Approximating a linear operator equation using a generalized Fourier series: applications

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The theory of generalized Fourier Series can be applied to the approximation of linear operator relationships. To demonstrate the computational results in using this approach several example problems where analytic solutions or quasi-analytic solutions exist are used for the testing the generalized Fourier Series technique. Applications include two-dimensional problems involving the Laplace and Poisson equations, tests for variation in results due to inner product weighting factors, and application to nonhomogeneous domain problems.

INTRODUCTION

INNER PRODUCTS FOR THE SOLUTION OF LINEAR OPERATOR EQUATIONS

The general setting for solving a linear operator equation with boundary values by means of an inner product is as follows: let Ω be a region in R^m with boundary Γ and denote the closure of Ω by $\mathrm{cl}(\Omega)$. Consider the Hilbert space $L^2(\mathrm{cl}(\Omega), d\mu)$, which has inner product $(f,g) = \int fg\, d\mu$ and where $d\mu$ is the measure defined below. (This is a real Hilbert space. For the complex version, use the complex conjugate of the function g in the integral.) The way to construct the necessary inner product for the development of a generalized Fourier Series is to choose the measure μ correctly; that is let μ be one measure μ_1 on Ω and another measure μ_2 on Γ. One natural choice for a plane region would be for μ_1 to be the usual two-dimensional Lebesque measure dV on Ω and for μ_2 to be the usual arc length measure ds on Γ. Then an inner product is given by[1]

$$(f,g) = \int_\Omega fg\, dV + \int_\Gamma fg\, ds \tag{1}$$

Consider a boundary value problem consisting of an operator L defined on domain $D(L)$ contained in $L^2(\Omega)$ and mapping into $L^2(\Omega)$, and a boundary condition operator B defined on a domain $D(B)$ in $L^2(\Omega)$ and mapping it into $L^2(\Gamma)$. The domains of L and B have to be chosen at least so that for f in $D(L)$, Lf is in $L^2(\Omega)$, and for f in $D(B)$, Bf is in $L^2(\Gamma)$. For example we could have $Lf = \nabla^2 f$, and $Bf(s)$ equal the almost everywhere (a.e.) radial limit of f at the point s on Γ, with appropriate domains.

The next step is to construct an operator T mapping its domain $D(T) = D(L) \cap D(B)$ into $L^2(\mathrm{cl}(\Omega))$ by[2]

$$\begin{aligned} Tf(x) &= Lf(x) \text{ for } x \text{ in } \Omega \\ Tf(s) &= Bf(s) \text{ for } s \text{ on } \Gamma \end{aligned} \tag{2}$$

From (2), there exists a single operator T on the Hilbert space $L^2(\mathrm{cl}(\Omega))$ which incorporates both the operator L and the boundary conditions B, and which is linear if both L and B are linear. An application of this procedure using the Complex Variable Boundary Element Method (CVBEM) is given in Hromadka *et al.*[3] In that study, $Lf = \nabla^2 f$ and Bf is the radial limit of f on Γ.

Consider the inhomogeneous equation $Lf = g_1$ with the inhomogeneous boundary conditions $Bf = g_2$. Then define a function g on $\mathrm{cl}(\Omega)$ by

$$\begin{aligned} g &= g_1 \text{ on } \Omega \\ g &= g_2 \text{ on } \Gamma \end{aligned} \tag{3}$$

Then if the solution exists for the operator equation

$$Tf = g \tag{4}$$

the solution f satisfies $\nabla^2 f = g_1$ on Ω, and $f = g_2$ on Γ in the usual sense of meaning that the radial limit of f is g_2 on Γ. One way to attempt to solve the equation $Tf = g$ is to look at a subspace D_n of dimension n, which is contained in $D(T)$, and to try to minimize $\| Th - g \|$ over all the h in D_n such as developed in Hromadka *et al.*[4]

PURPOSE OF PAPER

In this paper, the application of the approximation procedure to a set of simple linear operator problems is presented. Thirteen simple but detailed example problems are used to illustrate the approximation results obtained by the method when applied to practical problems. The theoretical development of the numerical method is included in a companion paper.[5] This paper focuses on the topics of weighting factor selection and modeling sensitivity, effects of additional basis functions on computational accuracy, and the effects on modeling results due to the addition of collocation points.

SENSITIVITY OF COMPUTATIONAL RESULTS TO VARIATIONS IN THE INNER PRODUCT WEIGHTING FACTOR

The inner product uses a weighting factor, ϵ, to weight the approximation effort in satisfying the PDE and BC values by

$$(u \cdot v) = \epsilon \int_{\Gamma} u \cdot v \, d\Gamma + (1-\epsilon) \int_{\Omega} Lu \, Lv \, d\Omega$$

The effects of varying ϵ between 0 and 1 is shown in the following simple application.

Example problem no. 1

Let $\phi = 2x^2y + y^2 + x + 6$ where $\nabla^2\phi = 2 + 4y$ on the unit square domain. Figure 1 depicts the problem domain and boundary conditions.

