Approximating a linear operator equation using a generalized Fourier series: development

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Many important engineering problems fall into the category of being linear operators, with supporting conditions. In this paper, a new inner-product and norm is developed which enables the numerical modeler to approximate such engineering problems by developing a generalized Fourier series. The resulting approximation is the 'best' approximation in that a least-squares (L^2) error is minimized simultaneously for fitting both the problem's boundary conditions and satisfying the linear operator relationship (the governing equations) over the problem's domain (both space and time). Because the numerical technique involves a well-defined inner-product, error evaluation is readily available using Bessel's inequality. Minimization of the approximation error is subsequently achieved with respect to a weighting of the inner product components, and the addition of basis functions used in the approximation.

INNER PRODUCTS FOR THE SOLUTION OF LINEAR OPERATOR EQUATIONS

The general setting for solving a linear operator equation with boundary values by means of an inner product is as follows: let Ω be a region in R^m with boundary Γ and denote the closure of Ω by $cl(\Omega)$. Consider the Hilbert space $L^2(cl(\Omega), d\mu)$, which has inner product $(f,g) = \int fg\, d\mu$. (This is a real Hilbert space. For the complex version, use the complex conjugate of the function g in the integral.) The way to construct the necessary inner product for the development of a generalized Fourier Series is to choose the measure μ correctly; that is let μ be one measure μ_1 on Ω and another measure μ_2 on Γ. One natural choice for a plane region would be for μ_1 to be the usual two-dimensional Lebesgue measure dV on Ω and for μ_2 to be the usual arc length measure ds on Γ. Then an inner product is given by:[1]

$$(f,g) = \int_{\Omega} fg\, dV + \int_{\Gamma} fg\, ds \tag{1}$$

Consider a boundary value problem consisting of an operator L defined on domain $D(L)$ contained in $L^2(\Omega)$ and mapping into $L^2(\Omega)$, and a boundary condition operator B defined on a domain $D(B)$ in $L^2(\Omega)$ and mapping it into $L^2(\Gamma)$. The domains of L and B have to be chosen so at least for f in $D(L)$, Lf is in $L^2(\Omega)$, and for f in $D(B)$, Bf is in $L^2(\Gamma)$. For example we could have $Lf = \nabla^2 f$, and $Bf(s)$ equal the almost everywhere (a.e.) radial limit of f at the point s on Γ, with appropriate domains.

The next step is to construct an operator T mapping its domain $D(T) = D(L) \cap D(B)$ into $L^2(cl(\Omega))$ by:[2]

$$\begin{aligned} Tf(x) &= Lf(x) \text{ for } x \text{ in } \Omega \\ Tf(s) &= Bf(s) \text{ for } s \text{ on } \Gamma \end{aligned} \tag{2}$$

Accepted December 1986. Discussion closes August 1987.

From (2) there exists a single operator T on the Hilbert space $L^2(cl(\Omega))$ which incorporates both the operator L and the boundary conditions B, and which is linear if both L and B are linear.

Consider the inhomogeneous equation $Lf = g_1$ with the inhomogeneous boundary conditions $Bf = g_2$. Then define a function g on $cl(\Omega)$ by:

$$\begin{aligned} g &= g_1 \text{ on } \Omega \\ g &= g_2 \text{ on } \Gamma \end{aligned}$$

Then if the solution exists for the operator equation:

$$Tf = g$$

the solution f satisfies $\nabla^2 f = g_1$ on Ω, and $f = g_2$ on Γ in the usual sense of meaning that the radial limit of f is g_2 on Γ. One way to attempt to solve the equation $Tf = g$ is to look at a subspace D_n of dimension n, which is contained in $D(T)$, and to try to minimize $\| Th - g \|$ over all the h in D_n such as developed in Hromadka *et al.*[3]

PURPOSE OF PAPER

In this paper, the mathematical development of the approximation procedure is presented. Three simple but detailed example problems are included to illustrate the subtle concept employed in the method, and to demonstrate the progression of steps used in the development of an associated computer program. Extension of the methods to a computer program for the approximation of boundary value problems of the two-dimensional Laplace equation is contained in a companion paper. Generalization of the computer program to other linear operator problems is the focus of a final paper.

