Developing Accurate Solutions of Potential Problems Using an Approximate Boundary and a Boundary Element Method

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ABSTRACT

The Complex Variable Boundary Element Method or CVBEM provides an easy-to-use numerical analogue of two-dimensional potential problems. The boundary integral approach develops a two-dimensional approximation function which solves the Laplace equation exactly over the problem domain, but satisfies the boundary conditions, generally, only at nodal points which are located on the problem boundary. The approximation function, however, achieves the problem boundary condition values at coordinates which define an approximate boundary. Should the approximate boundary coincide with the true problem boundary, the exact solution to the boundary value problem has been achieved. Because the approximate boundary is a visual representation of approximation error, the analyst can develop accurate CVBEM approximations by use of a light-pen and interactive CRT response to the CVBEM software.

INTRODUCTION

The Complex Variable Boundary Element Method or CVBEM has been shown to be a useful tool for the numerical analysis of Laplace or Poisson equation boundary value problems (Hromadka, 1984). The numerical procedure is to discretize the boundary Γ by nodal points into boundary elements, and then specify a continuous global trial function $G(\zeta)$ on Γ as a function of the nodal values. Using the Cauchy integral, the resulting integral equation is

$$\hat{\omega}(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{G(\zeta)d\zeta}{\zeta - z_0} \tag{1}$$

where $\hat{\omega}(z_0)$ is the CVBEM approximation for $z_0 \in \Omega$; and Ω is a two-dimensional simply connected domain enclosed by the simple closed contour Γ.

Because $G(\zeta)$ is continuous on Γ, then $\hat{\omega}(z)$ is analytic over Ω and can be rewritten as the sum of two harmonic functions

$$\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z) \tag{2}$$

Thus both $\hat{\phi}(z)$ and $\hat{\psi}(z)$ exactly satisfy the Laplace equation over Ω.

Approximation error occurs due to $\hat{\omega}(z)$ not satisfying the boundary conditions on Γ exactly. However, an approximate boundary $\hat{\Gamma}$ can be developed which represents the location of points where $\hat{\omega}(z)$ does equal the specified boundary conditions such as level curves (see Fig. 1). Consequently, the CVBEM approximation error can be interpreted as a transformation of $\Gamma \to \hat{\Gamma}$ where the ultimate objective is to have $\hat{\Gamma}$ coincident with Γ. Because all the error of approximation is due to the incorrect boundary element trial functions, accuracy is increased by the addition of boundary nodal points where approximation error is large (i.e., adaptive integration).

In this paper, a computer interactive technique is reported which graphically displays Γ and $\hat{\Gamma}$ so that the numerical analyst can readily specify additional nodal points on the CRT screen. In this fashion, the user interacts with the CVBEM to locate the necessary nodal point additions until $\hat{\Gamma}$ and Γ are within an acceptable level of tolerance. For example, the tolerance may be the allowable construction limits specified for a shaft (torsion problem) for use in aircraft design.

As $\hat{\Gamma}$ approaches Γ geometrically, the analyst is assured by the Maximum Modulus Theorem that the maximum approximation error occurs on Γ and that the governing partial differential equation (Laplace) is solved exactly. Consequently, the final product is the exact solution for a problem geometry which is within the construction tolerance of the design.

THEORETICAL BACKGROUND OF THE CVBEM

A complete presentation of the CVBEM development, case studies, mathematical proofs of convergence and existence, and several FORTRAN computer programs are given in Hromadka (1984). In order to develop the geometric interpretation of modeling error associated with the approximative boundary concept, a brief development of the CVBEM numerical technique is presented in the following.

Let Ω be a simply connected two-dimensional domain (i.e. no holes within Ω) enclosed by a simple closed contour Γ (e.g. Mathews, 1982). Let $\phi(x,y)$ be a two-dimensional harmonic function over $\Omega \cup \Gamma$; that is,

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0 , \quad (x,y) \in \Omega \cup \Gamma \tag{3}$$

Then there exists a simply connected domain Ω^* such that $\Omega \cup \Gamma$ is a proper subset of Ω^* and $\phi(x,y)$ is harmonic over Ω^*.

