SOME ADVANCES IN CVBEM MODELING
OF TWO-DIMENSIONAL POTENTIAL FLOW

by

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For Presentation at the
ASCE Specialty Conference
Advancement in Aerodynamics
Fluid Mechanics and Hydraulics

Minneapolis, Minnesota
June 3-6, 1986
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ABSTRACT: Of interest in flow modeling is the exact solution of classes of
partial differential equations such as the Laplace and Poisson equations. The
Complex-Variable Boundary-Element Method (CVBEM)—a new method in
computational hydraulics—exactly satisfies the class of potential equations
within the domain of interest and makes only two approximations along the
boundary: discretization of the boundary and use of trial functions.
Improvement of the solution accuracy and efficiency is among the most
important factors for enhancing the CVBEM modeling capability. Two new
schemes of boundary integration by trial functions have been investigated.
The first scheme uses variable-definition trial function, which affords the
modeler the capability of "fine tuning" without enlarging the size of solution
matrix. The second scheme expands the CVBEM into a generalized Fourier
series and eliminates the need for matrix solution entirely. These schemes
seem to offer great potential for effective CVBEM modeling.

INTRODUCTION

As a new member in the family of boundary element methods, the
Complex-Variable Boundary-Element Method or CVBEM has been shown to be
an effective and valuable technique for modeling two-dimensional potential
flows. The majority of boundary-element formulations have dealt with a
real-variable integration along the boundary of a real domain (1,2).
However, if the flow is two-dimensional, a boundary integration based on
the Cauchy integral formula leads to a faster and more efficient computer
algorithm (2). Among the first to apply such an approach were Hunt and
Isaacs (8), Hromadka and Guymon (4), and Vinje and Brevig (14).

Systematic and intensive research on the CVBEM soon followed (5,6). A
comprehensive account of the first-stage CVBEM development was later given
in a volume, Hromadka and Lai (7). The second stage of CVBEM study,
"advances in CVBEM modeling," was started immediately. Research topics and
activities involved in the second stage are laid down by Lai and Hromadka
(10), which include both ongoing projects and future plans.

These advanced modeling activities can generally be classified into three
categories: (a) expanded CVBEM applications through combined use of the
basic CVBEM with other existing solution methods (II); (b) advanced CVBEM
modeling through improvement of the solution techniques (i.e. improvement of
the numerical algorithms); and (c) enhanced CVBEM simulation through modi-
fication of computer algorithms, expansion of model components, adaptation
to the real-world complexities, and update, revision, or renovation of general
modeling techniques. A series of reports, each focusing on one of the afore-
mentioned areas, can be compiled; some of these have already been written
(3,11), some are in preparation, and some are planned for future publication.

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This paper is intended to address category (b) as described above. Several items falling into this category have been listed in Lai and Hromadka (10) under the section title "Development in CVBEM modeling." As the selection of trial functions affects most significantly the accuracy of CVBEM approximations, improvement in this matter is a most serious concern to the CVBEM modeler. This paper reports some recent improvements made in this area, accompanied by two illustrative examples.

THE PRINCIPLES OF THE COMPLEX-VARIABLE BOUNDARY-ELEMENT METHOD

In a simply connected complex region, \( \Omega \), with a simple closed boundary, \( \Gamma \) (see Fig. 1), the Cauchy integral formula

\[
\omega(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\xi)d\xi}{\xi - z}
\]

(1)

defines a single-valued analytic function, \( \omega \), at any interior point, \( z \), in terms of that function integrated along the boundary \( \Gamma \). Thus, the field of two-dimensional potential flow, which may be represented by complex potential \( \omega(z) = \Phi(z) + i \psi(z) \), \( \Phi = \) potential function, \( \psi = \) stream function, can be well described by Eq. 1 if proper boundary conditions are given. The counter part of Eq. 1 in computer modeling, can be expressed as

\[
\hat{\omega}(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \frac{G_n(\xi_j)d\xi}{\xi_j - z}
\]

(2)

Evidently, two basic approximations are involved in Eq. 2 from Eq. 1; the discretization of the boundary \( \Gamma \) into \( m \) boundary elements, \( \Gamma_j \) (Fig. 1), and the replacement of \( \omega(z) \) on the boundary by a continuous global trial function, \( G_n(z) \). These basic approximations result in an approximation function \( \hat{\omega}(z) \), (\( H_n \) approximation function), which is different from \( \omega(z) \). With a proper definition of \( G_n \) (see the following section), it can be proven that, for \( \delta = \max \{ |\Gamma_j| \} \), \( \hat{\Gamma} \) = \( \Gamma \); (i) \( \lim_{\delta \to 0} \hat{\Gamma} = \Gamma \); (ii) \( \lim_{\delta \to 0} G_n(z) = \omega(z), z \in \Gamma \);

(iii) \( \hat{\omega}(z) \) is analytic in \( \Omega \) and converges at \( \hat{\Gamma} \) (Cauchy principal values exist); (iv) \( \lim_{\delta \to 0} \hat{\omega}(z) = \omega(z), z \in \Omega \); and (iv) \( \hat{\omega}(z) \) is continuous on \( \partial \Omega \).