DEFINITION OF INNER-PRODUCT AND NORM

Given a linear operator relationship:

$$L(\phi) = h \text{ on } \Omega,\ \phi = \phi_b \text{ on } \Gamma \tag{3}$$

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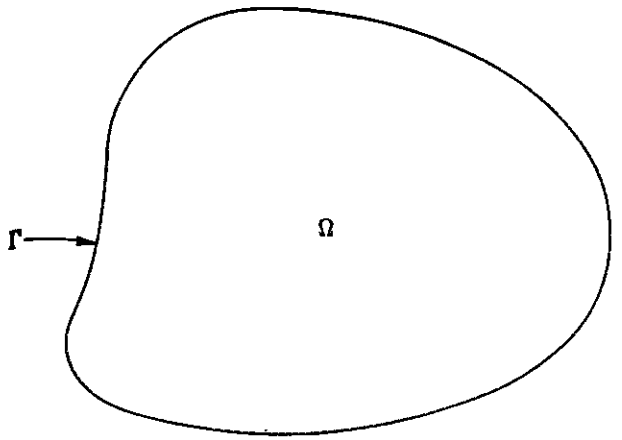

Figure 1. Definition of problem domain, Ω, and boundary, Γ. (Note: ϕ_b can include the temporal term boundary of the initial condition specification)

defined on the problem domain Ω with auxilliary conditions of $\phi = \phi_b$ on the boundary Γ (see Fig. 1). Here Ω may represent both time and space, and ϕ_b may be both initial and boundary conditions. It is assumed that the working space is sufficiently restricted (see following) such that ϕ is a unique almost everywhere (a.e.) solution to (3).

Choose a set of m linearly independent functions $\langle f_j \rangle^m$, and let S^m be the m-dimensional space spanned by the elements of $\langle f_j \rangle^m$. Here, the elements of $\langle f_j \rangle^m$ will be assumed to be functions of the dependent variables appearing in (3).

An inner-product is defined for elements of S^m by (u, v) where for $u, v \in S^m$

$$(u, v) = \int_\Gamma uv \, d\Gamma + \int_\Omega Lu \, Lv \, d\Omega \tag{4}$$

It is seen that (u, v) is indeed an inner-product, because for elements u, v, w in S^m

(i) $(u, v) = (v, u)$
(ii) $(ku, v) = k(u, v)$, for L a linear operator
(iii) $(u + v, w) = (u, w) + (v, w)$ for L a linear operator
(iv) $(u, u) = \int_\Gamma (u)^2 \, d\Gamma + \int_\Omega (Lu)^2 \, d\Omega \geqslant 0$
(v) $(u, u) = 0 \Rightarrow u = 0$ a.e. on Γ, and $Lu = 0$ a.e. over Ω

The above restrictions on the operator L imply that L is linear (see (ii) and (iii) in the above definition); if $Lu = 0$ a.e. over Ω and $u = 0$ a.e. on Γ, this must imply that the solution $u = [0]$, where $[0]$ is the zero element over $\Omega \cup \Gamma$; and for the inner-product to exist, the integrals must exist. For the inner-product of (4) to exist, the integrands must be finite. Additionally, each element $u \in S^m$ must satisfy $\int_\Gamma u^2 \, d\Gamma < \infty$.

For the above restrictions of L and the space S^m, the inner-product is defined and a norm "$\| \ \|$" immediately follows:

$$\| u \| \equiv (u, u)^{1/2} \tag{5}$$

The generalized Fourier series approach can now be used to obtain the "best" approximation $\phi_m \in S^m$ of the function ϕ using the newly defined inner-product and corresponding norm presented in (4) and (5).

The next step in developing the generalized Fourier series is to construct a new set of functions $\langle g_j \rangle^m$ which are the orthonormal representation of the $\langle f_j \rangle^m$.

ORTHONORMALIZATION PROCESS

The functions $\langle g_j \rangle^m$ can be obtained by the well-known Gramm–Schmidt procedure[4] using the newly defined norm of (4). That is:

$$g_1 = f_1 / \| f_1 \|$$
$$\vdots$$
$$g_m = [f_m - (f_m, g_1) g_1 - \ldots - (f_m, g_{m-1}) g_{m-1}] / \| f_m - (f_m, g_1) g_1 - \ldots - (f_m, g_{m-1}) g_{m-1} \|$$

Hence, the elements of $\langle g_j \rangle^m$ satisfy the convenient properties that

$$(g_j, g_k) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} \tag{7}$$

In a subsequent section, a simple one-dimensional problem illustrates the orthonormalization procedure of (6).

The elements $\langle g_j \rangle^m$ also form a basis for S^m but, because of (7), can be directly used in the development of a generalized Fourier series where the computed coefficients do not change as the dimension m of $\langle g_j \rangle^m$ increases. That is, as the number of orthonormalized elements increases in the approximation effort, the previously computed coefficients do not change. Each element $\phi_m \in S^m$ can now be written as:

$$\phi_m = \sum_{j=1}^{m} \gamma_j g_j, \quad \phi_m \in S^m \tag{8}$$

where γ_j are unique real constants.