There exists a harmonic function $\psi(x,y)$ conjugate to $\phi(x,y)$ which also satisfies the Laplace equation of (3) over Ω^* and additionally satisfies the Cauchy-Rieman conditions of

$$\frac{\partial\phi(x,y)}{\partial x} = \frac{\partial\psi(x,y)}{\partial y} , \quad \frac{\partial\phi(x,y)}{\partial y} = -\frac{\partial\psi(x,y)}{\partial x} \tag{4}$$

Let $z = x + iy$ be a complex variable over Ω^*. Then both $\phi(x,y)$ and $\psi(x,y)$ can be written in terms of $\phi(z)$ and $\psi(z)$ such that an analytic function $\omega(z)$ is defined over Ω^* by

$$\omega(z) = \phi(z) + i\psi(z) \tag{5}$$

where to simply notation, (5) can be rewritten as $\omega = \phi + i\psi$, $z \in \Omega^*$.

Equation (5) represents a relationship between two conjugate harmonic functions generally called the potential (ϕ) and stream functions (ψ).

The Cauchy integral theorem equates values of $\omega(z_0)$ for $z_0 \in \Omega$ to a line integral of $\omega(\zeta)$ for $\zeta \in \Gamma$ by

$$\omega(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta)d\zeta}{\zeta - z} \tag{6}$$

To illustrate the development of a CVBEM approximation function, $\hat{\omega}(z)$, consider $\omega(z)$ to be defined over Ω^* with $\Omega \cup \Gamma$ interior of Ω^*. Subdivide Γ into m boundary elements Γ_j such as shown in Fig. 2. Nodal points are specified at each element endpoint (here, a linear polynomial CVBEM approximation is being developed). At each node, determine nodal values of $\omega(z)$ by

$$\omega(z_j) \equiv \omega_j = \phi(z_j) + i\psi(z_j) \equiv \phi_j + i\psi_j ; \quad j = 1,2,\cdots,m \tag{7}$$

Then a global trial function of $\omega(z)$ is determined for $z \in \Gamma$ by

$$G(z) = \sum_{j=1}^{m} \delta_j \left[\omega_j N_j(z) + \omega_{j+1} N_{j+1}(z) \right] \tag{8}$$

Where the $N_j(z)$ are linear basis functions (see Fig. 3); and $\delta_j = 1$ for $z \in \Gamma_j$, and $\delta_j = 0$ for $z \notin \Gamma_j$. Substituting $G(z)$ in place of $\omega(\zeta)$ in (6) determines a CVBEM approximation $\hat{\omega}(z)$ of $\omega(z)$

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta) d\zeta}{\zeta - z} \tag{9}$$

Letting $||\Gamma_m|| = \max |z_{j+1} - z_j|$, $j = 1, 2, \cdots, m$, then it is seen (without proof) that

$$\lim_{||\Gamma_m|| \to 0} G(\zeta) = \omega(\zeta), \quad \zeta \in \Gamma \tag{10}$$

and therefore

$$\lim_{||\Gamma_m|| \to 0} (\omega(z) - \hat{\omega}(z)) = \lim_{||\Gamma_m|| \to 0} \frac{1}{2\pi i} \int_{\Gamma} \frac{(\omega(\zeta) - G(\zeta)) d\zeta}{\zeta - z} = 0 \tag{11}$$

Thus the error of approximation, $e(z)$, is defined by

$$e(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\omega(\zeta) - G(\zeta)) d\zeta}{\zeta - z} \tag{12}$$

Because $G(\zeta)$ is continuous on Γ then $\hat{\omega}(z)$ is analytic over Ω which implies both $\hat{\phi}(z)$ and $\hat{\psi}(z)$, where $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, are potential functions over Ω.

In practice, $\phi(z)$ is known on Γ_ϕ and $\psi(z)$ is known on a separate contour on Γ_ψ where $\Gamma = \Gamma_\phi \cup \Gamma_\psi$. Thus $\hat{\omega}(z)$ is not completely defined without estimates for the unknown nodal values. To obtain such estimates, the real (or imaginary) parts of $\hat{\omega}(z)$ are collocated to the m known nodal values, resulting in m equations for the m unknown nodal values. Using these m nodal value estimates along with the m known nodal values supplies the $\hat{\omega}(z)$ integral function with sufficient data to determine the CVBEM approximation of (9).

CVBEM APPROXIMATION ERROR

Generally, numerical approximation errors in solving potential problems is of two forms: (i) errors due to not satisfying the governing equation over Ω, and (ii) errors due to not satisfying the boundary conditions continuously on Γ. For the CVBEM, (and for other boundary integral equation methods), the first type of approximation error is eliminated due to both $\hat{\phi}$ and $\hat{\psi}$ being potential functions. But $\hat{\omega}(z)$ does not usually satisfy the boundary conditions continuously on Γ (if it did, then $\hat{\omega}(z) = \omega(z)$). The next step in the CVBEM analysis is to work with $\hat{\omega}(z)$ in order that $\hat{\omega}(z) \to \omega(z)$.

