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SOME ADVANCES IN CVBEM MODELING OF TWO-DIMENSIONAL POTENTIAL FLOW

by

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SOME ADVANCES IN CVBEM MODELING OF TWO-DIMENSIONAL POTENTIAL FLOW

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ABSTRACT: Of interest in flow modeling is the exact solution of classes of partial differential equations such as the Laplace and Poisson equations. The Complex-Variable Boundary-Element Method (CVBEM)—a new method in computational hydraulics—exactly satisfies the class of potential equations within the domain of interest and makes only two approximations along the boundary: discretization of the boundary and use of trial functions. Improvement of the solution accuracy and efficiency is among the most important factors for enhancing the CVBEM modeling capability. Two new schemes of boundary integration by trial functions have been investigated. The first scheme uses variable-definition trial function, which affords the modeler the capability of "fine tuning" without enlarging the size of solution matrix. The second scheme expands the CVBEM into a generalized Fourier series and eliminates the need for matrix solution entirely. These schemes seem to offer great potential for effective CVBEM modeling.

INTRODUCTION

As a new member in the family of boundary element methods, the Complex-Variable Boundary-Element Method or CVBEM has been shown to be an effective and valuable technique for modeling two-dimensional potential flows. The majority of boundary-element formulations have dealt with a real-variable integration along the boundary of a real domain (1,12). However, if the flow is two-dimensional, a boundary integration based on the Cauchy integral formula leads to a faster and more efficient computer algorithm (2). Among the first to apply such an approach were Hunt and Isaacs (8), Hromadka and Guymon (4), and Vinje and Brevig (14).

Systematic and intensive research on the CVBEM soon followed (5,6). A comprehensive account of the first-stage CVBEM development was later given in a volume, Hromadka and Lai (7). The second stage of CVBEM study, "advances in CVBEM modeling," was started immediately. Research topics and activities involved in the second stage are laid down by Lai and Hromadka (10), which include both ongoing projects and future plans.

These advanced modeling activities can generally be classified into three categories: (a) expanded CVBEM applications through combined use of the basic CVBEM with other existing solution methods (11); (b) advanced CVBEM modeling through improvement of the solution techniques (i.e. improvement of the numerical algorithms); and (c) enhanced CVBEM simulation through modification of computer algorithms, expansion of model components, adaptation to the real-world complexities, and update, revision, or renovation of general modeling techniques. A series of reports, each focusing on one of the aforementioned areas, can be compiled; some of these have already been written (3,11), some are in preparation, and some are planned for future publication.

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This paper is intended to address category (b) as described above. Several items falling into this category have been listed in Lai and Hromadka (10) under the section title "Development in CVBEM modeling." As the selection of trial functions affects most significantly the accuracy of CVBEM approximations, improvement in this matter is a most serious concern to the CVBEM modeler. This paper reports some recent improvements made in this area, accompanied by two illustrative examples.

THE PRINCIPLES OF THE COMPLEX-VARIABLE BOUNDARY-ELEMENT METHOD

In a simply connected complex region, Ω, with a simple closed boundary, Γ (see Fig. 1), the Cauchy integral formula

$$\omega(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\omega(\zeta)d\zeta}{\zeta - z} \tag{1}$$

defines a single-valued analytic function, ω, at any interior point, z, in terms of that funciton integrated along the boundary Γ. Thus, the field of two-dimensional potential flow, which may be represented by complex potential $\omega(z) = \phi(z) + i\,\psi(z)$, ϕ = potential function, ψ = stream function, can be well described by Eq. 1 if proper boundary conditions are given. The counter part of Eq. 1 in computer modeling, can be expressed as

$$\hat{\omega}(z) = \frac{1}{2\pi i}\sum_{j=1}^{m}\int_{\Gamma_j}\frac{G_n(\zeta)d\zeta}{\zeta - z} \tag{2}$$