GENERALIZED FOURIER SERIES

The ultimate objective is to find the element $\phi_m \in S^m$ such that $\| \phi_m - \phi \|$ is a minimum. That is, we want $\| \phi_m - \phi \|^2$ to be a minimum, where

$$\| \phi_m - \phi \|^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(L \sum_{j=1}^{m} \gamma_j g_j - L\phi \right)^2 d\Omega \tag{9}$$

Remembering that L is a linear operator, and $L\phi = f$ by the problem definition, of (3) we have that (9) can be rewritten as:

$$\| \phi_m - \phi \|^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(\sum_{j=1}^{m} \gamma_j L g_j - f \right)^2 d\Omega \tag{10}$$

Thus, minimizing $\| \phi_m - \phi \|^2$ is equivalent to minimizing the error or approximating the boundary conditions and the error of approximating the governing operator relationship in a least-square (or L^2) sense. Because the $\langle g_j \rangle^m$ are orthonormalized and the inner-product $(\ ,\ )$ is well-defined, the coefficients γ_j of (8) are immediately determined by the generalized Fourier constants, γ_j^*, where:

$$\gamma_j^* = (g_j, \phi), \quad j = 1, 2, \ldots, m \tag{11}$$

Thus

$$\phi_m^* = \sum_{j=1}^{m} \gamma_j^* g_j = \sum_{j=1}^{m} (g_j, \phi)\, g_j \tag{12}$$

is the "best" approximation of ϕ, in the space S^m.

EXAMPLE PROBLEM NO. 1 (DIFFERENTIAL EQUATION)

To illustrate the previous developments, a simple one-dimensional torsion problem is studied. In this example, four polynomials (linearly independent functions) are used as a basis:

$$\langle f_j \rangle^4 = \langle 1, x, x^2, x^3 \rangle$$

with the problem defined by, for $0 \leqslant x \leqslant 1$:

$$\frac{\partial^2 \phi}{\partial x^2} = -2, \quad \phi(x=0) = 1 \text{ and } \phi(x=1) = 2$$

Here $L = \partial^2/\partial x^2$, $h = -2$, and ϕ_b is given by the two point values at $x = 0, 1$. The inner-product of (4) is now given as:

$$(u, v) = \int_\Gamma uv \, d\Gamma + \int_\Omega Lu\, Lv \, d\Omega$$

$$= uv|_{x=0} + uv|_{x=1} + \int_\Omega \frac{\partial^2(u)}{\partial x^2} \frac{\partial^2(v)}{\partial x^2} \, d\Omega \tag{13}$$

The four-dimension space S^4 is the set of all functions (polynomials) such that $\phi_4(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$ where the C_i are real constants.

The orthonormalization of the $\langle f_j \rangle^4$ proceeds as follows:

For element g_1:

$$(f_1, f_1) = (1)(1)|_{x=0} + (1)(1)|_{x=1} + \int_{x=0}^{1} \frac{\partial^2(1)}{\partial x^2} \frac{\partial^2(1)}{\partial x^2} \, dx = 2$$

and

$$g_1 = f_1/\|f_1\| = 1/\sqrt{2} = \sqrt{2}/2$$

For element g_2:

$$(f_2, g_1) = (x, \sqrt{2}/2) = (x)(\sqrt{2}/2)|_{x=0} + (x)(\sqrt{2}/2)|_{x=1} + \int_0^1 \frac{\partial^2(x)}{\partial x^2} \frac{\partial^2(\sqrt{2}/2)}{\partial x^2} dx$$

$$= \sqrt{2}/2$$

$$\hat{g}_2 = f_2 - (f_2, g_1) g_1 = x - (\sqrt{2}/2)(\sqrt{2}/2) = x - 1/2$$

$$(\hat{g}_2, \hat{g}_2) = (x - 1/2)(x - 1/2)|_{x=0} + (x - 1/2)(x - 1/2)|_{x=1} + \int_0^1 \frac{\partial^2(x - 1/2)}{\partial x^2} \frac{\partial^2(x - 1/2)}{\partial x^2} dx$$

$$= 1/2$$

$$\therefore g_2 = \hat{g}_2/\|\hat{g}_2\| = (x - 1/2)/(\sqrt{2}/2) = (2x - 1)/\sqrt{2}$$