First, define $\langle 1, x, y, xy, x^2 \rangle$ to be the set of basis functions, and use $\epsilon = 0$. The resulting approximation function is $\hat{\phi} = 2x^2$.

Applying the linear operator ∇^2 on $\hat{\phi}$, we obtain $\nabla^2\hat{\phi} = 4$. The graph of $\nabla^2\phi = 2 + 4y$ on the unit square domain is depicted in Fig. 2. From Fig. 2, it is seen that indeed for $\epsilon = 0$, the best approximation for $\nabla^2\phi$ is $\nabla^2\phi = 4$ which coincides to $\nabla^2\hat{\phi} = 4$. Thus, the $\nabla^2\hat{\phi}$ approximation satisfies the linear operator on the least square sense (L^2) for $\epsilon = 0$ for the given set of basis functions.

Second, we choose $\langle 1, x, y \rangle$ as the set of basis functions and use $\epsilon = 1$. Then the resulting approximation function is

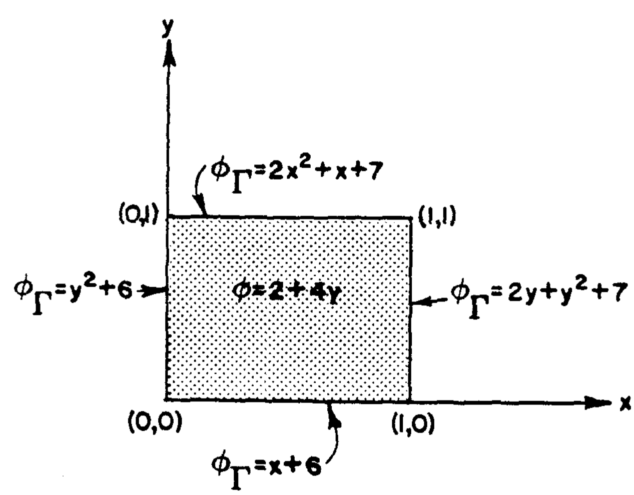

Figure 1. Domain and boundary conditions for example problem no. 1

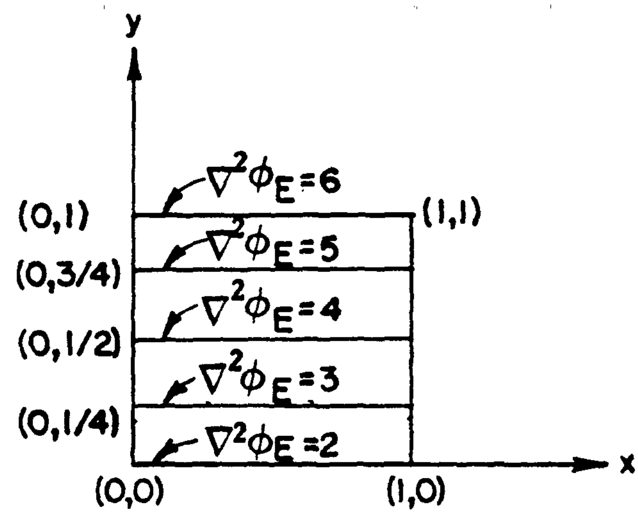

Figure 2

$\hat{\phi} = 5.4583 + 2x + 1.75y$. For comparison, the least squares method can be used to minimize the function

$$\chi = \int_{\Gamma} (k_1 + k_2x + k_3y - \phi_e)^2 \, d\Gamma$$

with respect to parameters k_1, k_2, and k_3 along the boundary of the unit square domain. From Fig. 1, one can write χ as follows:

$$\chi = \int_{x=0}^{1} (k_1 + k_2x - x - 6)^2 \, dx + \int_{y=0}^{1} (k_1 + k_2 + k_3y - 2y - y^2 - 7)^2 \, dy + \int_{x=0}^{1} (k_1 + k_2x + k_3 - 2x^2 - x - 7)^2 \, dx + \int_{y=0}^{1} (k_1 + k_3y - y^2 - 6)^2 \, dy$$

Minimizing χ with respect to k_1, k_2, and k_3, we obtain

$k_1 = 5.4583$

$k_2 = 2.0$

$k_3 = 1.75$

which verifies that the approximation function is the least square fit (L^2) with respect to the problem boundary conditions.

Example problem no. 2

Let us consider a Poisson problem $\nabla^2\phi = 2 + 2y + 12x^2$ on a unit square domain with boundary conditions $\phi_b = x^2 + y^2$ (Fig. 3).