Evidently, two basic approximations are involved in Eq. 2 from Eq. 1; the discretization of the boundary Γ into m boundary elements, Γ_j (Fig. 1), and the replacement of $\omega(z)$ on the boundary by a continuous global trial function, $G_n(z)$. These basic approximations result in an approximation function $\hat{\omega}(z)$, (H_n approximation function), which is different from $\omega(z)$. With a proper definition of G_n (see the following section), it can be proven that, for $\delta = \max\left|\Gamma_j\right|$, $\hat{\Gamma} = \bigcup_{j=1}^{m}\Gamma_j$; (i) $\lim_{\delta=0}\hat{\Gamma} = \Gamma$, $\lim_{\delta=0} G_n(z) = \omega(z)$, $z \in \Gamma$;

(ii) $\hat{\omega}(z)$ is analytic in Ω, and converges at $\hat{\Gamma}$ (Cauchy principal values exist); (iii) $\lim_{\delta=0}\hat{\omega}(z) = \omega(z)$, $z \in \Omega$; and (iv) $\hat{\omega}(z)$ is continuous on $\Omega \cup \hat{\Gamma}$. (7)

For CVBEM algorithm development based on the foregoing fundamental principles, one can easily perceive that both the accuracy and the efficiency of CVBEM modeling depend upon the soundness and degree of approximations rendered by $\hat{\Gamma}$ and G_n. The effect of the $\hat{\Gamma}$ approximation is a straight forward trade-off between finer discretization for better accuracy and fewer elements for easier computation. The effect of G_n, on the other hand, is more involved and complicated requiring some technical considerations and mathematical scrutiny. Advanced modeling through the improvement of trial functions seems to be a main concern of the CVBEM modeler, and is the theme of this paper.

THE TRIAL FUNCTION AND THE CVBEM DEVELOPMENT

To define the trial function G_n in the $\hat{\omega}$ development (see Eq. 2), a complex polynomial of degree n is generally used. The polynomial is defined on each boundary element but joined together at each vertex node to form a continuous global function. Among many possible choices for n, the linear

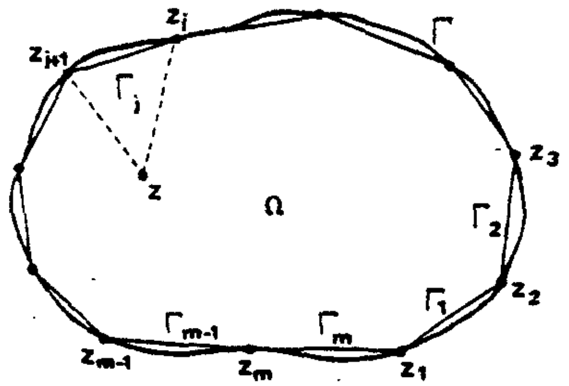

Fig. 1. CVBEM Boundary Discretization:
Γ_j = Boundary element linking nodes j and j+1;
z_j = Nodal coordinate for node j, ($z_{m+1} = z_1$);
Γ = Natural boundary

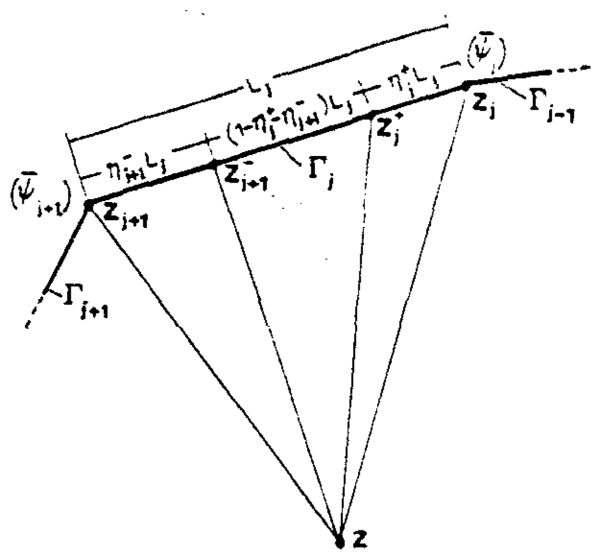

Fig. 2. Boundary-Element Geometry for Variable-Definition Trial Function

function (n=1) is, by far, most frequently used. The linear global trial function $G_1(z)$ can be defined as