Probably the easiest form of error to study is the development of the approximate boundary $\hat{\Gamma}$ which represents the locations where $\hat{\omega}(z)$ achieves the desired boundary values of $\omega(z)$ (Hromadka, 1985). Generally, the boundary conditions are constant values of ϕ or ψ along boundary elements, i.e., $\phi = \phi_j$ for $z \in \Gamma_j$ or $\psi = \psi_k$, for $z \in \Gamma_k$. This set of m nodal values $\{\phi_j, \psi_k\}$ are level curves of $\omega(z)$. The approximate boundary $\hat{\Gamma}$ is determined by locating those points where $\hat{\phi} = \phi_j$ and $\hat{\psi} = \psi_k$. Due to the collocation process, $\hat{\Gamma}$ intersects Γ at least at each nodal point location, z_j, $j = 1, 2, \cdots, m$.

To determine $\hat{\Gamma}$, each element Γ_j is further subdivided by interior points (specified by the program user) where $\hat{\omega}(z)$ is to be evaluated. At each element interior point, $\hat{\omega}(z)$ is calculated from the line integral of (9) and the values of $\hat{\phi}$ and $\hat{\psi}$ are determined. If the appropriate $\hat{\phi}$ (or $\hat{\psi}$) matches the boundary condition on Γ_j, then $\hat{\Gamma}$ intersects Γ at that point. Otherwise, subsequent points are evaluated by marching pointwise along a line perpendicular to Γ_j until the boundary condition value is reached. For point locations interior of Ω, Eq. (9) is used. For points exterior of $\Omega \cup \Gamma$, an analytic continuation of (9) is used.

In this fashion, a set of points are determined where $\hat{\omega}(z)$ equals the desired ϕ_j or ψ_k values. The contour $\hat{\Gamma}$ is estimated by then connecting these points by straight lines. Because $\hat{\Gamma}$ and Γ intersect at least at nodal point locations, $\hat{\Gamma}$ appears as a plot which typically oscillates about the Γ contour.

GRAPHICAL DISPLAY TECHNIQUE

Using a CRT graphical display software package, the CVBEM software is designed to operate in the following analysis steps:

1. Using a light-pen or batch-file data entry, the coordinates of the problem boundary are entered by means of nodal points. This initial data entry of nodal coordinates results in the first CVBEM approximation function.

2. Boundary condition values are entered at nodal points. Boundary conditions are assumed to vary linearly between nodes.

3. The CVBEM approximation function is developed.

4. The approximate boundary corresponding to the nodal point distribution is developed.

5. The approximate boundary $\hat{\Gamma}$ is plotted on the graphics CRT super-imposed on the true problem boundary Γ. A magnification factor may be entered to enlarge the discrepancy between $\hat{\Gamma}$ and Γ.

6. Using the light-pen, the analyst locates additional nodes on Γ for the subsequent development of a new CVBEM approximation function and a new $\hat{\Gamma}$.

7a. Go to Step 3 to repeat the CVBEM analysis.

7b. Otherwise, if $\hat{\Gamma}$ and Γ are within an acceptable tolerance, the CVBEM analysis effort is complete.

APPLICATION

To illustrate the above procedure, the approximate boundary technique is used to develop a highly accurate numerical solution of heat flow in saturated soil due to freezing conduit (-10°C) located beneath the soil surface. Of concern in this problem is the precise location of the 0°-C isotherm which represents the steady-state freezing front within the soil. The ultimate location of the freezing front leads to the design of the neighboring roadway embankment to survive the freezing-thawing cycle of the algid climate.

The problem definition for this problem is shown in Fig. 4. Also shown in Fig. 4 are the initial nodal point locations used to develop the approximate boundary.

Figure 5 shows the approximate boundary resulting from the initial CVBEM solution. From the figure, $\hat{\Gamma}$ and Γ are very close, with the largest discrepancy located near the freezing conduit. Consequently, only near the conduit are additional nodes required.

Using the light-pen, two nodes are added on the freezing conduit.
Figure 6 shows the resulting approximate boundary superimposed on the true
problem boundary. The maximum departure between $\hat{\Gamma}$ and Γ is less than 2 cm,
which is an acceptable tolerance for this analysis.

Using the nodal distribution of Fig. 4, modified by Fig. 6, the CVBEM
approximation provides the exact solution to the given boundary value
problem with the true problem boundary Γ transformed to the approximate
boundary, $\hat{\Gamma}$.