By choosing the following set of basis functions $\langle x^2, x^3, x^4, y^2, y^3, y^4, xy^2, yx^2, x^2y^2 \rangle$, we obtain the approximation function $\hat{\phi}_b = x^2 + y^2$ for $\epsilon = 1$ and $\hat{\phi}_\Omega = x^2 + \frac{1}{3}y^3 + x^4$ for $\epsilon = 0$ with $\nabla^2\hat{\phi}_\Omega = 2 + 2y + 12x^2$. This verifies the two extreme values of the weighting factor $\epsilon = 0$ and 1 in satisfying the PDE on the domain and satisfying the BC values on the boundary, respectively.

SOLVING TWO-DIMENSIONAL POTENTIAL PROBLEMS

In the following are application problems and computed results in solving the Laplace equation in two dimension. Because the Laplace equation is a linear operator, the L^2 approach is used.

Example problem no. 3

The flow of an idea fluid around a 90° bend can be expressed by the analytic function

$$\omega = z^2$$
$$= (x^2 - y^2) + 2xyi$$

Since the state variable function $\phi = (x^2 - y^2)$ and the stream function $\psi = (2xy)$ are of polynomial forms, the approximation functions for the state variable and stream functions are found to result in the exact solutions regardless of the value of the weighting factor ϵ, $(0 < \epsilon < 1)$. Figure 4 depicts the problem domain and nodal point placement used for this application.

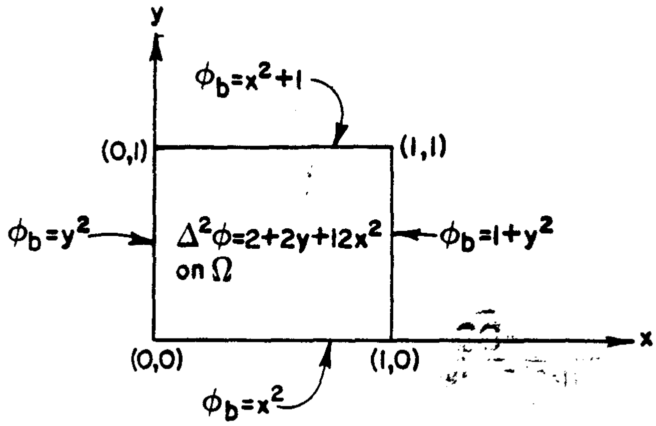

Figure 3. Domain and boundary conditions for example problem no. 2

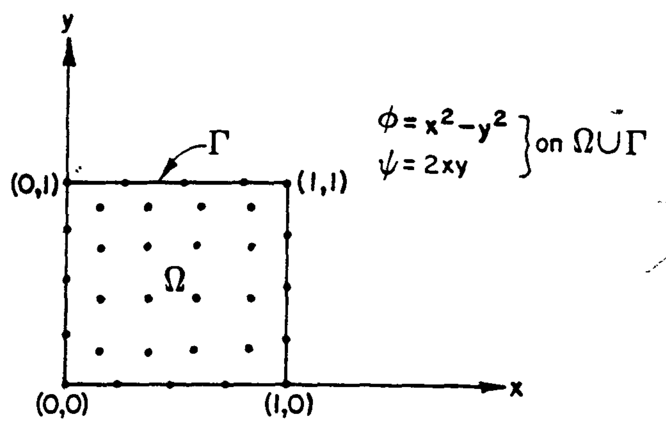

Figure 4. Domain and nodal placement for example problem no. 3

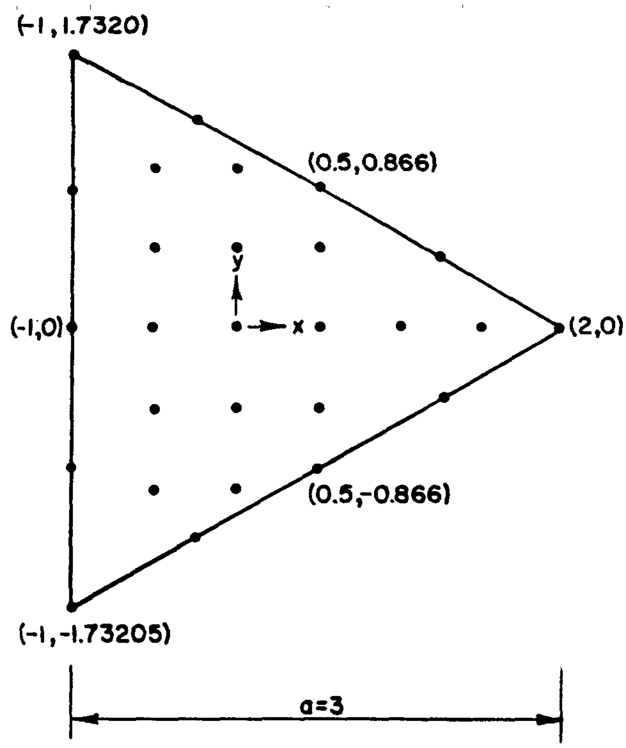

Figure 5. Domain and nodal placement for example problem no. 4

Example problem no. 4

Consider the St. Venant torsion problem for an equilateral triangular section of Fig. 5. The analytic solution for $\phi(x,y)$ is given by

$$\phi(x,y) = (x^3 - 3xy^2)/2a + 2a^2/27$$

Figure 5 depicts the nodal placement and boundary conditions for the case of $a = 3$. The L^2 approximation function is found to result in the exact solution.