$$G_1(z) = \sum_{j=1}^{m} N_j(z)\,\bar{\omega}_j \tag{3}$$

in which $\bar{\omega}_j$ is a specified nodal value at nodal point z_j on $\Gamma_{j-1} \cap \Gamma_j$ i.e. $\bar{\omega}_j = \bar{\omega}(z_j) = \bar{\Phi}(z_j) + i\bar{\Psi}(z_j)$, and $N_j(z)$ is a linear basis function defined as

$$N_j(z) = \begin{cases} (z - z_{j-1}) \,/\, (z_j - z_{j-1}), & z \in \Gamma_{j-1} \\ (z_{j+1} - z)/(z_{j+1} - z_j), & z \in \Gamma_j \\ 0, & z \notin \Gamma_{j-1} \cup \Gamma_j \end{cases} \tag{4}$$

Trial functions of other degrees can also be developed accordingly.(7)

To obtain $\hat{\omega}(z) = \hat{\Phi}(z) + i\hat{\Psi}(z) = \hat{\Phi}(x,y) + i\hat{\Psi}(x,y)$, from the given potential flow problem, the following algorithm has been developed. Assume at each node either $\bar{\Phi}(z_j)$ or $\bar{\Psi}(z_j)$ is known but not both (a typical condition in engineering problems). Let the interior point z ($z \in \Omega$) in Eq. 2 approach to an arbitrary node j. With the assurance of the existence of a Cauchy principal value, the circuit integration can be completed, which results in $\hat{\omega}(z_j)$. By repeating the same process to all nodes, and then by equating the real and imaginary parts on both sides of each resulting equation, 2m real-variable equations may be obtained. Then the m unknown nodal values, $\bar{\Psi}(z_j)$ or $\bar{\Phi}(z_j)$, can be evaluated by using only m out of the 2m equations, either by collocating known nodal values explicitly or collocating unknowns implicity. The process involves solution of an m x m matrix. (7)

General strategy in improving the accuracy of CVBEM solutions includes (a) finer discretization of the boundary, which is equivalent to increasing the number of nodes, and , (b) use of a higher degree polynomial for the trial function. The first approach results in a rapidly expanded square matrix, that is fully populated, whereas, the second approach leads to a greatly complicated solution algorithm. Consequently, it is desirable to derive different forms of global trial function that are not heavily subject to the above mentioned computational difficulties or constraints.

Two new schemes are proposed in the subsequent sections: one uses a variable-definition trial function to increase the solution accuracy without enlarging the square matrix, and the other relies on the Fourier series expansion to completely eliminate the matrix solution.

VARIABLE-DEFINITION TRIAL FUNCTION

Instead of using only one trial function for each boundary element Γ_j, a combination of trial functions, herein called a variable-definition trial function, may be used within a single element. For instance, a combination of constants and a linear function within a Γ_j may be tried.

Referring to Fig. 2, two non-negative weighting factors η_j^+ and η_{j+1}^- are selected on the boundary element Γ_j such that $\eta_j^+ + \eta_{j+1}^- \leq 1$. The straight-line element Γ_j with length L_j is then divided into three segments at points z_j^+ and z_{j+1}^-, with lengths, $\eta_j^+ L_j$, $(1 - \eta_j^+ - \eta_{j+1}^-)L_j$, and $\eta_{j+1}^- L_j$. Furthermore, the $\omega(z)$-value on Γ_j is distributed in such a way that $\omega(z) = \bar{\omega}_j$ and $\omega(z) = \bar{\omega}_{j+1}$ in the two end segments close to z_j and z_{j+1}, respectively, and linearly varied in the mid-segment, i.e. $\omega(z) = N_j^{\pm}(z)\,\bar{\omega}_j + N_{j+1}^{\pm}(z)\,\bar{\omega}_{j+1} = [(z_{j+1}^- - z)\bar{\omega}_j + (z - z_j^+)\bar{\omega}_{j+1}]/(z_{j+1}^- - z_j^+)$. The net effect of these weightings is an increase in nodal value influence over the corresponding element.