OTHER BOUNDARY ELEMENT METHODS

The presented approximate boundary technique may be developed for
other boundary integral methods. Typically, the procedure to be used
is to expand the boundary integral approximation function into a finite
sum of potential functions (usually, logarithm and polynomial functions
form the basis of the approximation function expansion) which are re-
defined with respect to the argument function (the angle θ measured with
respect to each nodal point on the boundary) so that the approximation
function may be evaluated on both sides of the problem boundary; that is,
an analytic continuation is developed in the vicinity of each node to
the exterior of the problem domain.

The approximate boundary is then developed for each problem by the
procedures presented in the previous section.

Because the approximate boundary provides an easy-to-understand re-
presentation of approximation error, its incorporation into general purpose
boundary element codes should be considered. The metric for approximation
error is simply the "closeness-of-fit" between $\hat{\Gamma}$ and Γ. Because $\hat{\Gamma} \rightarrow \Gamma$ as
the number of nodes becomes large (placed by the preceding technique),
eventually the boundary integral provides a "probable prototype" where $\hat{\Gamma}$
is closer to Γ than required by the construction tolerance of Γ. That is,
$\hat{\Gamma}$ may represent a more realistic representation of the actual constructed
domain than the mathematical idealization of Γ.

CONCLUSIONS

The approximate boundary technique is used with the CVBEM to
develop highly accurate numerical solutions of potential problems.
Because the approximate boundary provides an easy-to-use error evaluation
procedure, it's use in general purpose boundary element method codes
should be considered.

REFERENCES

1. Hromadka II, T. V., "The Complex Variable Boundary Element Method,"
    Springer-Verlag (1984).
2. Hromadka II, T. V., "Four Methods of Locating Collocation Points for
    the CVBEM," Engineering Analysis (1985). in-press
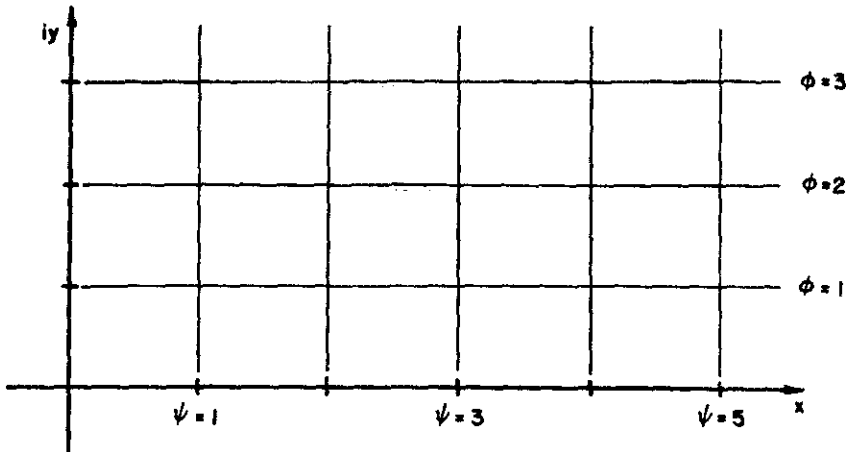3. Mathews, J. H., "Basic Complex Variables," Allyn & Bacon, Inc., (1982).

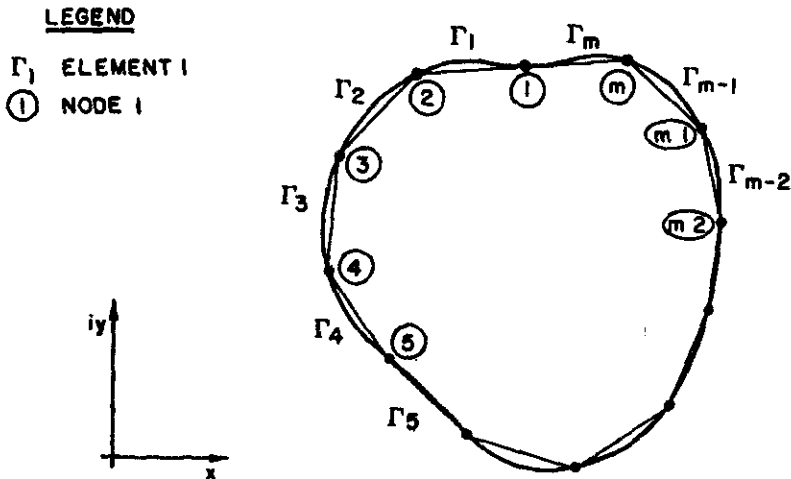Fig. 1. Level Curves of an Analytic Function
(Example Shown: w(z) = z)