Example problem no. 5

Figure 6 depicts the nodal placement and boundary conditions for a mechanical gear problem. The resulting L^2 approximation function is

$$\hat{\phi} = 9.518 + 0.3131x + 0.1812y + 0.7686xy + 0.2219x^2$$
$$- 0.2228y^2 - 1.018x^2y + 0.3397y^3 - 0.2463x^3y$$
$$+ 0.2124x^2y^2 + 0.2419xy^3$$

for a weighting factor of $\epsilon = 0.5$. Table 1 compares the results from a Complex Variable Boundary Element Method or CVBEM[6] model and the L^2 approximation function for the interior nodal points, and the defined operator relationship.

Example problem no. 6

Ideal fluid flow around a cylinder has the analytic function definition of

$$\omega(z) = z + \frac{1}{z}$$

The state variable (potential) function can be expressed as

$$\phi(x,y) = x\left(1 + \frac{1}{x^2 + y^2}\right)$$

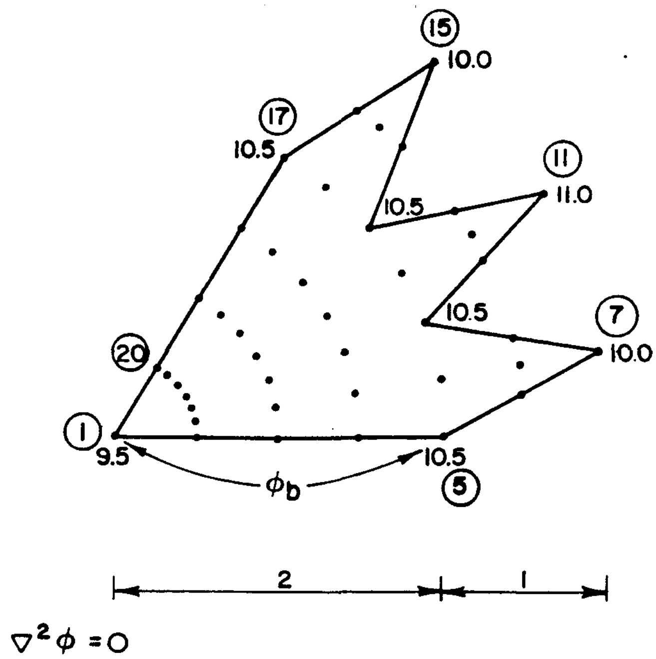

Figure 6. Domain and boundary conditions for example problem no. 5

Table 1. Comparison of L^2 and CVBEM computational results

Interior nodal points				
x	y	CVBEM	$\hat{\phi}$	$\nabla^2\hat{\phi}$
0.321	0.383	9.7842	9.752	8.03×10^{-5}
0.643	0.766	10.0391	10.04	8.56×10^{-4}
0.964	1.149	10.2610	10.28	6.36×10^{-4}
1.286	1.532	10.3668	10.39	-5.79×10^{-4}
1.607	1.915	10.2417	10.31	-2.79×10^{-3}
0.492	0.087	9.857	9.752	2.35×10^{-4}
0.985	0.174	10.0992	10.04	1.19×10^{-3}
1.477	0.261	10.2749	10.28	1.18×10^{-3}
1.970	0.347	10.3659	10.39	-2.98×10^{-3}
2.462	0.434	10.2439	10.31	-1.76×10^{-3}
0.470	0.171	9.8476	9.767	-3.55×10^{-4}
0.940	0.342	10.0944	10.05	-1.32×10^{-3}
1.410	0.513	10.2830	10.27	-4.50×10^{-3}
0.433	0.250	9.8329	9.772	-5.88×10^{-4}
0.866	0.50	10.0839	10.05	-2.26×10^{-3}
1.299	0.75	10.2856	10.27	-6.7×10^{-3}
1.732	1.0	10.4917	10.42	-1.39×10^{-2}
2.165	1.25	10.7489	10.63	-2.39×10^{-2}
0.383	0.321	9.8119	9.767	-4.34×10^{-4}
0.766	0.643	10.0659	10.05	-1.49×10^{-3}
1.149	0.964	10.2806	10.27	-4.87×10^{-3}

and the stream function can be expressed as

$$\psi(x,y) = y\left(1 - \frac{1}{x^2+y^2}\right)$$

where

$$\omega(z) = \phi(x,y) + i\psi(x,y)$$

The L^2 approximation functions ($\epsilon = 0.5$) are

$$\hat{\phi} = 1.704 + 0.9795x - 2.587y + 2.115xy - 1.009x^2 + 1.003y^2 - 1.289xy^2 + 0.431x^3 + 0.1452x^3y + 0.2258x^2y^2 - 0.1461xy^3$$

and

$$\hat{\psi} = -1.704 + 2.587x + 1.02y - 2.115xy - 1.003x^2 + 1.009y^2 + 1.289x^2y - 0.4309y^3 + 0.1461x^3y - 0.2257x^2y^2 - 1.452xy^3$$

for state variable and stream functions, respectively. Figure 7 shows the approximation relative error between a CVBEM model and the L^2 approximation function values.