For CVBEM algorithm development based on the foregoing fundamental principles, one can easily perceive that both the accuracy and the efficiency of CVBEM modeling depend upon the soundness and degree of approximations rendered by \( \hat{\Gamma} \) and \( G_n \). The effect of the \( \hat{\Gamma} \) approximation is a straightforward trade-off between finer discretization for better accuracy and fewer elements for easier computation. The effect of \( G_n \), on the other hand, is more involved and complicated requiring some technical considerations and mathematical scrutiny. Advanced modeling through the improvement of trial functions seems to be a main concern of the CVBEM modeler, and is the theme of this paper.

THE TRIAL FUNCTION AND THE CVBEM DEVELOPMENT

To define the trial function \( G_n \) in the \( \omega \) development (see Eq. 2), a complex polynomial of degree \( n \) is generally used. The polynomial is defined on each boundary element but joined together at each vertex to form a continuous global function. Among many possible choices for \( n \), the linear
Fig. 1. CVBEM Boundary Discretization:

\[ \Gamma_j = \text{Boundary element linking nodes } j \text{ and } j+1; \]
\[ z_j = \text{Nodal coordinate for node } j, \quad (z_m = z_1); \]
\[ \Gamma = \text{Natural boundary} \]

The linear global trial function \( \phi_1(z) \) can be defined as

\[ \phi_1(z) = \sum_{j=1}^{m} N_j(z) \bar{\omega}_j \]  

in which \( \bar{\omega}_j \) is a specified nodal value at nodal point \( z_j \) on \( \Gamma_j \cap \Gamma' \), i.e., \( \bar{\omega}_j = \phi(z_j) + i\psi(z_j) \), and \( N_j(z) \) is a linear basis function defined as

\[ N_j(z) = \begin{cases} 
\frac{(z - z_{j-1})}{(z_j - z_{j+1})}, & z \in \Gamma_j \\
\frac{(z_{j+1} - z)}{(z_{j+1} - z_j)}, & z \in \Gamma_j \\
0, & \text{elsewhere}
\end{cases} \]  

Trial functions of other degrees can also be developed accordingly. (7)

To obtain \( \phi(z) = \Phi(z) + i\psi(z) = \Phi(x,y) + i\psi(x,y) \), from the given potential flow problem, the following algorithm has been developed. Assume at each node either \( \Phi(z_j) \) or \( \psi(z_j) \) is known but not both (a typical condition in engineering problems). Let the interior point \( z \) (in \( \Omega \)) in Eq. 2 approach to an arbitrary node \( j \). With the assurance of the existence of a Cauchy principal value, the circuit integration can be completed, which results in \( \omega(z_j) \). By repeating the same process to all nodes, and then by equating the real and imaginary parts on both sides of each resulting equation, \( 2m \) real-variable equations may be obtained. Then the \( m \) unknown nodal values, \( \bar{\omega}(z_j) \) or \( \bar{\psi}(z_j) \), can be evaluated by using only \( m \) out of the \( 2m \) equations, either by collocating known nodal values explicitly or collocating unknowns implicitly. The process involves solution of an \( m \times m \) matrix. (7)

General strategy in improving the accuracy of CVBEM solutions includes (a) finer discretization of the boundary, which is equivalent to increasing the number of nodes, and (b) use of a higher degree polynomial for the trial function. The first approach results in a rapidly expanded square matrix, that is fully populated, whereas, the second approach leads to a greatly complicated solution algorithm. Consequently, it is desirable to derive different forms of global trial functions that are not heavily subject to the above mentioned computational difficulties or constraints.

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Two new schemes are proposed in the subsequent sections: one uses a variable-definition trial function to increase the solution accuracy without
enlarging the square matrix, and the other relies on the Fourier series
expansion to completely eliminate the matrix solution.

**VARIABLE-DEFINITION TRIAL FUNCTION**

Instead of using only one trial function for each boundary element \( \Gamma_j \),
a combination of trial functions, herein called a variable-definition trial
function, may be used within a single element. For instance, a combination
of constants and a linear function within a \( \Gamma_j \) may be tried.