Similarly for element g_3:

$$g_3 = (x^2 - x)/2$$

Element g_4 is more involved and is derived in detail for illustration:

$$(f_4, g_1) = \sqrt{2}/2; \; (f_4, g_2) = \sqrt{2}/2; \; (f_4, g_3) = 3$$

$$\hat{g}_4 = x^3 - (f_4, g_1) g_1 - (f_4, g_2) g_2 - (f_4, g_3) g_3$$

$$= x^3 - \frac{3}{2}x^2 + \frac{x}{2}$$

$$(\hat{g}_4, \hat{g}_4) = \left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right)^2\Big|_{x=0} + \left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right)^2_{x=1} + \int_0^1 \left[\frac{\partial^2 \left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right)}{\partial x^2}\right]^2 dx$$

$$= 0 + 0 + \int_0^1 (6x - 3)^2 \, dx = 3$$

$$\therefore g_4 = \hat{g}_4/\|g_4\| = (x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x)/\sqrt{3}$$

Hence, the orthonormal vectors $\langle g_j \rangle^4$ are:

$$\langle g_j \rangle^4 = \langle \sqrt{2}/2, (2x-1)/\sqrt{2}, (x^2-x)/2, (2x^3 - 3x^2 + x)/2\sqrt{3} \rangle$$

Now, any element $\phi_4 \in S^4$ is of the form

$$\phi^4 = \sum_{j=1}^{4} \gamma_j g_j$$

The norm $\|\phi_4 - \phi\|$ is a minimum when $\gamma_j = \gamma_j^*$ where γ_j^* are the generalized Fourier series coeffcients determined from:

$$\gamma_j^* = (g_j, \phi)$$

That is,

$$\gamma_j^* = \int_\Gamma g_j \phi_b \, d\Gamma + \int_\Omega Lg_j \, L\phi \, d\Omega$$

$$= \int_\Gamma g_j \phi_b \, d\Gamma + \int_\Omega Lg_j f \, d\Omega$$

where for simplicity, the Lg_j are given by:

$$\langle Lg_j \rangle^4 = \langle 0, 0, 1, (6x - 3)/\sqrt{3} \rangle$$

Remembering that $h = -2$ by the problem definition, we solve for the γ_j^* as follows:

$$\gamma_1^* = (g_1, \phi) = \left(\frac{\sqrt{2}}{2}\right)(\phi_b)\Big|_{x=0} + \left(\frac{\sqrt{2}}{2}\right)(\phi_b)\Big|_{x=1} + 0$$

$$= 3\sqrt{2}/2$$

$$\gamma_2^* = (g_2, \phi) = \left(\frac{2x-1}{\sqrt{2}}\right)(\phi_b)\Big|_{x=0} + \left(\frac{2x-1}{\sqrt{2}}\right)(\phi_b)\Big|_{x=1} + 0$$

$$= \sqrt{2}/2$$

$$\gamma_3^* = (g_3, \phi) = 0 + 0 + \int_0^1 Lg_3 f \, dx = \int_0^1 (1)(-2)\, dx = -2$$

$$\gamma_4^* = (g_4, \phi) = 0 + 0 + \int_0^1 Lg_4 f \, dx = \int_0^1 \left(\frac{6x-3}{\sqrt{3}}\right)(-2)\, dx = 0$$

Thus, the best approximation in S^4 is given by:

$$\phi_4^* = \sum_{j=1}^{4} \gamma_j^* g_j = 1 + 2x - x^2$$

It is readily seen that $L\phi_4^* = -2 = h$, and ϕ_4^* satisfies the problem boundary conditions.

DISCUSSION

From the example problem, a best approximation of a linear operator relationship is obtained by a generalized Fourier series development which minimizes, in a least-squares (L^2) sense, the error of approximation.

Because the generalized Fourier series approach is used, several advantages over a matrix solution (for the generalized Fourier series coefficients) are obtained:

1. Elimination of the need for solving large, fully populated, matrices such as occurs when solving the normal equations.
2. Elimination of the instability which typically arises in a matrix solution for Fourier coefficients (i.e. higher powers of the expansion basis functions assumed).
3. The generalized Fourier series coefficients do not change as additional functions are added (i.e. as the dimension m of the space S^m is increased).
4. Generalized Fourier series theory applies; hence, error analysis can be conducted using Bessel's inequality as discussed in the next section.

APPROXIMATION ERROR EVALUATION

Due to the generalized Fourier series approach and the definition of the inner-product, Bessel's inequality applies. That is, for any dimension m:

$$(\phi, \phi) \geq \sum_{j=1}^{m} (g_j, \phi)^2 = \sum_{j=1}^{m} \gamma_j^{*2} \tag{14}$$

where

$$(\phi, \phi) = \int_\Gamma (\phi)^2 \, d\Gamma + \int_\Omega (L\phi)^2 \, d\Omega = \int_\Gamma \phi^2 \, d\Gamma + \int_\Omega f^2 \, d\Omega \tag{15}$$

Equation (15) is readily evaluated and forms an upper bound to the sum of $(g_j, \phi)^2$ as the dimension m increases. Consequently, one may interact with the approximation effort by carefully adding functions to the $\langle f_j \rangle^m$ in order to best reduce the difference computed by Bessel's inequality. In a following section, Bessel's inequality will be used to define an objective function (noted by χ) which will be subsequently minimized by determining a weighting factor ϵ to be used in the inner product of (4).