Example problem no. 7

Ideal fluid flow around a cylinder in a 90° bend has the analytic solution of $\omega(z) = z^2 + z^{-2}$. The state variable function ϕ and stream function ψ can be expressed, respectively, as

$$\phi = (x^2 - y^2)\left[1 + \frac{1}{(x^2+y^2)^2}\right]$$

and

$$\psi = 2xy\left[1 - \frac{1}{(x^2+y^2)^2}\right]$$

The L^2 approximation functions ($\epsilon = 0.5$) for the state variable function $\hat{\phi}$ and the stream function $\hat{\psi}$ are given

$$\hat{\phi} = 3.197x - 3.197y - 1.828x^2 + 1.828y^2 + 1.894x^2y - 1.894xy^2 + 0.6294x^3 - 0.6294y^3 + 0.4612x^3y - 0.4612xy^3$$

and

$$\hat{\psi} = -3.192 + 3.389x + 3.389y - 4.311xy + 1.352x^2 + 1.352y^2 + 2.121x^2y + 2.121xy^2 - 0.7136x^3 - 0.7136y^3 - 0.7536x^2y^2 + 0.1259x^4 + 0.1259y^4$$

on the domain shown in Fig. 8. In comparison of the approximation to exact values it was found that the relative error is high along the circumference of the cylinder. If we included some singular terms, e.g. $1/x$, $1/y$, $1/x^2$, $1/y^2, \ldots$, into the set of basis function, the approximation function for the state variable $\hat{\phi}$ becomes

$$\hat{\phi} = 1.892x - 1.892y - 1.432x^2 + 1.432y^2 + 0.4409x^2y - 0.4409xy^2 - \frac{0.434}{x} + \frac{0.434}{y} + \frac{0.1237}{x^2} - \frac{0.1237}{y^2} - \frac{0.01721}{x^3} + \frac{0.01721}{y^3}$$

By increasing the boundary nodal points from 32 to 60 and the terms of the set of basis functions, we obtain respectively the approximation functions

$$\hat{\phi}_1 = 3.324x - 3.324y - 2.162x^2 + 2.162y^2 + 0.3721x^2y - 0.3721xy^2 + 0.6896x^3 - 0.6896^3 - \frac{2.336}{x} + \frac{2.336}{y} + \frac{0.9113}{x^2} - \frac{0.9113}{y^2} - \frac{0.1609}{x^3} + \frac{0.1609}{y^3} + \frac{0.01315}{x^4} - \frac{0.01315}{y^4}$$