The contribution I_j, such a variable-definition global trial function $G(z)$ can bring forth to element Γ_j is, from Eq. 2,

$$I_j = \frac{1}{2\pi i}\int_{\Gamma_j} \frac{G(\zeta)d\zeta}{\zeta - z} \tag{5}$$

Now, if $\phi(z)$ is known on Γ_j, then a trial function can be selected to exactly calculate $\phi(z)$ contribution for Γ_j (with η_j^+ and η_{j+1}^- properly adjusted). For $\psi(z)$ unknown on Γ_j, use the trial function specified on Γ_j as shown in Fig. 2 (using the η_j^+ and η_{j+1}^- that are properly calibrated). Then the integral approximation is,

$$\int_{\Gamma_j} \frac{\psi(\zeta)d\zeta}{\zeta - z} = \int_{z_j}^{z_j^+} \bar{\Psi}_j \frac{d\zeta}{\zeta - z} + \int_{z_j^+}^{z_{j+1}^-} [N_j^{\pm}(\zeta)\bar{\Psi}_j + N_{j+1}^{\pm}(\zeta)\bar{\Psi}_{j+1}] \frac{d\zeta}{\zeta - z}$$
$$+ \int_{z_{j+1}^-}^{z_{j+1}} \bar{\Psi}_{j+1} \frac{d\zeta}{\zeta - z} \tag{6}$$

Using this technique, a global matrix system of order m x m (Because there is no increase in the number of elements or of vertex nodal points, the value of m, and thus the size of the solution matrix, remains unchanged) can be formed and the unknown nodal values, $\bar{\Psi}_j$, computed. With the known $(\bar{\Phi}_j)$ and the evaluated $(\bar{\Psi}_j)$ nodal values available, the approximation function $\hat{\omega}(z)$ can be readily computed.

An application of the variable-definition trial function to CVBEM solution of an ideal-fluid flow around a cylinder is illustrated in Fig. 3. The problem boundary Γ and domain Ω, with strategically placed 21 nodal points, are shown in Fig. 3a. The boundary conditions and other pertinent data are also provided in the figure. The numerical solution was first carried out using $\eta_j^+ = \eta_j^- = 0.50$, (corresponding to defining a constant nodal value over each boundary element). The comparison of the computed approximative boundary $\hat{\Gamma}$ and the given boundary Γ, displayed in Fig. 3b,