EXAMPLE PROBLEM NO. 2 (VOLTERA INTEGRAL)

To further illustrate the approximation method, a Voltera integral equation (such as occurs in developing unit hydrographs from watershed rainfall-runoff data) is considered where:

$$q(t) = \int_0^t i(t-s)\, \phi(s)\, ds, \quad 0 \leq t \leq 2$$

where for simplicity the effective rainfall intensity is given by the constant value:

$$i(t-s) = 1$$

and the runoff hydrograph flowrate $q(t)$ is given by:

$$q(t) = \begin{cases} t^3, & 0 \leq t \leq 1 \\ -2t^2 + 7t - 4, & 1 \leq t \leq 2 \end{cases}$$

In this class of problem, neither boundary (nor initial) conditions are involved, hence the inner product of (4) becomes:

$$(u, v) = \int_\Omega Lu \, Lv \, d\Omega$$

$$= \int_{t=0}^{2} \left[\int_0^t i(t-s) u(s)\, ds \int_0^t i(t-s) v(s)\, ds \right] dt \tag{16}$$

By assumption $i(t-s) = 1$, and the inner product reduces to

$$(u, v) = \int_{t=0}^{2} \left[\int_0^t u(s)\, ds \int_0^t v(s)\, ds \right] dt$$

Three elements are considered for basis functions $\langle f_j \rangle^3$, namely the polynomials $\langle 1, s, s^2 \rangle$. The orthonormalized elements $\langle g_j \rangle^3$ are determined in the following:

g_1:

$$Lf_1 = \int_0^t (1)\, ds = t$$

$$\therefore \; (f_1, f_1) = \int_0^2 t^2 \, dt = 8/3; \; \|f_1\| = 2\sqrt{\tfrac{2}{3}}$$

and

$$g_1 = f_1 / \|f_1\| = \sqrt{\tfrac{3}{8}}$$

g_2:

$$Lf_2 = \int_0^t s \, ds = t^2/2$$

$$Lg_1 = \int_0^t \sqrt{\tfrac{3}{8}}\, ds = t\sqrt{\tfrac{3}{8}}$$

$$\therefore\ (f_2, g_1) = \int_0^2 Lf_2\, Lg_1\, \mathrm{d}t = \int_0^2 \left(\frac{t^2}{2}\right)(t\sqrt{\tfrac{3}{8}})\,\mathrm{d}t = \sqrt{\tfrac{3}{2}}$$

Now

$$\hat{g}_2 = f_2 - (f_2, g_1)\, g_1 = s - 3/4$$

$$L\hat{g}_2 = \int_0^t (s - 3/4)\,\mathrm{d}s = \frac{t^2}{2} - \frac{3t}{4}$$

$$\therefore\ (\hat{g}_2, \hat{g}_2) = \int_0^2 \left(\frac{t^2}{2} - \frac{3t}{4}\right)^2 \mathrm{d}t = \frac{1}{10}$$

$$\therefore\ g_2 = \hat{g}_2 / \|\hat{g}_2\| = (s - \tfrac{3}{4})\sqrt{10}$$

g_3:

Analogous to the above:

$$(f_3, g_1) = \tfrac{16}{5}\sqrt{\tfrac{1}{6}}$$

$$(f_3, g_2) = \sqrt{10}/5.625$$

$$\therefore\ \hat{g}_3 = f_3 - (f_3, g_1)\, g_1 - (f_3, g_2)\, g_2$$

$$= s^2 + 0.5\bar{3} - 1.\bar{7}s$$

where the overbar notation indicates repetitive digits. Finally:

$$g_3 = \hat{g}_3 / \|\hat{g}_3\| = 10.5234s^2 - 18.708s + 5.6125$$

The generalized Fourier coefficients are determined as before by:

$$\gamma_1^* = (g_1, \phi) = \int_\Omega Lg_1\, L\phi\, \mathrm{d}\Omega$$

$$= \sqrt{\tfrac{3}{8}} \int_0^1 (t)(t^3)\,\mathrm{d}t + \sqrt{\tfrac{3}{8}} \int_1^2 (t)(-2t^2 + 7t - 4)\,\mathrm{d}t$$

$$= 1.8575$$

$$\gamma_2^* = (g_2, \phi) = 0.21082$$

$$\gamma_3^* = (g_3, \phi) = -0.325$$

Thus the best approximation is developed (for the defined inner product of (16)) by:

$$\phi_3 = -3.42s^2 + 6.7467s - 1.1865$$

For this example problem, the exact solution is determined by taking the derivative of the $q(t)$ function (rewritten in terms of the variable s):

$$\phi(s) = \begin{cases} 3s^2, & 0 \leqslant s \leqslant 1 \\ -4s + 7, & 1 \leqslant s \leqslant 2 \end{cases}$$