Fig. 2. Modeling Γ by Boundary Elements Γ_j
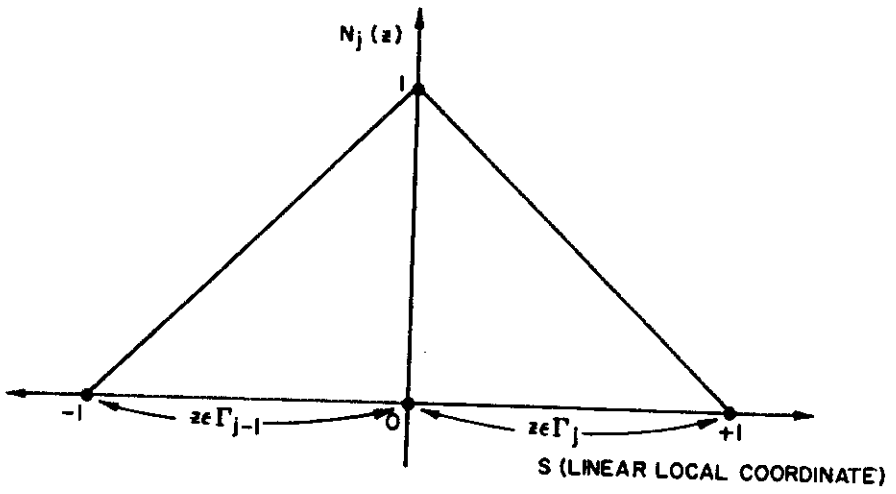
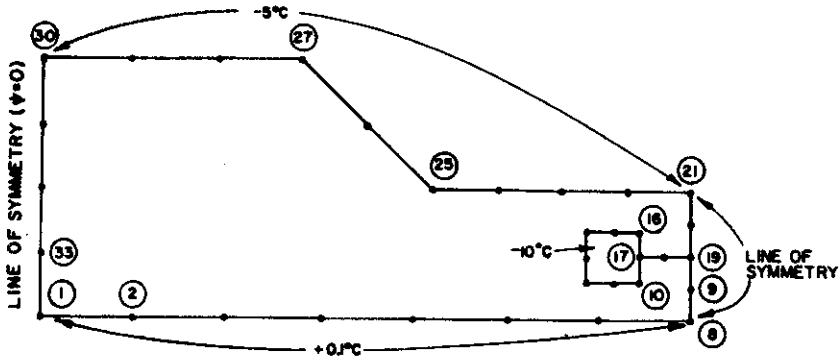Fig. 3. Linear Basis Function

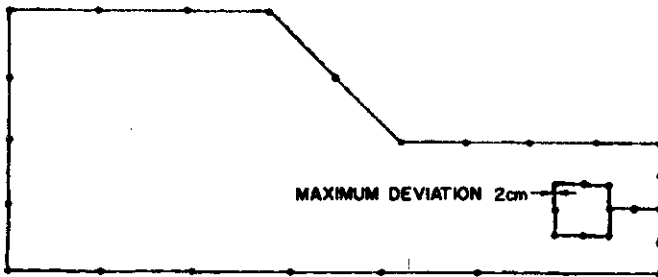Fig. 4 Freezing Conduit Application Problem Geometry

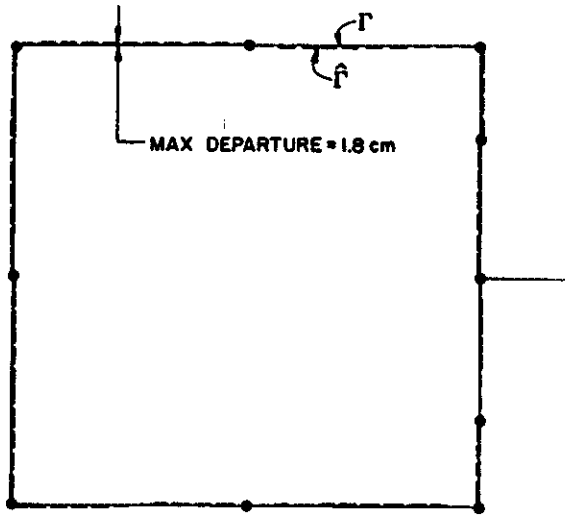Fig. 5 Initial CVBEM Solution Approximate Boundary

Fig. 6 Additional Nodes on Freezing Conduit
Reduces Approximation Error