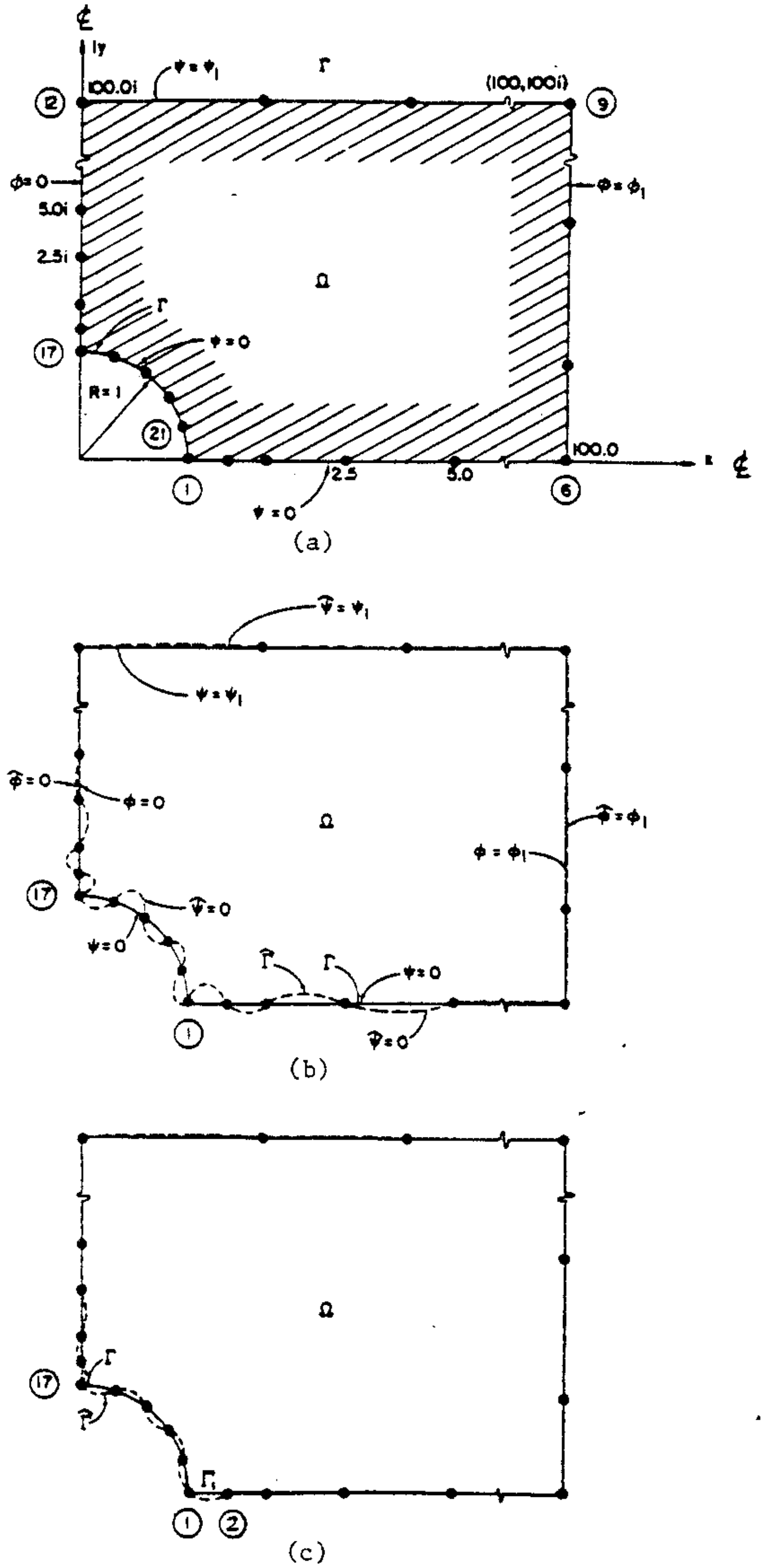

Fig. 3. Ideal-Fluid Flow Around a Cylinder: (a) Boundary conditions and nodal point placement, (b) Approximative boundary for $\eta^{\pm}_{j}=0.5$, (c) Approximative boundary using $\eta^{\pm}_{j}=0$ except for Nodes 1 and 17. (Note: Displacements are magnified tenfold.)

indicated that numerical integration error is most significant near the cylinder. Subsequently, $\eta_j^+ = \eta_j^- = 0$ were tried (corresponding to a linear trial funcion over each boundary element), which indicated that further adjustment was needed near node 1 or 17. The choice of $\eta_2^- = 0.95$ for Γ_1 and an equivalent choice for Γ_{16} were finally made. The improvement in computational accuracy is reflected in the closer fit between $\hat{\Gamma}$ and Γ as shown in Fig. 3c

Likewise, in other problems, the optimum simulation results may be obtained by properly adjusting the weighting factors $\eta_j^+$and η_j^-. More mathematical and technical details including practical error analysis are reported separately (3). In short, the basic idea of the variable-definition trial function is to provide the modeler with the capability of fine tuning without enlarging the matrix size, an important factor in simulation by a microcomputer.