Figure 2 compares the exact solution $\phi(s)$ to the approximation function $\phi_3(s)$ developed from using only three polynomial basis functions.

It is noted that although the example problems Nos. 1 and 2 are different linear operator relationships (i.e. a PDE and a Voltera integral), the approximation method and procedures are identical.

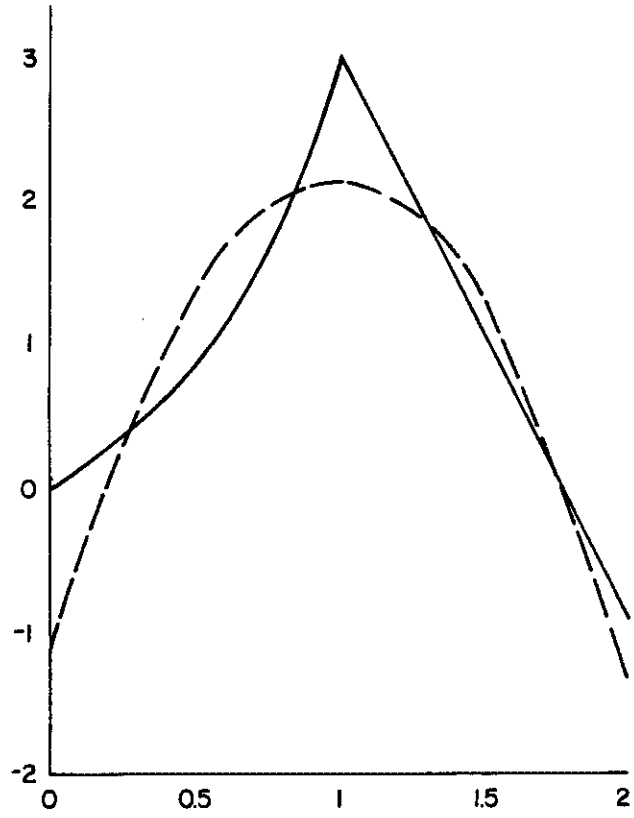

Figure 2. Best approximation unit hydrograph (dashed) and exact unit hydrograph (solid line)

Additionally, Bessel's inequality (14) can be used to evaluate the error of approximation for this problem as follows:

$$(\phi, \phi) = \int_{t=0}^{2} (L\phi)^2\,\mathrm{d}t = \int_{t=0}^{2} [q(t)]^2\,\mathrm{d}t$$

$$= \int_0^1 (t^3)^2\,\mathrm{d}t + \int_1^2 (-2t^2 + 7t - 4)^2\,\mathrm{d}t = 3.6095$$

In comparison,

$$\sum_{j=1}^{m} \gamma_j^{*2} = (1.8575)^2 + (0.21082)^2 + (-0.325)^2$$

$$= 3.6003 \leqslant (\phi, \phi)$$

That is, although the generalized Fourier coefficients provide for the best approximation from space S^m, the error of approximation χ given by:

$$\chi = (\phi, \phi) - \sum_{j=1}^{m} \gamma_j^{*2} \tag{17}$$

is nonzero and, therefore, the addition of additional elements to $\langle f_j \rangle^m$ (increasing the dimension of S^m) will necessarily add more positive values to the sum the γ_j^{*2}, resulting in a decrease in χ. Should $\chi = 0$, then $\|\phi - \phi_m\| = 0$ and

$\phi - \phi_m = [0]$, the zero element, and $\phi = \phi_m$ a.e. For instance, example problem No. 1 results in:

$$(\phi, \phi) = \int_\Gamma (\phi)^2 \, d\Gamma + \int_\Omega (L\phi)^2 \, d\Omega$$

$$= (\phi_b)^2|_{x=0} + (\phi_b)^2|_{x=1} + \int_{x=0}^{1} (f)^2 \, dx$$

$$= (1) + (4) + (4) = 9$$

In comparison,

$$\sum_{j=1}^{m} \gamma_j^{*2} = \sum_{j=1}^{4} \gamma_j^{*2}$$

$$= \left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (-2)^2 + (0)^2 = 9$$

Thus, $\chi = 0$, which indicates that the approximation ϕ_4 equals the exact solution ϕ a.e. (almost everywhere). Of course for this example, $\phi = \phi_4$ identically over Ω, and the a.e. statement can be dropped.

