Complex boundary element solution of flow field problems without matrices

T. V. HROMADKA II

Associate Professor of Mathematics, California State University, Fullerton, California, USA

C. C. YEN

Hydrologist, Williamson and Schmid, Irvine, California, USA

Instead of developing a square order m matrix system for m boundary nodes by collocating the boundary integral equation at nodal point boundary values, the Complex Variable Boundary Element Method is now expanded as a generalised Fourier series – eliminating the matrix solution entirely. Boundary conditions are approximated in a 'mean-square' error sense in that a new vector space norm is defined which is analogous to the L^2 norm, and then minimised by the selection of complex coefficients to be associated to each nodal point located on the problem boundary, Γ. For engineering problems where the boundary condition values and their first derivative are piecewise continuous on Γ (i.e. Dirichlet conditions), the new CVBEM approximation converges almost everywhere (*ae*) on Γ as guaranteed by well-known generalised Fourier series theory.

1. INTRODUCTION

1.1. Objectives of paper

The objective in using the Complex Variable Boundary Element Method (or CVBEM) is to approximate analytic complex functions. More specifically, if ω is a two-dimensional complex function which is analytic over a simply connected domain Ω with boundary values $\omega(\zeta)$ for $\zeta \in \Gamma$ (Γ is a simple closed contour), then the real (ϕ) and imaginary (ψ) parts of $\omega = \phi + i\psi$ both satisfy the Laplace equation over Ω. Thus, two-dimensional flow field problems can be solved numerically by the CVBEM.

The development of the CVBEM for engineering applications is detailed in Hromadka.[1] Generally speaking, the CVBEM is a boundary integral technique and, consequently, a literature review of this class of numerical methods can be found in other books such as Lapidus and Pinder.[2]

However in this paper, the CVBEM departs from the other boundary integral methods by using a new unpublished technique in satisfying boundary conditions. Instead of developing a square order m matrix system for m boundary nodes by collocating the boundary integral equation at nodal point boundary values, the CVBEM is now expanded as a generalised Fourier series – eliminating the matrix solution entirely. Boundary conditions are approximated in a 'mean-square' error sense in that a new vector space norm is defined which is analogous to the L^2 norm, and then minimised by the selection of complex coefficients to be associated to each nodal point located on the problem boundary, Γ. For engineering problems where the boundary condition values and their first derivative are piecewise continuous on Γ (i.e. Dirichlet conditions), the new CVBEM approximation converges almost everywhere (*ae*) on Γ as guaranteed by well-known generalised Fourier series theory.

In this paper the new CVBEM approach will be developed with mathematical rigor before presenting applications of the numerical technique. In order to keep the paper concise, the development of CVBEM approach, the definition of the new inner-product used, the definition of the working vector spaces, proofs of convergence of the generalised Fourier series expansion, and the proof of boundary condition convergence are all presented in standard Theorem/Proof form.

The final section of the paper illustrates the new numerical technique by solving several flow field problems where solutions are known.

1.2. Notation

Let ω be an analytic function over Ω.

The following notation is used in this paper:

Ω	= convex, simply connected domain
Γ	= simple closed contour forming the boundary of Ω
ζ, z	$= \zeta \in \Gamma, z \in \Omega; \zeta = \mathrm{Re}^{i\theta}$ for $0 \leqslant \theta < 2\pi$
ϕ	$= \mathrm{Re}\,\omega, \omega = \phi + i\psi$
ψ	$= \mathrm{Im}\,\omega$
$\lvert\omega\rvert$	$= (\phi^2 + \psi^2)^{1/2}$
$\mathrm{d}\mu$	$= \lvert \mathrm{d}\zeta \rvert, \zeta \in \Gamma$
$\lVert\omega\rVert_p$	$= (\int_\Gamma \lvert\omega(\zeta)\rvert^p \,\mathrm{d}\mu)^{1/p}$
$\{z_j\}$	= nodal points defined on Γ
Γ_j	= boundary element (line segment) connecting z_j, z_{j+1}
Γ_ϕ, Γ_ψ	$= \Gamma_\phi \cup \Gamma_\psi = \Gamma$ and $\Gamma_\phi \cap \Gamma_\psi$ at two points of Γ. Here ϕ is known on Γ_ϕ and ψ is known on Γ_ψ where $\omega = \phi + i\psi$