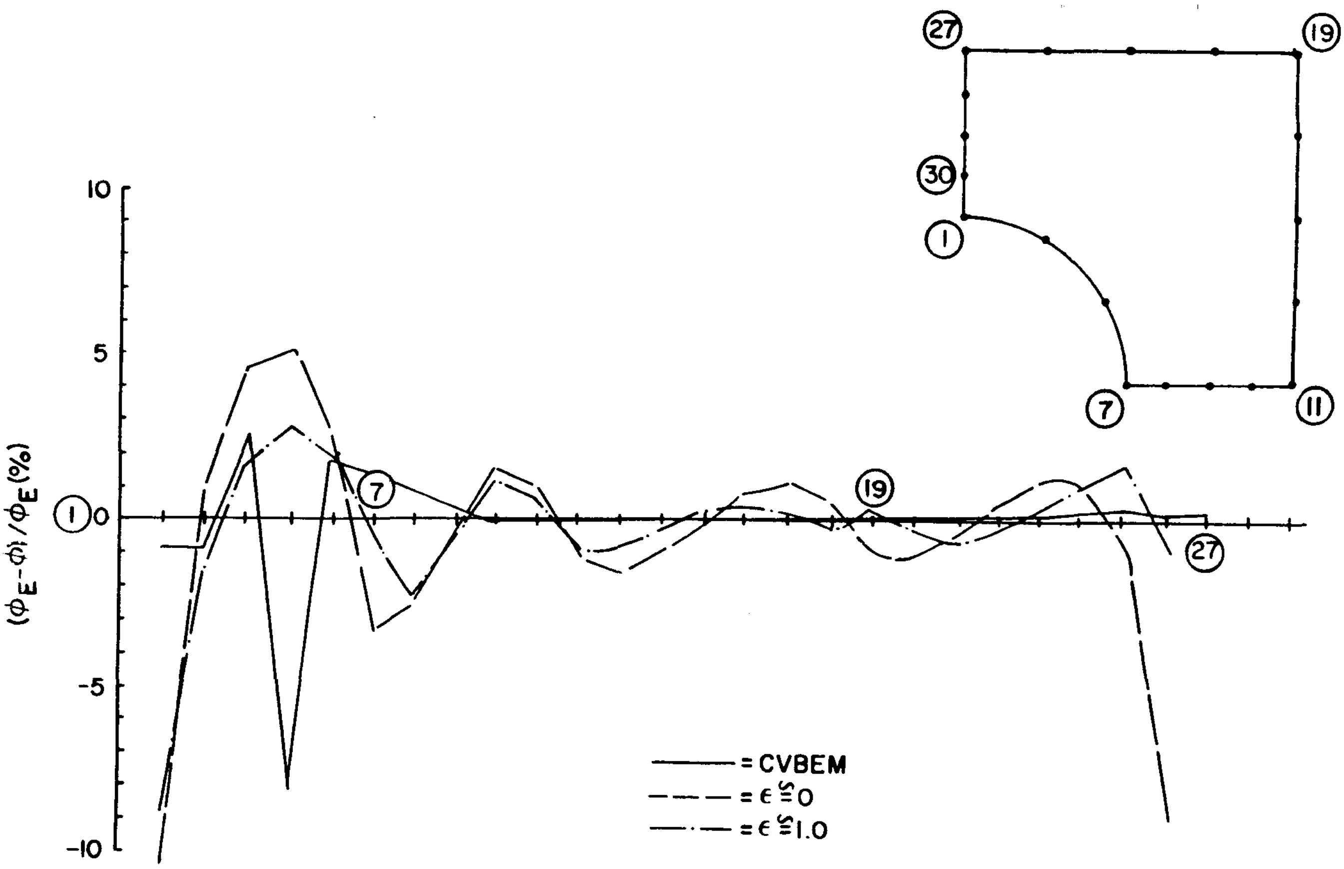

Figure 7. Domain, nodal placement and relative error for example problem no. 6

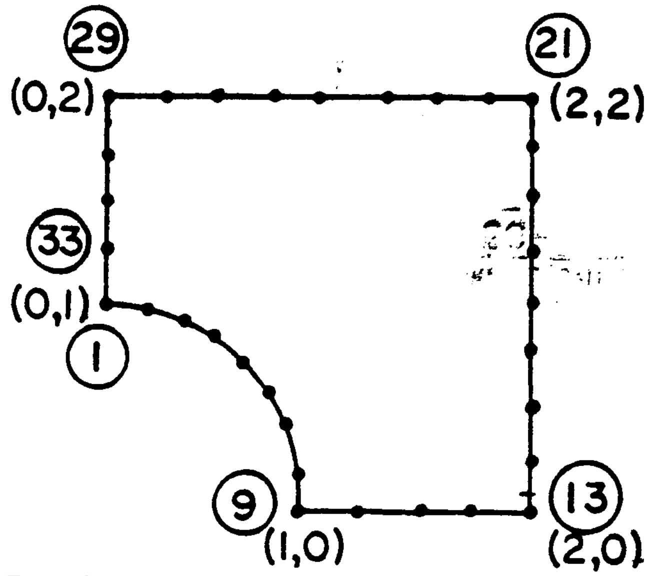

Figure 8

and

$$\hat{\phi}_2 = -0.01232 + 3.905x - 3.937y + 0.3525xy - 2.892x^2 + 2.736y^2 + 2.686x^2y - 3.14xy^2 + 1.126x^2 - 0.895y^3 - 1.017x^3y + 0.299x^2y^2 + 0.9077xy^3 - 0.05475x^4 + 0.0745x^4y + 0.049x^3y^2 - 0.1336x^2y^3$$

The relative error ($\phi_{EXACT} - \hat{\phi}/\phi_{EXACT}$) of the approximation function $\hat{\phi}_1$ is smaller than the approximation function $\hat{\phi}_2$.

Example problem no. 8

Figure 9 depicts the flow net for soil-water flow through a homogeneous soil as computed by the CVBEM. The L^2 approximation function of the state variable function $\hat{\phi}$ is given by

$$\hat{\phi} = 24.0 - 0.6549x - 0.0989y - 1.864xy - 0.06852x^2 + 0.118y^2 + 2.4 \times 10^{-3}x^2y + 1.676 \times 10^{-3}xy^2 + 1.878 \times 10^{-3}x^3 - 3.129 \times 10^{-4}y^3$$

for $\epsilon = 0.5$. Table 2 shows the interior nodal point values for both the CVBEM model and the L^2 approximation function.

APPLICATION TO OTHER LINEAR OPERATORS

The L^2 techniques can be applied to other linear operators. Hromadka and Pinder[5] examine a linear integral equation along with two PDE problems. This section considers nonhomogeneous problems involving potential theory.