THE WEIGHTED INNER PRODUCT

In the inner product of (4), equal weight is given to the various requirements imposed on the best approximation function ϕ_m from the space S^m spanned by the m linearly independent basis functions $\langle f_j \rangle^m$. Namely, the L^2 error in satisfying the linear operator relationship over Ω is considered by equal importance as the L^2 error in satisfying the problem's boundary (and initial) conditions, (of course for the Voltera integral example problem, only one term is used in the inner product definition and the concerns as to weighting factors is no longer needed).

Due to the limitations of computer power, only a finite number of basis functions can be used for approximation purposes, and so an argument is made to weight the terms which compose the inner product differently. For $0 < \epsilon < 1$, one weighting of (4) is simply:

$$(u, v) = \epsilon \int_\Gamma uv \, d\Gamma + (1 - \epsilon) \int_\Omega Lu \, Lv \, d\Omega \tag{18}$$

In (18), an ϵ-value close to 1 would force the approximation function ϕ_n of S^m to focus upon satisfying the problem's boundary conditions rather than satisfying the linear operator. Similarly, and ϵ-value close to 0 would focus the ϕ_m approximation towards satisfying the linear operator relationship and ignore the boundary conditions.

It is noted that (18) is still an inner product for a given choice of ϵ, and will be used to develop the generalized Fourier series using the previously presented procedures. And as the dimension S^m increases, the Bessel's inequality still applies in that $\chi = \chi_\epsilon$, and

$$\chi_\epsilon = 0 \Rightarrow \| \phi_m - \phi \|_\epsilon = 0 \tag{19}$$

In (19), ϵ-notation has been added to clarify that all norms, inner products, and even the orthonormalized basis functions are now functions of ϵ for $0 < \epsilon < 1$. However, for ease of presentation in the following text, the ϵ-notation is omitted although it is implied that all relationships are now dependent on the ϵ-value used in the weighting of the inner product components.

The selection of the 'optimum' ϵ-value to be used in (18) depends on the rule assigned for optimization. In this paper, ϵ is chosen which minimizes the Bessel's inequality relationship:

$$\chi_\epsilon = (\phi, \phi)_\epsilon - \sum_{j=1}^{m} \gamma_{\epsilon_j}^{*2} \tag{20}$$

$$= (\phi, \phi)_\epsilon - \sum_{j=1}^{m} (\phi, g_{\epsilon_j})_\epsilon^2 \tag{21}$$

In (20) and (21) it is stressed that all terms depend on ϵ. The inner product weighting ϵ-value is chosen which minimizes χ_ϵ of (21).

EXAMPLE NO. 3 (WEIGHTED INNER PRODUCT)

To illustrate the inner product weighting concept, example problem No. 1 is restudied with only one basis function, $f_1 = x^2$. It is recalled that $L\phi = \partial^2\phi/\partial x^2$, $h = -2$, and $\phi(x = 0) = 1$, $\phi(x = 1) = 2$. Proceeding as before, and dropping the ϵ subscript notation, therefore:

$$(f_1, f_1) = \epsilon \int_\Gamma (f_1)^2 \, d\Gamma + (1 - \epsilon) \int_\Omega (Lf_1)^2 \, d\Omega$$

$$= \epsilon(x^2)^2|_{x=0} + \epsilon(x^2)^2|_{x=1} + (1 - \epsilon) \int_0^1 (-2)^2 \, dx$$

$$= 4 - 3\epsilon$$

$$\therefore \quad \| f_1 \| = \sqrt{4 - 3\epsilon}$$

and

$$g_1 = f_1 / \| f_1 \| = x^2 / \sqrt{4 - 3\epsilon}$$

The only Fourier coefficient γ_1^* is computed as:

$$\gamma_1^* = (\phi, g_1) = \epsilon \left(\frac{x^2}{\sqrt{4 - 3\epsilon}} \right) (1) \Big|_{x=0}$$

$$+ \epsilon \left(\frac{x^2}{\sqrt{4 - 3\epsilon}} \right) (2) \Big|_{x=1}$$

$$+ (1 - \epsilon) \int_0^1 \left(\frac{2}{\sqrt{4 - 3\epsilon}} \right) (-2) \, dx = \frac{(6\epsilon - 4)}{\sqrt{4 - 3\epsilon}}$$