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(ω, ω)	$= \int_{\Gamma_\phi} \phi^2 \, d\mu + \int_{\Gamma_\psi} \psi^2 \, d\mu$
$\|\omega\|$	$= (\omega, \omega)^{1/2}$
L	= length of Γ
l	$= \max \|z_{j+1} - z_j\|$, for nodes $\{z_j\} \in \Gamma$
z_c	= centroid of Ω oriented such that $z_c = 0 + 0i$
δ	= a co-ordinate reduction factor, $0 \leqslant \delta < 1$
Γ_δ	$= \{\delta\zeta, \zeta \in \Gamma\}$
Ω_δ	$= \{\delta z, z \in \Omega\}$
$\bar{\Omega}$	$= \Omega \cup \Gamma$
$\bar{\Omega}_\delta$	$= \Omega_\delta \cup \Gamma_\delta$
Λ	= number of angle points on Γ
ω_j	= nodal value $\omega(z_j)$, $\omega_j = \phi_j + i\psi_j$
θ_j	= branch-cut angle of $\ln_j(z - z_j)$
$N_j(\zeta)$	= linear basis function defined on $\zeta \in \Gamma$
$\bar{\omega}_j$	approximate nodal value at $z_j \in \Gamma$

1.3. Definition of working space, W_Ω

Let Ω be a simply connected convex domain with a simple closed piecewise linear boundary Γ and with its centroid located at $0 + 0i$. Then $\omega \in W_\Omega$ has the properties

(i) $\omega(z)$ is analytic over Ω

(ii) $\lim_{\delta \to 1} \int_\Gamma |\omega(\delta\zeta)|^2 \, d\Gamma \leqslant M < \infty$

1.4. Definition of the function $\|\omega\|$

A key element in the CVBEM development of this paper is the definition of a norm and inner-product. In the following sections, insight into the new norm function is presented by an analogy to the well known $L^2(\Gamma)$ norm and inner-product.

The symbol $\|\omega\|$ for $\omega \in W_\Omega$ is notation for

$$\|\omega\| = \left[\int_{\Gamma_\phi} (\mathrm{Re}\,\omega)^2 \, d\mu + \int_{\Gamma_\psi} (\mathrm{Im}\,\omega)^2 \, d\mu \right]^{1/2}$$

The symbol $\|\omega\|_p$ for $\omega \in W_\Omega$ is notation for

$$\|\omega\|_p = \left[\int_\Gamma |\omega(\zeta)|^p \, d\mu \right]^{1/p}, \quad p \geqslant 1$$

Of importance is the case of $p = 2$:

$$\|\omega\| = \left[\int_\Gamma |\omega(\zeta)|^2 \, d\phi \right]^{1/2} = \left[\int_\Gamma \left([\mathrm{Re}\,\omega]^2 + [\mathrm{Im}\,\omega]^2 \right) d\mu \right]^{1/2}$$

1.5. Almost everywhere (ae) equality

A property or function which applies everywhere on a set E except for a subset E' in E such that $m(E') = 0$ is said to apply almost-everywhere (*ae*). Because sets of Lebesque measure zero have no effect on integration, almost-everywhere (*ae*) equality on Γ indicates the same class of element. Thus for $\omega \in W_\Omega$, $[\omega] = \{\omega \in W_\Omega : \omega(\zeta)$ are equal *ae* for $\zeta \in \Gamma\}$. For example, $[0] = \{\omega \in W_\Omega : \omega(\zeta) = 0$ *ae*, $\zeta \in \Gamma\}$. When understood, the notation '[ ]' will be dropped.

1.6. Theorem (relationship of $\|\omega\|$ to $\|\omega\|_2$)

Let $\omega \in W_\Omega$. Then $\|\omega\|_2^2 = \|\omega\|^2 + \|i\omega\|^2$.

Proof

Let $\omega = \phi + i\psi$. Then

$$\|\omega\|_2^2 = \int_\Gamma |\omega(\zeta)|^2 \, d\mu$$

$$= \int_\Gamma (\phi^2 + \psi^2) \, d\mu$$

$$= \int_{\Gamma_\phi} \phi^2 \, d\mu + \int_{\Gamma_\psi} \phi^2 \, d\mu + \int_{\Gamma_\phi} \psi^2 \, d\mu + \int_{\Gamma_\psi} \psi^2 \, d\mu$$

$$= \|\omega\|^2 + \int_{\Gamma_\phi} (-\psi)^2 \, d\mu + \int_{\Gamma_\psi} \phi^2 \, d\mu$$

$$= \|\omega\|^2 + \|i\omega\|^2$$

1.7. Theorem

Let $\omega \in W_\Omega$. Then $\|\omega\|_2 = 0 \Rightarrow \|\omega\| = 0$.

Proof

$\|\omega\|_2 = 0$ implies $\|\omega\|_2^2 = \|\omega\|^2 + \|i\omega\|^2 = 0$.

1.8. Theorem

Let $\omega \in W_\Omega$. Then $\|\omega\| \leqslant \|\omega\|_2$.