FOURIER SERIES SOLUTION

Instead of developing a square matrix system for boundary nodes, which has been the case for all previous approaches, the CVBEM may be expanded as a generalized Fourier series, thus eliminating the matrix solution entirely. Through the use of Lebesque integrals and through generalized mathematical theorems [for background reading see Korn and Korn (9)], it can be proved that the complex flow field approximated by $\hat{\omega}(z)$ is amenable for solution by the Fourier series expansion. In other words, the problem flow fields possess the properties that are essential for the Fouries series application such as that $\hat{\omega}(z)$ is in an inner-product space, that Bessel's inequality applies, that the orthonormal vector technique is applicable, and that the CVBEM converges to the boundary values or to the midpoint values if discontinuous.

For a given potential flow problem, Eq. 2, with a linear trial function G_1, can be expanded and transformed to the form (see (7) for details)

$$\hat{\omega}_m(z) = \sum_{j=1}^{m} c_j\,(z-z_j)\,\ln\,(z-z_j) + R_1\,(z). \tag{7}$$

in which $c_j = a_j + ib_j$, a_j and b_j are real numbers; and $R_1(z)$ is a first degree complex polynomial resulting from the 2π–circuit along Γ about point z. By writing $R_1(z) = c_{m+1}(1)+c_{m+2}z$, $R_1(z)$ in Eq. 7 can be included in the preceding summation as the (m+1)th and (m+2)th terms. This permits Eq. 7 to be rearranged in the form (h_j may be identified by comparing with Eq. 7)

$$\hat{\omega}_m(z) = \sum_{j=1}^{m+2}(a_j\,h_j + ib_j\,h_j) = \sum_{j=1}^{2(m+2)} \gamma_j\,f_j \tag{8}$$

in which γ_j are real-number coefficients such that $\gamma_{2j-1} = a_j$ and $\gamma_{2j} = b_j$; and $f_{2j-1} = h_j$ and $f_{2j} = if_{2j-1} = ih_j$. The $\{f_j\}$ are then orthonormalized by the Gram-Schmidt procedure (9) to the set of function $\{g_j\}$ using the definition of inner-product, that is, $g_1 = f_1/\|f_1\|, \cdots, g_m = [f_m - (f_m,g_1)g_1 - \cdots - (f_m,g_{m-1})g_{m-1}]/\|f_m-(f_m,g_1)g_1-\cdots-(f_m,g_{m-1})g_{m-1}\|$. With respect to $\{g_j\}$,

$$\hat{\omega}_m(z) = \sum_{j=1}^{2(m+2)} \hat{\gamma}_j\,g_j(z) \tag{9}$$

in which $\hat{\gamma}_j$ are generalized Fourier coefficients to be determined. The value of $\|\omega-\hat{\omega}\|$ is minimized when $\hat{\gamma}_j = (\omega,g_j)$. By back substitution, the γ_j corresponding to the $\{f_j\}$ can be evaluated, which leads to the solution of the CVBEM approximator $\hat{\omega}(z)$, satisfying the given boundary conditions of ϕ and/or ψ in mean.

The Fourier series solution technique has been applied to a steady irrotational flow in a branching channel. The boundary conditions are