Referring to Fig. 2, two non-negative weighting factors \( \eta^+_j \) and \( \eta^-_{j+1} \) are
selected on the boundary element \( \Gamma_j \) such that \( \eta^+_j + \eta^-_{j+1} < 1 \). The straight-line
element \( \Gamma_j \) with length \( L_j \) is then divided into three segments at points \( z_j^+ \)
and \( z_{j+1}^- \) with lengths, \( \eta^+_j L_j \), \( (1 - \eta^+_j - \eta^-_{j+1})L_j \), and \( \eta^-_{j+1} L_j \). Furthermore, the
\( \omega(z) \)-value on \( \Gamma_j \) is distributed in such a way that \( \omega(z) = \overline{\omega}_j \) and \( \omega(z) = \overline{\omega}_{j+1} \)
in the two end segments close to \( z_j \) and \( z_{j+1}^- \), respectively, and linearly varied
in the mid-segment, i.e., \( \omega(z) = N^+_j(z) \overline{\omega}_j + N^-_{j+1}(z) \overline{\omega}_{j+1} = \left( z - z_j^+ \right) \overline{\omega}_j + \right( z - z_{j+1}^- \right) \overline{\omega}_{j+1} / \left( z_j^+ - z_{j+1}^- \right) \). The net effect of these weightings is an increase in nodal
value influence over the corresponding element.

The contribution \( I_j \), such a variable-definition global trial function \( G(z) \)
can bring forth to element \( \Gamma_j \) is, from Eq. 2,

\[
I_j = \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{G(\zeta) d\zeta}{\zeta - z}
\]

(5)

Now, if \( \Phi(z) \) is known on \( \Gamma_j \), then a trial function can be selected to
exactly calculate \( \Phi(z) \) contribution for \( \Gamma_j \) (with \( \eta^+_j \) and \( \eta^-_{j+1} \) properly adjusted).

For \( \Phi(z) \) unknown on \( \Gamma_j \), use the trial function specified on \( \Gamma_j \) as shown in
Fig. 2 (using the \( \eta_j \) and \( \eta_{j+1} \) that are properly calibrated). Then the integral
approximation is,

\[
\int_{\Gamma_j} \frac{\Phi(z) d\zeta}{\zeta - z} = \int_{z_j^+}^{z_j} \frac{d\zeta}{\zeta - z} \overline{\omega}_j + \int_{z_{j+1}^-}^{z_{j+1}^+} \left[ N^+_j(\zeta) \overline{\omega}_j + N^-_{j+1}(\zeta) \overline{\omega}_{j+1} \right] \frac{d\zeta}{\zeta - z}
\]

(6)

Using this technique, a global matrix system of order \( m \times m \) (Because there
is no increase in the number of elements or of vertex nodal points, the value
of \( m \), and thus the size of the solution matrix, remains unchanged) can be
formed and the unknown nodal values, \( \overline{\omega}_j \), computed. With the known \( \overline{\omega}_j \),
and the evaluated \( \overline{\omega}_j \) nodal values available, the approximation function \( \omega(z) \)
can be readily computed.

An application of the variable-definition trial function to CVBEM
solution of an ideal-fluid flow around a cylinder is illustrated in Fig. 3.
The problem boundary \( \Gamma \) and domain \( \Omega \), with strategically placed 21 nodal
points, are shown in Fig. 3a. The boundary conditions and other pertinent
data are also provided in the figure. The numerical solution was first
carried out using \( \eta_j^+ = \eta_j^- = 0.50 \), (corresponding to defining a constant nodal
value over each boundary element). The comparison of the computed
approximative boundary \( \Gamma \) and the given boundary \( \Gamma \), displayed in Fig. 3b,
Fig. 3. Ideal-Fluid Flow Around a Cylinder: (a) Boundary conditions and nodal point placement, (b) Approximative boundary for $\eta_i=0.5$, (c) Approximative boundary using $\eta_i=0$ except for Nodes 1 and 17. (Note: Displacements are magnified tenfold.)
indicated that numerical integration error is most significant near the
cylinder. Subsequently, \( \eta^* = \eta_j^* = 0 \) were tried (corresponding to a linear trial
funcion over each boundary element), which indicated that further adjustment
was needed near node 1 or 17. The choice of \( \eta_2^* = 0.95 \) for \( \eta_1 \) and an equiva-
 lent choice for \( \eta_1 \) were finally made. The improvement in computational
accuracy is reflected in the closer fit between \( \bar{\Gamma} \) and \( \Gamma \) as shown in Fig. 3c.
Likewise, in other problems, the optimum simulation results may be
obtained by properly adjusting the weighting factors \( \eta_1^* \) and \( \eta_j^* \). More math-
eatical and technical details including practical error analysis are reported
separately (3). In short, the basic idea of the variable-definition trial function is to provide
the modeler with the capability of fine tuning without enlarging the matrix size, an important factor in simulation by a microcomputer.