Proof

Let $\omega = \phi + i\psi$. Using Theorem 1.6,

$\|\omega\|_2^2 = \|\omega\|^2 + \|i\omega\|^2$

Because $\|i\omega\|^2 \geqslant 0$, then $\|\omega\|^2 \leqslant \|\omega\|_2^2$.

2. MATHEMATICAL DEVELOPMENT

2.1. Discussion: a note on Hardy spaces

The H^p spaces (or Hardy spaces) are well documented in the literature (e.g. Duren[5]). Of special interest are the $E^p(\Omega)$ spaces of complex valued functions. If $\omega \in E^2(\Omega)$, then ω satisfies the conditions of Definition 1.3 for W_Ω, where $\|\omega(\delta\zeta)\|_2$ is bounded as $\delta \to 1$. Finally, if $\omega \in E^2(\Omega)$ then the Cauchy integral representation of $\omega(z)$ for $z \in \Omega$ applies. It is seen that $W_\Omega \subset E^2(\Omega)$.

2.2. Theorem (boundary integral representation)

Let $\omega \in W_\Omega$ and $z \in \Omega$. Then

$$\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta) \, d\zeta}{\zeta - z}$$

Proof

For $\omega \in W_\Omega$, then $\omega \in E^2(\Omega)$ and the result follows immediately.

2.3. Almost everywhere (ae) equivalence

For $\omega \in W_\Omega$, functions $x \in W_\Omega$ which are equal to ω *ae* on Γ represent an equivalence class of functions which may be noted as $[\omega]$. Therefore, functions x and y in W_Ω are in the same equivalence class when

$$\int_\Gamma |x - y| \, d\mu = 0$$

For simplicity, $\omega \in W_\Omega$ is understood to indicate $[\omega]$. This follows directly from the boundary integral representation of $\omega(z)$ for $z \in \Omega$, and the fact that integrals over sets of measure zero have no effect on the integral value.

2.4. Theorem (uniqueness of zero element in W_Ω)

Let $\omega \in W_\Omega$ and $\phi = 0$ *ae* on Γ_ϕ and $\psi = 0$ *ae* on Γ_ψ. Then $\omega = [0] \in W_\Omega$.

Proof

Let $z \in \Omega$. Then

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\Gamma_\psi} \frac{\phi(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\Gamma_\phi} \frac{i\psi(\zeta)\,d\zeta}{\zeta - z}$$

due to $\phi(\zeta) = 0$ *ae* on Γ_ϕ and $\psi(\zeta) = 0$ *ae* on Γ_ψ. Therefore an equivalent function $\omega^* = \phi^* + i\psi^*$ can be formed where $\omega^* \in W_\Omega$ and

$$\phi^*(z) = \begin{cases} 0, & z \in \Gamma_\phi \\ \phi(z), & z \in \Gamma_\psi \end{cases}$$

$$\psi^*(z) = \begin{cases} 0, & z \in \Gamma_\psi \\ \psi(z), & z \in \Gamma_\phi \end{cases}$$

and $\omega(z) = \omega^*(z)$ for all $z \in \Omega$.

By use of the Riemann Mapping Theorem and Caratheodory's Extension of the Riemann Mapping Theorem, any two bounded simply connected domains can be conformally mapped onto each other with a one-to-one correspondence (from the continuous extension of the conformal mapping) of the boundary points. In a recent textbook, Mathews[3] applies the well-known Cauchy-Riemann equations and shows that for the case of ϕ^* being constant on Γ_ϕ and $\partial\phi^*/\partial n = 0$ on Γ_ψ that ω^* is a constant complex number over Ω. By continuity, $\phi^* = 0$ over Ω and, by a similar argument, $\psi^* = 0$ over Ω. Thus $\omega^* = 0$ over Ω and, consequently, $\omega = 0$ over Ω. Hence, $\omega = [0]$.

2.5. Theorem (W_Ω is a vector space)

W_Ω is a linear vector space over the field of real numbers.

Proof

This follows directly from the character of analytic functions. The zero element has already been noted by [0] in Theorem 2.4.

2.6. Theorem (definition of the innter-product)

Let $x, y, z \in W_\Omega$. Define a real-valued function (x, y) by

$$(x,y) = \int_{\Gamma_\phi} \mathrm{Re}\,x\;\mathrm{Re}\,y\;d\mu + \int_{\Gamma_\psi} \mathrm{Im}\,x\;\mathrm{Im}\,y\;d\mu$$

Then (,) is an inner-product over W_Ω.

Proof

It is obvious that $(x, y) = (y, x)$; $(kx, y) = k(x, y)$ for k real; $(x + y, z) = (x, z) + (y, z)$; and $(x, x) \geq 0$. By theorem 2.4, $(x, x) = 0$ implies $\mathrm{Re}\,x = 0$ *ae* on Γ_ϕ and $\mathrm{Im}\,x = 0$ *ae* on Γ_ψ and $x = [0] \in W_\Omega$.