Example problem no. 9

Consider the linear operator, $\nabla^2\phi$,

$$\nabla^2\phi = T_{xx}\frac{\partial^2\phi}{\partial x^2} + T_{yy}\frac{\partial^2\phi}{\partial y^2}$$

on the unit square domain with boundary condition

$\phi_b = 6 + y^2 + x^2y$

(Fig. 10).

The L^2 approximation functions for different T_{xx} and T_{yy} values are shown in Table 3.

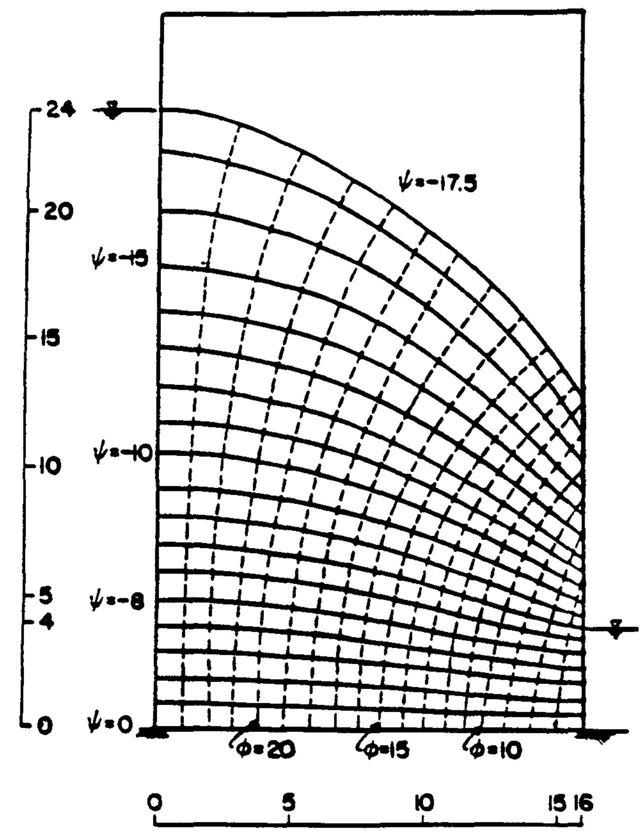

Figure 9. Streamlines and potentials for soil-water flow through a homogeneous soil for example problem no. 8

Table 2. CVBEM and L^2 computed results for example no. 8

x	y	CVBEM	$\hat{\phi}$
4	4	19.9233	20.14
4	8	20.3408	20.35
4	12	20.9110	20.91
4	16	21.5258	21.70
4	20	22.1359	22.61
8	4	15.6543	15.34
8	8	16.5817	16.04
8	12	17.8253	17.30
8	16	19.0937	19.01
8	20	20.2656	21.04
12	4	10.7715	10.11
12	8	12.5019	11.6
12	12	14.7428	13.87
12	16	16.8199	16.8

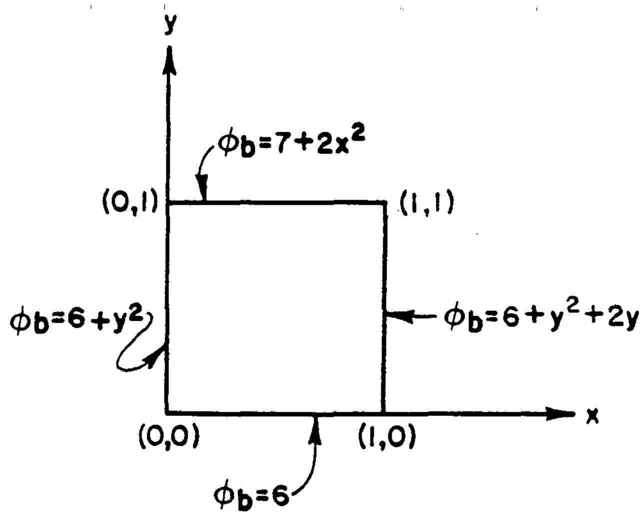

Figure 10

Example problem no. 10 – Poisson problem

Consider

$$\phi = 6 + y^2 + 2x^2y$$

on the unit square domain (Fig. 1) with the linear operator

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

such that

$$\nabla^2\phi = 2 + 4y$$

The L^2 approximation functions are listed in Table 4 for the various values of the weighting factor, ϵ.

Example problem no. 11

Consider a linear operator,

$$\nabla^2\phi = T_{xx}\frac{\partial^2\phi}{\partial x^2} + T_{yy}\frac{\partial^2\phi}{\partial y^2}$$

on the domain of example problem no. 10, we obtain

$$\nabla^2\phi = 2T_{xx} + 2 \cdot T_{yy} \cdot y$$

The L^2 approximation functions for different values of T_{xx} and T_{yy} are listed in Table 5 for $\epsilon = 0.5$. The maximum relative error on the boundary is of magnitude 10^{-4} and on the interior is of magnitude 10^{-5} for the $T_{xx} \neq T_{yy}$.