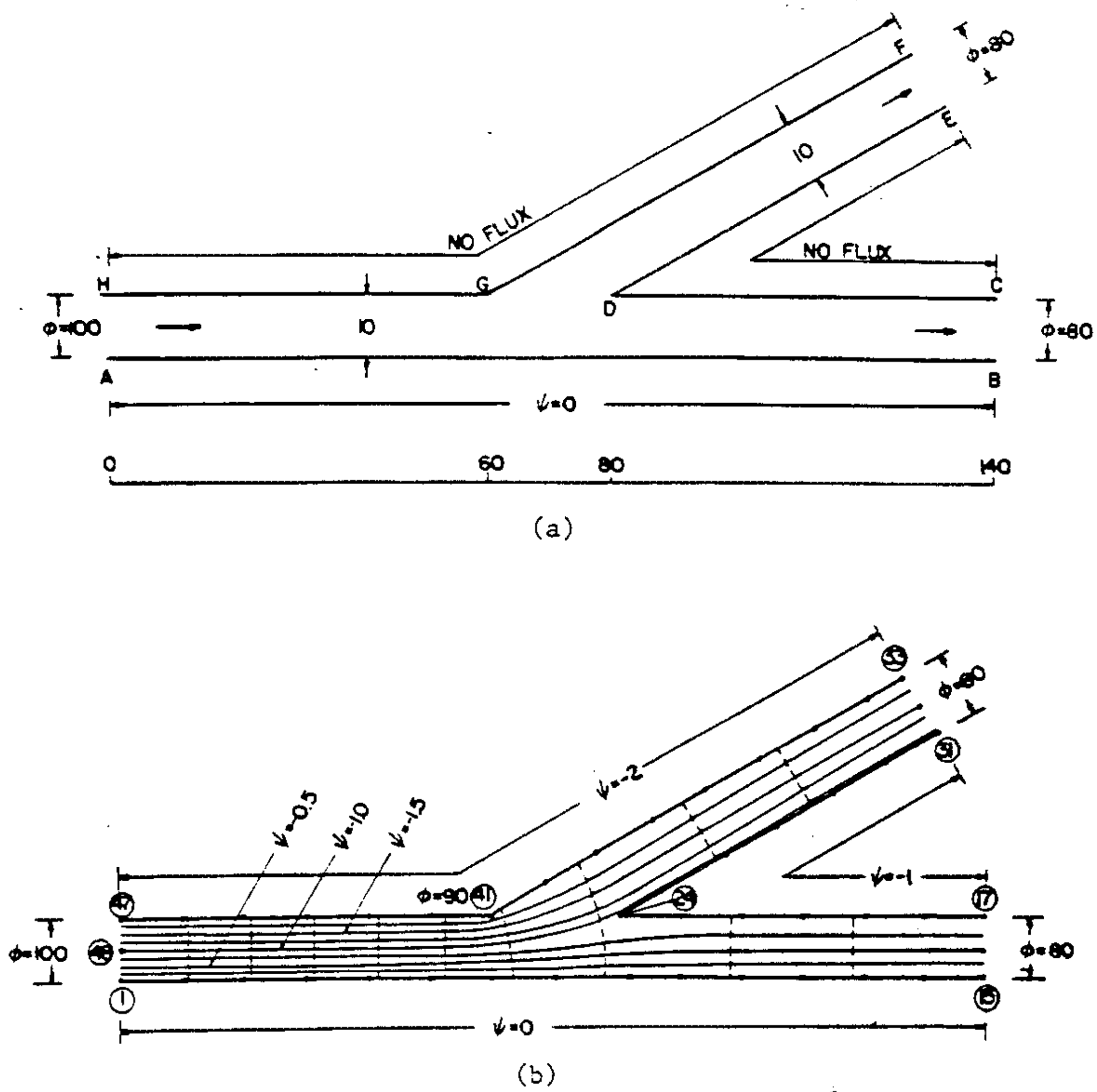

Fig. 4. Flow in a Branching Channel: (a) Boundary conditions, (b) Computed results.

specified at up- and down-stream ends as shown in Fig. 4a. The computed results are plotted in Fig. 4b, with solid lines representing streamlines and dashed lines equipotential lines. The computer algorithm for this approach is efficient, and the FORTRAN program implemented on a 64K personal computer, is easy to use. The rigorous theorem proofs, the detail mathematical development, and the computer algorithm formulation, accompanied by several more problem examples will be reported separately. Schultz (13) has applied a least-squares method to breaking waves and has reported that the approach appears to be more economical and robust than collocation approaches.