Three theorems follow immediately from the above.

2.7. Theorem (W_Ω is an inner-product space)

For the defined inner-product, W_Ω is an inner-product space over the field of real numbers.

2.8. Theorem ($\|\omega\|$ is a norm on W_Ω)

A norm is defined by $\|x\| = (x, x)^{1/2}$ for $x \in W_\Omega$.

2.9. Theorem

Let $x \in W_\Omega$ and $\|x\| = 0$. Then $x = [0]$.

3. THE CVBEM AND W_Ω

3.1. Definition

Let the number of angle points of Γ be noted as Λ. By a nodal partition P_n of Γ, $m = \Lambda(n - 1)$ nodes $\{z_j\}$ are defined on Γ such that a node is located at each angle-point of Γ and the remaining nodes are distributed on Γ. Nodes are numbered sequentially in a counterclockwise direction along Γ. The scale of P_n is indicated by l where $l = \max |z_{j+1} - z_j|$. Thus n nodes are equally spaced along each line segment.

3.2. Definition

A boundary element Γ_j is the line segment joining nodes z_j and z_{j+1}.

3.3. Theorem

Let P_n be defined on Γ. Then

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j$$

where $m = \Lambda(n - 1)$.

Proof

Follows from Γ being piecewise linear, and the construction of P_n.

3.4. Definition

A linear basis function $N_j(\zeta)$ is defined for $\zeta \in \Gamma$ by

$$N_j(\zeta) = \begin{cases} (\zeta - z_{j-1})/(z_j - z_{j-1}), & \zeta \in \Gamma_{j-1} \\ (z_{j+1} - \zeta)/(z_{j+1} - z_j), & \zeta \in \Gamma_j \\ 0, & \zeta \notin \Gamma_{j-1} \cup \Gamma_j \end{cases}$$

The values of $N_j(\zeta)$ is found to be real and bounded as indicated by the next theorem.

3.5. Theorem

Let $N_j(\zeta)$ be defined for node $z_j \in \Gamma$. Then $0 \leq N_j(\zeta) \leq 1$.

3.6. Definition

Let P_n be defined on Γ with $m \geq \Lambda$ and with scale l. At each node z_j, define nodal values $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$ where $\bar{\phi}_j$ and $\bar{\psi}_j$ are real numbers. A global trial function $G_m(\zeta)$ is defined on Γ by

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\,\bar{\omega}_j$$

3.7. Theorem

$G_m(\zeta)$ is continuous on Γ.

3.8. Discussion

As a result of $\omega(\zeta) \in L^2(\Gamma)$, then $\omega(\zeta)$ is measurable on Γ and for every $\epsilon > 0$ there exists a continuous complex-valued function $g(\zeta)$ such that

$$\| \omega(\zeta) - g(\zeta) \|_1 < \epsilon/2$$

Choosing $G_m(\zeta)$ to approximate $g(\zeta)$ by

$$\| G_m(\zeta) - g(\zeta) \|_1 < \epsilon/2$$

then

$$\| \omega(\zeta) - G_m(\zeta) \|_1 \leqslant \| \omega(\zeta) - g(\zeta) \|_1 + \| g(\zeta) - G_m(\zeta) \|_1 < \epsilon$$

3.9. Theorem

Let $\omega \in W_\Omega$. For $\epsilon > 0$ there exists a $G_m(\zeta)$ such that $\| \omega(\zeta) - G_m(\zeta) \|_1 < \epsilon$.

Proof

Follows from the discussion in 3.8.

3.10. Discussion

The CVBEM approximation function $\hat{\omega}_m(z)$ is developed from the singular integral for a partition P_n of Γ by

$$\hat{\omega}_m(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G_m(\zeta)\, d\zeta}{\zeta - z}, \quad z \in \Omega \tag{1}$$

where

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\, \bar{\omega}_j$$

is the global trial function chosen to achieve

$$\| \omega(\zeta) - G_m(\zeta) \|_1 < \epsilon$$

for $\omega \in W_\Omega$ and $\epsilon > 0$. Expanding G_m in the integrand gives

$$\hat{\omega}_m(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_\Gamma \frac{N_j(\zeta)\, \bar{\omega}_j\, d\zeta}{\zeta - z}, \quad z \in \Omega \tag{2}$$