FOURIER SERIES SOLUTION

Instead of developing a square matrix system for boundary nodes, which
has been the case for all previous approaches, the CVBEM may be expanded
as a generalized Fourier series, thus eliminating the matrix solution entirely.
Through the use of Lebesque integrals and through generalized mathematical
theorems (for background reading see Korn and Korn (9)), it can be proved
that the complex flow field approximated by \( \hat{\omega}(z) \) is amenable for solution by
the Fourier series expansion. In other words, the problem flow fields possess
the properties that are essential for the Fourier series application such as that
\( \hat{\omega}(z) \) is in an inner-product space, that Bessel's inequality applies, that the
orthonormal vector technique is applicable, and that the CVBEM converges to
the boundary values or to the midpoint values if discontinuous.

For a given potential flow problem, Eq. 2, with a linear trial function
\( G_1 \), can be expanded and transformed to the form (see (7) for details)

\[
\hat{\omega}_m(z) = \sum_{j=1}^{m} c_j (z-z_j) \ln (z-z_j) + R_1(z).
\]  

(7)

in which \( c_j = a_j + ib_j \), \( a_j \) and \( b_j \) are real numbers; and \( R_1(z) \) is a first
degree complex polynomial resulting from the \( 2\pi \)-circuit about \( \Gamma \) about point
\( z \). By writing \( R_1(z) = c_{m+1}(1)+c_{m+2}z \), \( R_1(z) \) in Eq. 7 can be included in the
preceding summation as the \( (m+1) \)th and \( (m+2) \)th terms. This permits Eq. 7 to
be rearranged in the form \( h_j = 1 \) may be identified by comparing with Eq. 7)

\[
\hat{\omega}_m(z) = \sum_{j=1}^{m+2} (a_j h_j + ib_j h_j) \gamma_j f_j \quad (8)
\]

in which \( \gamma_j \) are real-number coefficients such that \( \gamma_{j-1} = a_j \) and \( \gamma_{j-1} = b_j \);
and \( f_{j+1} = h_j \) and \( f_{j} = -if_{j-1} = ih_j. \) The \( \{f_j\} \) are then orthonormalized by the
Gram-Schmidt procedure (9) to the set of function \( \{g_j\} \) using the
definition of inner-product, that is, \( g_1 = f_1/\|f_1\|; \ldots; g_{m} = [f_m - (f_m g_1) g_1 - \ldots = (f_m g_{m-1}) g_{m-1}] / \| f_m - (f_m g_1) g_1 - \ldots - (f_m g_{m-1}) g_{m-1} \| \). With respect to \( \{g_j\},

\[
\hat{\omega}_m(z) = \sum_{j=1}^{2(m+2)} \gamma_j g_j(z).
\]  

(9)

in which \( \gamma_j \) are generalized Fourier coefficients to be determined. The value of
\( \| \omega - \hat{\omega} \| \) is minimized when \( \gamma_j = (\omega g_j) \). By back substitution, the \( \gamma_j \) corre-
sponding to the \( \{f_j\} \) can be evaluated, which leads to the solution of the
CVBEM approximant \( \hat{\omega}(z) \), satisfying the given boundary conditions of \( \psi \)
and/or \( \psi \) in mean.

The Fourier series solution technique has been applied to a steady
irrotational flow in a branching channel. The boundary conditions are
Fig. 4. Flow in a Branching Channel: (a) Boundary conditions, (b) Computed results.

specified at up- and down-stream ends as shown in Fig. 4a. The computed results are plotted in Fig. 4b, with solid lines representing streamlines and dashed lines equipotential lines. The computer algorithm for this approach is efficient, and the FORTRAN program implemented on a 64K personal computer, is easy to use. The rigorous theorem proofs, the detail mathematical development, and the computer algorithm formulation, accompanied by several more problem examples will be reported separately. Schultz (13) has applied a least-squares method to breaking waves and has reported that the approach appears to be more economical and robust than collocation approaches.

CONCLUSIONS
Improvement of the solution technique is among the most important factors for advancing the CVBEM modeling capability. The solution accuracy
of the CVBEM depends heavily on the type of trial function and the method of the boundary integration used. Two new schemes of boundary integration by trial functions have been investigated. The first scheme uses a variable-definition trial function, which is found to afford the modeler the capability of delicate adjustment or "fine tuning" to increase the solution accuracy without enlarging the matrix size. The second scheme expands the CVBEM into a generalized Fourier series, approximates boundary conditions in a "mean-square" error sense, and thereby eliminates the need for matrix solution entirely. These schemes seem to show substantial improvement over the previous schemes and offer potential for more effective CVBEM modeling.

REFERENCES
Key Words