Example problem no. 12

Consider the example problem no. 10 on the problem domain of example problem no. 4 (Fig. 5), the L^2 approximation function provides the exact solution regardless the value of the weighting factor ($0 < \epsilon < 1$).

Table 3. L^2 function coefficients and relative errors for example no. 9

T_{xx}	T_{yy}	1	x	y	xy	x^2	y^2	x^2y	xy^2	x^3	y^3	Maximum relative error on boundary	Maximum relative error on operator relationship
1	1	5.856	0.8996	0.6616	0.2092	−0.8946	0.8959	1.791	≈0	≈0	−0.5967	10^{-2}	10^{-3}
2	1	5.878	0.8277	0.3158	0.4275	−0.8277	1.656	1.572	≈0	≈0	−1.048	10^{-2}	10^{-3}
1	2	5.849	0.8657	0.8637	0.1007	−0.8657	0.4333	1.899	≈0	≈0	−0.3165	10^{-2}	10^{-3}

Example problem no. 13

Consider

$$\phi = \sin x + yx + 10 + \cos^2 y$$

on a unit square domain with a linear operator $\nabla^2\phi$, such that

$$\nabla^2\phi = 2 - 4\cos^2 y - \sin x$$

The function ϕ can be expanded into an infinite series as

$$\phi = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} + yx + 10 + \left(1 + \frac{1}{2}\sum_{k=1}^{\infty} (-1)^k \frac{(2y)^{2k}}{(2k)!}\right)$$

or

$$\phi = 11 + x + yx - \frac{x3}{3!} + \frac{x5}{5!} - \frac{x7}{7!} + \ldots - y^2 + \frac{y4}{3} - \frac{2y6}{45} + \ldots$$

Table 6 lists the approximation functions for different nodal densities placements and basis function sets. The maximum relative error along the boundary is of the magnitude of order 10^{-4}. In all cases, $\epsilon = 0.5$.

Table 4. L^2 solution of Poisson problem

ϵ	$\hat{\phi}$
0	$x^2 + 2x^2y$ or $y^2 + 2x^2y$
$0 < \epsilon < 1$	$6 + y^2 + 2x^2y$
$\epsilon = 1$	$6 + y^2 + 2x^2y$

Table 5. Example problem no. 11. L^2 approximation coefficients

T_{xx}	T_{yy}	$\hat{\phi}$
1	1	$6 + y^2 + 2x^2y$
2	1	$6.014 - 8.5 \times 10^{-2}x - 0.305y + 0.171yy + 8.5 \times 10^{-2}x^2 + 1.829y^2 + 1.829x^2y - 0.553y^3$
1	2	$5.993 + 3.37 \times 10^{-2}x + 0.174y - 6.75 \times 10^{-2}xy - 3.37 \times 10^{-2}x^2 + 0.517y^2 + 2.067x^2y + 0.3221y^3$
2	2	$6 + y^2 + 2x^2y$

Table 6. Example problem no. 13. L^2 approximation results

Number of nodes		
Γ	Ω	$\hat{\phi}$
8	16	$11 + 0.999x + 0.049y + xy - 0.013x^2 - 1.29y^2 + 0x^2y + 0xy^2 - 0.14x^3 + 0.53y^3$
8	25	$11 + 0.996x + 0.044y + xy - 0.011x^2 - 1.28y^2 + 0x^2y + 0xy^2 - 0.14x^3 + 0.53y^3$
16	16	$11 + x + 0.047y + xy - 0.016x^2 - 1.29y^2 + 0.9 \times 10^{-3}x^2y - 0.64 \times 10^{-4}xy^2 - 0.14x^3 + 0.53y^3$
16	25	$11 + 0.998x + 0.041y + xy - 0.013x^2 - 1.28y^2 - 0.86 \times 10^{-4}x^2y - 0.43 \times 10^{-5}xy^2 - 0.14x^3 + 0.53y^3$
16	25	$11 + 0.992x + 0.0008654y + 0.9995xy + 6.07 \times 10^{-3}x^2 - 1.07y^2 + 4.62 \times 10^{-4}x^2y + 5.52 \times 10^{-4}xy^2 - 0.1837x^3 + 0.1806y^3 - 7.82 \times 10^{-12}x^3y - 5.48 \times 10^{-4}x^2y^2 - 7.832 \times 10^{-12}xy^3 + 1.986x^4 + 1.729y^4$

CONCLUSIONS

The generalized Fourier Series method has been applied to several linear operator relationships to demonstrate the procedures involved. Sensitivity tests are shown which examine the weighting factor used, the number and choice of basis functions, and the number and placement of collocation points. Currently, the computer program used to develop the provided solutions is small (less than 300 lines of FORTRAN code) and can be accommodated on most personal computers.

Use of this new numerical method appears to open the door to providing highly accurate solutions to linear operator problems. However, the extension of the method to applications involving nonlinear operator relationships requires further research.

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