Appendix A shows that $\hat{\omega}_m(z)$ can be written as

$$\hat{\omega}_m(z) = R_1(z) + \sum_{j=1}^{m} c_j (z - z_j) \ln (z - z_j), \quad z \in \Omega \tag{3}$$

where $R_1(z)$ is a first degree complex polynomial resulting from the 2π-circuit along Γ about point z; the complex logarithm is with respect to point z (the branch cut is a ray originating from point $z \in \Omega$); and the c_j are complex constants $c_j = a_j + ib_j$ where the a_j and b_j are real numbers. The problem now can be restated as how to choose the best values for the c_j (and the $R_1(z)$ constants) such as to minimise a defined norm. Because ϕ is known only on Γ_ϕ and ψ is known only on Γ_ψ $(\omega = \phi + i\psi)$, $\| \omega - \hat{\omega}_m \|_2$ is undefined. Therefore, the constants will be chosen to minimise the newly defined norm $\| \omega - \hat{\omega}_m \|$ where the goal is $\| \omega - \hat{\omega}_m \| \to 0 \Rightarrow \hat{\omega}_m(z) \to \omega(z)$ for all $z \in \Omega$.

For development purposes, the $\ln(z - z_j)$ functions are replaced by $\ln_j(z - z_j)$ functions where logarithm branch cuts are rays from each z_j which lie exterior of $\bar{\Omega} - \{z_j\}$ (see Appendix A).

Letting $R_1(z) = c_{m+1} + c_{m+2} z$, the CVBEM approximation used is now defined as

$$\hat{\omega}_m(z) = \sum_{j=1}^{m+2} c_j T_j \tag{4}$$

where

$$T_j = \begin{cases} (z - z_j) \ln_j (z - z_j), & j = 1, 2, \ldots, m \\ 1 + 0i, & j = m + 1 \\ z, & j = m + 2 \end{cases}$$

By the use of the $\ln_j(z - z_j)$ functions, $\hat{\omega}_m(z)$ is analytic over $\bar{\Omega}$ except at the nodal points, and $\hat{\omega}_m(z)$ is continuous over $\bar{\Omega}$. In fact, $\hat{\omega}_m(z)$ is analytic everywhere except along the branch cuts.

If $c_j = a_j + ib_j$ is substituted into (4), the CVBEM approximation can be written with respect to real number coefficients γ_j as

$$\hat{\omega}_m(z) = \sum_{j=1}^{2(m+2)} \gamma_j f_j \tag{5}$$

where the f_j functions are given by

$$\left. \begin{aligned} f_1 &= (z - z_1) \ln_1 (z - z_1) \\ f_2 &= i f_1 \\ &\vdots \\ f_{2m-1} &= (z - z_m) \ln_m (z - z_m) \\ f_{2m} &= i f_{2m-1} \\ f_{2m+1} &= 1 \\ f_{2m+2} &= i \\ f_{2m+3} &= z \\ f_{2(m+2)} &= iz \end{aligned} \right\} \tag{6}$$

3.11. Theorem (linear independence of nodal expansion functions)

The set of functions

$$\{(z - z_j) \ln_j (z - z_j),\ j = 1, 2, \ldots, m + 1\}$$

are linearly independent.

Proof

Using induction, let the first m functions are linearly independent, but the *(m + 1)-function is linearly dependent* on the other m functions. Then for complex constants c_j,

$$c_{m+1}(z - z_{m+1}) \ln_{m+1}(z - z_{m+1}) = \sum_{j=1}^{m} c_j (z - z_j) \ln_j (z - z_j)$$

Taking the second derivative with respect to z gives

$$\frac{c_{m+1}}{(z - z_{m+1})} = \sum_{j=1}^{m} \frac{c_j}{(z - z_j)}, \quad \text{for } z \neq z_k,\ k = 1, 2, \ldots, m + 1$$

Rearranging terms, the above implies that

$$c_{m+1}\prod_{k=1}^{m}(z-z_k)=(z-z_{m+1})\sum_{j=1}^{m}c_j\prod_{\substack{k=1\\k\neq j}}^{m}(z-z_k)$$

which is valid only if $c_k=0$ for each $k=1,2,\ldots,m+1$.

3.12. Discussion

From the previous theorem, the set of functions $\{T_j\}$ of (4) are also linearly independent and, more importantly in this development, the $\{f_j\}$ are linearly independent with respect to the real number field. Thus for a given number m of nodes on Γ, the functions $\{f_j; j=1,2,\ldots,2(m+2)\}$ forms a basis for the vector space spanned by the $\{f_j\}$, noted by $\hat{W}_\Omega^{2(m+2)}$. In this notation, m indicates the number of nodes defined on Γ (always, $m\geqslant\Lambda$), and the hat indicates the CVBEM approximation function vector space.

The CVBEM objective is to choose a $\hat{\omega}_m\in\hat{W}_\Omega^m$ which minimises $\|\omega-\hat{\omega}_m\|$ where $\omega\in W_\Omega$ and the nodes $\{z_j\}$ are fixed on Γ.