Thus,

$$\phi_1 = \gamma_1 g_1$$

$$= x^2 \left(\frac{6\epsilon - 4}{4 - 3\epsilon} \right); \quad \text{for } 0 < \epsilon < 1$$

The next step is to compute χ_ϵ:

$$(\phi, \phi) = \epsilon(\phi_b)^2|_{x=0} + \epsilon(\phi_b)^2|_{x=1} + (1 - \epsilon) \int_0^1 f^2 \, dx$$

$$= \epsilon(1)^2 + \epsilon(2)^2 + (1 - \epsilon) \int_0^1 (-2)^2 \, dx$$

$$= 4 + \epsilon; \text{ for } 0 < \epsilon < 1$$

$$\gamma_1^{*2} = (36\epsilon^2 - 48\epsilon + 16)/(4 - 3\epsilon)$$

$$\chi_\epsilon = (\phi, \phi) - \gamma_1^{*2}$$

$$= (4 + \epsilon) - (36\epsilon^2 - 48\epsilon + 16)/(4 - 3\epsilon)$$

$$= 4 + 13\epsilon + 16/(3\epsilon - 4)$$

$$= \epsilon(39\epsilon - 40)/(3\epsilon - 4)$$

Figure 3 displays the plot of χ_ϵ against ϵ for $0 < \epsilon < 1$. Because only one basis function $f_1 = x^2$ was chosen in this simple example, the weighting is focused toward satisfying the PDE or the boundary conditions as shown in Table 1. For this simple problem, $\phi_1 = kx^2$ where $k = (6\epsilon - 4)/(4 - 3\epsilon)$ from the above calculations. Table 1 summarizes the implications resulting from using various values of k in ϕ_1.

From Fig. 3 it is seen that χ_ϵ is minimum when $\epsilon = 0$. Obviously from Table 1, however, $\epsilon = 0$ would not be the optimum choice of ϵ due to the approximation only satisfying in a minimum least-squares (L^2) sense the PDE and neglecting the boundary conditions. For typical applications, ϵ is chosen when maximizes χ_ϵ. In this way, the 'largest' value of approximation error is being used to evaluate Bessel's inequality, which is then used to evaluate the reduction in approximation error as additional elements are added to the test collection of basis functions.

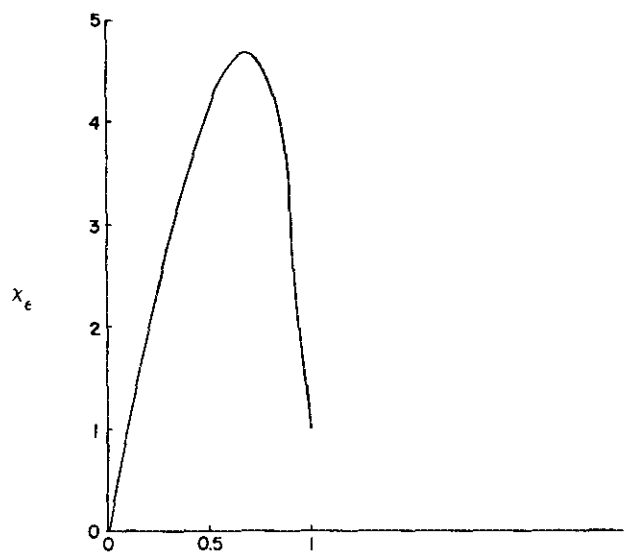

Figure 3. $\chi_\epsilon = (\phi, \phi) - \gamma_1^{*2}$ *for example problem No. 3*

Table 1. Inner product weighting implications for problem No. 3

ϵ	k (for $\phi = kx^2$)	Notes
0	−1.0	All weighting is focused toward satisfying $\partial^2\phi/\partial x^2 = -2$. Here, $\phi_1 = -x^2$
0.50	−0.40	An intermediate value for ϕ_1
1.0	+2.0	All weighting if focused towards satisfying $\phi(x = 0) = 1$ and $\phi(x = 1) = 2$. Here, $\phi_1 = 2x^2$

CONCLUSIONS

A new approximation method is presented which incorporates a classical generalized Fourier series expansion to solving a linear operator relationship. A new inner-product and norm is presented which enables the modeler to develop the 'best' approximation of the boundary conditions and the linear operator relationship in a least-squares (L^2) sense. Because the method is based on generalized Fourier series theory, Bessel's inequality applies and allows for a readily computed 'error of approximation'. By weighting the components of the inner product, a relationship based on the Bessel's inequality is developed which can be used to determine an optimum inner product weighting factor.

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