CONCLUSIONS

Improvement of the solution technique is among the most important factors for advancing the CVBEM modeling capability. The solution accuracy of the CVBEM depends heavily on the type of trial function and the method of the boundary integration used. Two new schemes of boundary integration by trial functions have been investigated. The first scheme uses a variable-definition trial function, which is found to afford the modeler the capability of delicate adjustment or "fine tuning" to increase the solution accuracy without enlarging the matrix size. The second scheme expands the CVBEM into a generalized Fourier series, approximates boundary conditions in a "mean-square" error sense, and therby eliminates the need for matrix solution entirely. These schemes seem to show substantial improvement over the previous schemes and offer potential for more effective CVBEM modeling.

REFERENCES

1. Brebbia, C.A., "New Development in Boundary Element Methods." Proc., 2nd Internat'l Sem., Univ. of Southampton, U.K., Mar., 1980. Comp. Mech. Center Pub.
2. Dold, J.W., and Peregrine, D.H., "Steep Unsteady Water Waves: An Efficient Computational Scheme," School of Math., Univ. of Bristol Internal Rep. AM-16-04.
3. Hromadka II, T.V., "Variable Trial Functions and the CVBEM," Engineering Analysis, CML Pub., Southampton, England. (in press)
4. Hromadka II, T.V., and Guymon, G.L., "Application of Boundary Integral Equation to Prediction of Freezing Fronts in Soils," Cold Regions Science and Technology, (6), 1982, pp. 115-121.
5. Hromadka II, T.V., and Guymon, G.L., "The Complex Variable Boundary Element Method: Development," Internat. Jour. Numerical Methods in Engineering Vol. 20, 1984, pp. 25-37.
6. Hromadka II, T.V., and Guymon, G.L., "Reducing Relative Error from the CVBEM by Proper Treatment of the Known Boundary Condition," Internat. Jour. Num. Methods in Eng., Vol. 20, 1984. pp. 2113-2120.
7. Hromadka II, T.V., and Lai, C., "The Complex Variable Boundary Element Method for Engineers and Scientists," (Springer Verlag; in process).
8. Hunt, B., and Isaacs, L.T., "Integral Equation Formulation For Ground Water Flow," Jour. of the Hyd. Div., ASCE, Vol. 107, No. HY10, 1981.
9. Korn, G.A., and Korn, T.M., "Mathematical Handbook," McGraw-Hill, New York, 1961.
10. Lai, C., and Hromadka II, T.V., "Modeling Hydraulic Problems Using the CVBEM and the Microcomputer," Proc., 1985 ASCE Hydraulics Div. Specialty Conf., Orlando, Florida, Aug., 1985.
11. Lai, C., and Hromadka T.V., "Modeling Complex Two-Dimensional Flows by the Complex-Variable Boundary-Element Method," Proc., Internat. Symp. on Refined Flow Modeling and Turbulence Measurements, Univ. of Iowa, Iowa City, Iowa, Sept., 1985.
12. Liggett, J.A., and Liu, P. L-F., "The Boundary Integral Equation Method for Porous Media Flow," George Allen and Unwin, London, 1983.
13. Schultz, W.W., "Integral Equation Algorithm for Breaking Waves," Proc., 11th IMCS World congress, Oslo, Aug., 1985.
14. Vinje, T., and Brevig, P, "Numerical Calculations of Forces from Breaking Waves," Hydrodynamics in Ocean Engineering, Norwegian Inst. Tech., 1981, pp. 547-565.

Key Words

Computational Fluid Mechanics, Computational Hydraulics, Complex Variables, Boundary Element Method, Two-Dimensional Flow, Potential Flow, Numerical Flow Modeling, Cauchy Integral Formula, Trial Functions, Fourier Series.