3.13. Theorem

Let $\omega\in W_\Omega$ and $z\in\Omega$. For every $\epsilon>0$ there exists a CVBEM approximation $\hat{\omega}_m$ such that $|\omega(z)-\hat{\omega}_m(z)|<\epsilon$.

Proof

Let $d=\min|\zeta-z|$, $\zeta\in\Gamma$. Then for a global trial function $G_m(\zeta)$ defined on Γ

$$|\omega(z)-\hat{\omega}_m(z)|=\left|\frac{1}{2\pi i}\int_\Gamma\frac{[\omega(\zeta)-G_m(\zeta)]\,\mathrm{d}\zeta}{\zeta-z}\right|$$

$$\leqslant\frac{1}{2\pi d}\|\omega-G_m\|_1\leqslant\frac{\sqrt{L}}{2\pi d}\|\omega-G_m\|_2$$

Choosing G_m (see section 3.10) such that $\|\omega-G_m\|_2<2\pi d\epsilon/\sqrt{L}$ (or $\|\omega-G_m\|_1<2\pi d\epsilon$) guarantees the desired result.

More insight as to the power of the CVBEM is provided by an analogy to convergence in measure:

3.14. Theorem

Let $\epsilon>0$. Then there exists a $0<\delta<1$ such that the

$$\int\int_{\Omega-\Omega_\delta}\mathrm{d}\Omega<\epsilon\quad\text{and}\quad\lim_{\substack{l\to 0\\m\to\infty}}|\omega(z)-\hat{\omega}_m(z)|=0$$

Proof

Choose $0<\delta<1$ such that the area of $\Omega-\Omega_\delta$ is less than ϵ. Let $d=(1-\delta)\min|\zeta|$, $\zeta\in\Gamma$ where $\omega\in W_\Omega$. Then by Theorem 3.13, the required result follows.

3.15. Discussion

The above theorems discuss the existence of a CVBEM approximation $\hat{\omega}_m(z)$ which converges in measure to $\omega(z)$. That is, for an arbitrarily small $(1-\delta)$-strip inside of Γ, $\hat{\omega}_m(z)\to\omega(z)$ for all $z\in\bar{\Omega}_\delta$ as $m\to\infty$ and $l\to 0$. To develop the CVBEM approximation $\hat{\omega}_m(z)$, the defined norm $\|x\|$ for $x\in W_\Omega$ is used.

To proceed, the $\{f_j\}$ are orthonormalised by the Gram-Schmidt procedure to the set of functions $\{g_j\}$ using the defined inner-product on W_Ω. That is $g_1=f_1/\|f_1\|$, $g_2=(f_2-(f_2,g_1)g_1)/\|f_2-(f_2,g_1)g_1\|$, and so forth. With respect to $\{g_j\}$,

$$\hat{\omega}_m(z)=\sum_{j=1}^{2(m+2)}\hat{\gamma}_j g_j(z)$$

where the $\hat{\gamma}_j$ are generalised Fourier coefficients to be determined. It is noted that the $g_k(z)$ are finite combinations of the f_j-functions. The value of $\|\omega-\hat{\omega}_m\|$ is minimised when $\hat{\gamma}_j=(\omega,g_j)$.

By back-substitution, the γ_j corresponding to the $\{f_j\}$ can be evaluated. In this fashion, the CVBEM approximator $\hat{\omega}_m(z)$ is developed for $\omega\in W_\Omega$ and the provided boundary conditions of ϕ defined on Γ_ϕ and ψ defined on Γ_ψ.

Because W_Ω is an inner-product space with the defined inner-product, Bessel's inequality applies.

4. THE SPACE W_Ω^A

4.1. Definition

A subspace of W_Ω are those elements which are analytic over $\bar{\Omega}$. Thus, $\omega\in W_\Omega^A$ implies ω is analytic over $\Omega\cup\Gamma$.

4.2. Theorem

W_Ω^A is a linear vector space over the field of real numbers.

Proof

Follows from the parent space W_Ω. However, it is noted that *ae* equality is unnecessary due to $\omega\in W_\Omega^A$ implies continuity over $\bar{\Omega}$.

4.3. Theorem

W_Ω^A is an inner-product space using the defined inner-product.

