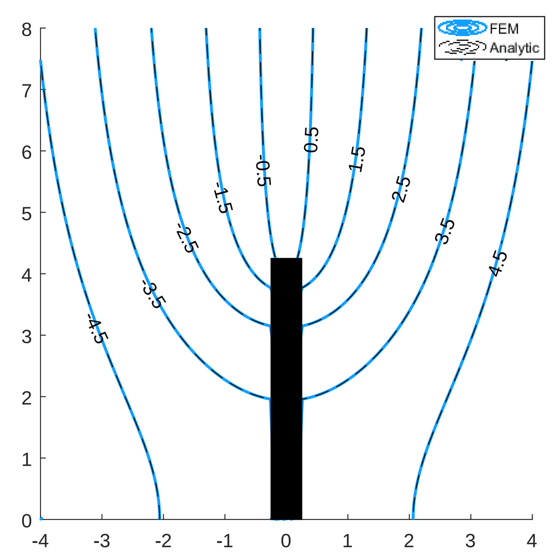

Figure 7: Potential Lines

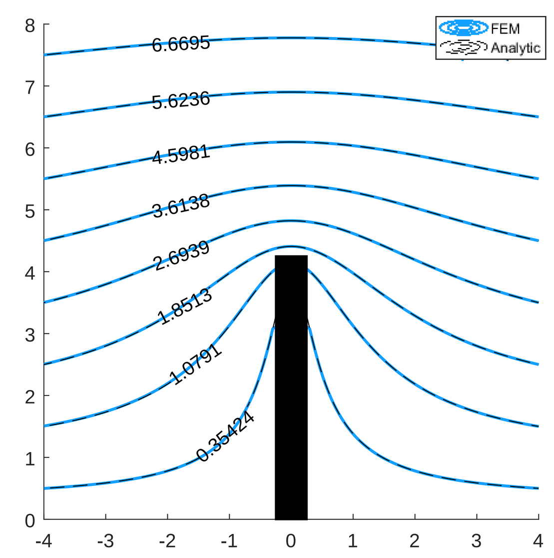

Figure 8: Streamlines

Comparison Between CVBEM and FEM Approximations

The following figures demonstrate the accuracy of both the CVBEM and FEM in predicting the source of a detected contaminant using back tracing. Figure 9 shows the entire problem domain. Note that the correct analytic streamline is shown as a dotted black line and the area near the true source is outlined in purple. Figure 10 depicts the analytic, CVBEM, and FEM streamlines that intersect the detected contaminant location near the target source in black, green, and blue respectively. From Figure 10, it is clear that the CVBEM model provides a much more accurate prediction of the actual source's location than the FEM.

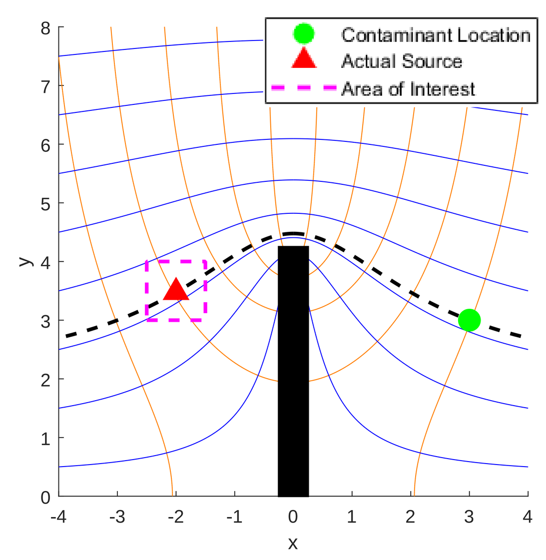

Figure 9: Example Problem

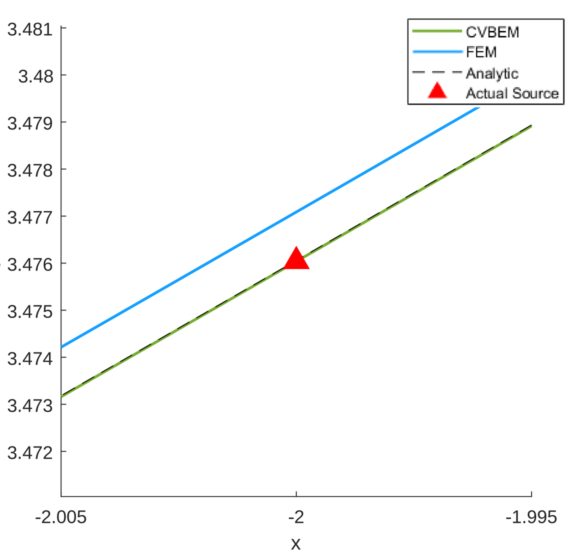

Figure 10: Target Streamlines

Conclusions

This project compared the accuracy of two numerical methods; namely, the CVBEM and FEM, in identifying the source of a groundwater contaminant given the location of detection and the boundary conditions of the groundwater flow regime. The CVBEM approximation of groundwater flow around a rectangular barrier was quite accurate with a maximum absolute error of 2.38×10^{-5} for a 40 node model with the NPA2. While the FEM exhibits similar performance when generating potential isocontours, FEM-generated streamlines are less accurate than their CVBEM equivalents as demonstrated by Figure 10. As a result, the CVBEM more reliably predicts the location of the LUST than the FEM does for this test problem.

Acknowledgements

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