Proof

Of interest is showing $(x,x)=0\Rightarrow x=0$. Green's theorem gives

$$\int_\Omega(\phi_x^2+\phi_y^2)\,\mathrm{d}\Omega=\int_\Gamma\phi\frac{\partial\phi}{\partial n}\mathrm{d}\Gamma+\int_\Omega\phi\nabla^2\phi\,\mathrm{d}\Omega$$

where ϕ_x and ϕ_y are partial derivatives of $\phi(x,y)$ in the x- and y-direction, and $\partial\phi/\partial n$ is a normal derivative along Γ. But $|\partial\phi/\partial n|=|\partial\psi/\partial s|$ where s is a tangential co-ordinate along Γ and the Cauchy-Riemann relations apply. Thus $\nabla^2\phi=0$ over Ω due to $\omega=\phi+i\psi$ and $\omega\in W_\Omega^A$. Also, $\phi=0$ on Γ_ϕ and $\partial\psi/\partial s=0$ on Γ_ψ by assumption. Thus

$$\int_\Omega(\phi_x^2+\phi_y^2)\,\mathrm{d}\Omega=0\quad\text{and}\quad\phi(x,y)$$

is constant over Ω. By continuity, $\phi=0$ over $\bar{\Omega}$. Similarly $\psi=0$ over $\bar{\Omega}$, and $\omega=0$.

4.4. Discussion

For $\omega\in W_\Omega^A$ and $z\in\Omega$, Cauchy's theorem gives immediately that

$$\omega(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z},\quad z\in\Omega\qquad(7)$$

Letting

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\,\omega_j$$

where

$$\omega_j = \omega(z_j)$$

then

$$\lim_{\substack{m\to\infty \\ l\to 0}} G_m(\zeta) = \omega(\zeta)$$

and

$$\omega(z) = \lim_{\substack{m\to\infty \\ l\to 0}} \frac{1}{2\pi i} \int_{\Gamma} \frac{G_m(\zeta)\,d\zeta}{\zeta - z}, \quad z \in \Omega \tag{8}$$

(A detailed proof of this convergence is in Appendix B.) Thus for $z \in \Omega$,

$$\omega(z) = c_0 + c_{-1}z + \sum_{j=1}^{\infty} c_j(z - z_j)\ln_j(z - z_j), \quad z \in \Omega \tag{9}$$

where now c_0 and c_{-1} are also complex constants. It can also be argued that the $c_0 + c_{-1}z$ terms can be eliminated entirely when using the infinite series expansion.

Because

$$\omega(z) = \lim_{\substack{m\to\infty \\ l\to 0}} \hat{\omega}_m(z)$$

over Ω, then the boundary values of the limiting CVBEM approximator (taking in the limit as $\delta\zeta \to \zeta$ for each $\zeta \in \Gamma$) equal the boundary values of $\omega \in W_\Omega^A$.

Writing the $\omega(z)$ function with respect to the Gramm-Schmidt orthonormalised functions $\{g_j\}$ of Section 3 (with respect to the defined inner-product)

$$\omega(z) = \sum_{j=1}^{\infty} (\omega, g_j)\,g_j(z), \quad z \in \Omega \tag{10}$$

4.5. Theorem

The set $\{g_j\}$ is complete.

Proof

Suppose $\omega \in W_\Omega^A$ and $(\omega, g_j) = 0$ for every j. Then from (10),

$$\omega(z) = \sum_{j=1}^{\infty} (\omega, g_j)\,g_j = 0, \quad z \in \Omega$$

Thus $\omega(z)$ is the zero element of W_Ω^A in that in the limit as $\delta\zeta \to \zeta$, $\phi = 0$ on Γ_ϕ and $\psi = 0$ on Γ_ψ where $\omega = \phi + i\psi$. Thus the set $\{g_j\}$ is complete.

4.6. Theorem

Let $\omega \in W_\Omega^A$. Then ω satisfies the Dirichlet conditions for generalised Fourier series.

Proof

By assumption, there are a finite number of line segments composing Γ_ϕ and Γ_ψ. Because ω is analytic on Γ, then the boundary condition functions $B(\zeta)$ on Γ_ϕ and $B'(\zeta)$ on Γ_ψ are both piecewise continuous on Γ.

4.7. Discussion: another look at W_Ω

By Theorem 4.6, the CVBEM will converge to the boundary values where continuous, and to the midpoint value of the discontinuity where discontinuous. Because $\hat{\omega}_m(z)$ is analytic over Ω as $m \to \infty$ (Appendix B), then also $\hat{\omega}_m(z) \to \omega(z)$ as $m \to \infty$. But by Definition 2.1, $\omega(\delta\zeta) \to \omega(\zeta)$ in $L^2(\Gamma)$. Due to $\omega(\delta\zeta)$ being analytic over Ω, we immediately have $\hat{\omega}_m(z)$ approximates $\omega(\delta\zeta)$ which, in turn, approximates $\omega(z)$ arbitrarily close in $L^2(\Gamma)$.

5. APPLICATIONS

5.1. Computer program

A FORTRAN computer program was prepared based on the least-square boundary fit described in the previous sections. Matrix solution routines are not needed due to the orthonormal vector technique. The program was prepared to accommodate analytic function equivalents for sources, sinks, flux boundary conditions (i.e. tangential derivatives of the stream function ψ), and dissimilar regions (up to five regions with different isotropic conduction or diffusion parameters).

5.2. Nodal point placement on Γ

The program initiates by modeling a user-defined nodal point placement (adequate to represent to geometry of Γ as a minimum) to develop a CVBEM approximation. Then, the user enters (by the CRT) x, y-co-ordinates for the next nodal location on Γ (i.e. adding another basis function) and the program computes Bessel's inequality. After several trials for the nodal location on Γ, Bessel's inequality is computed for each try and is subsequently minimized, and the optimum choice for the next node on Γ is made. In this fashion $\hat{\omega}_m(z) \to \omega(z)$ as $m \to \infty$. This procedure for locating a 'best' position for the next node to be added on Γ can be linearly programmed.

5.3. Flow-field (flow-net) development

By entering x, y-co-ordinates, $\hat{\omega}_m(z)$ values are computed and the flow-net can be plotted with respect to the approximation $\hat{\phi}_m(z)$ and $\hat{\psi}_m(z)$ values. Such flow-nets are included in the provided applications.

5.4. Approximate boundary development

Hromadka[1] details the 'approximate boundary' $\hat{\Gamma}$ technique for CVBEM error evaluation. The contour $\hat{\Gamma}$ represents the location where $\hat{\omega}_m(z)$ achieves the boundary conditions of $\omega(z)$ on Γ. That is, if the provided boundary conditions are level curves of $\omega(z)$ on Γ, then $\hat{\Gamma}$ represents the corresponding level curves of $\hat{\omega}_m(z)$. Hence if the approximate boundary $\hat{\Gamma}$ lies 'sufficiently close' to Γ, the analyst can conclude that an adequate approximation has been developed. This error evaluation technique is very useful due to the ease of interpretation. Even beginners can develop highly accurate CVBEM approximations by simply observing the relationship of $\hat{\Gamma}$ to Γ, and adding nodes to Γ where departures are considered unacceptable. In the included example problems, approximate boundaries are developed for each test problem.

5.5. Applications

Example 1. Ideal fluid flow around a cylindrical corner has the analytic solution of $\omega(z) = z^2 + z^{-2}$. Figure 1(a) depicts the problem geometry and specified boundary conditions. Figure 1(b) and 1(c) show the error plots in

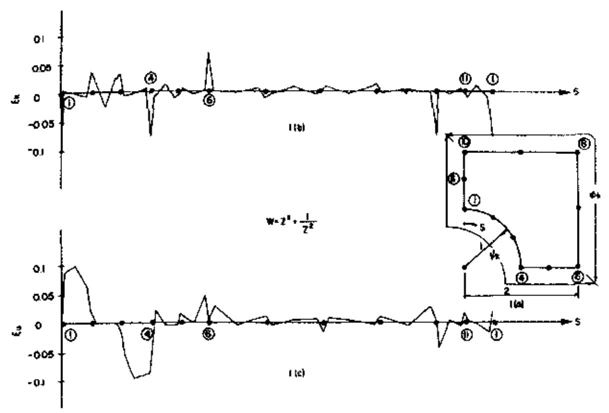

Figure 1. Error plots for ideal fluid flow around a cylindrical corner

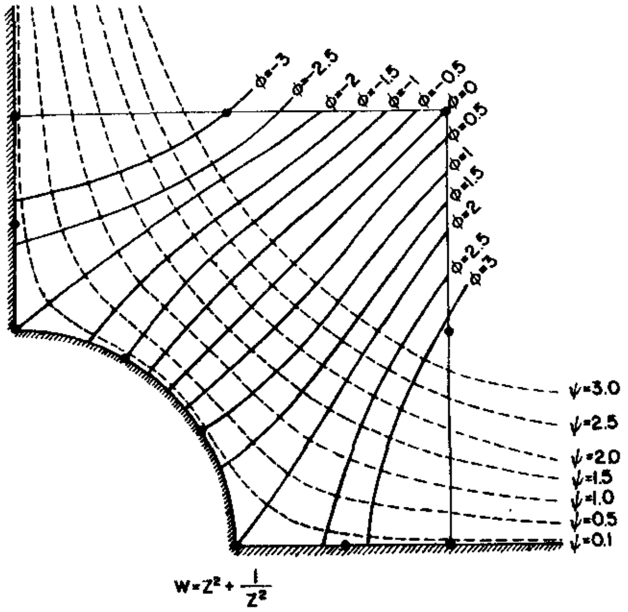

Figure 2. Computed flow net for ideal fluid flow around a cylindrical corner

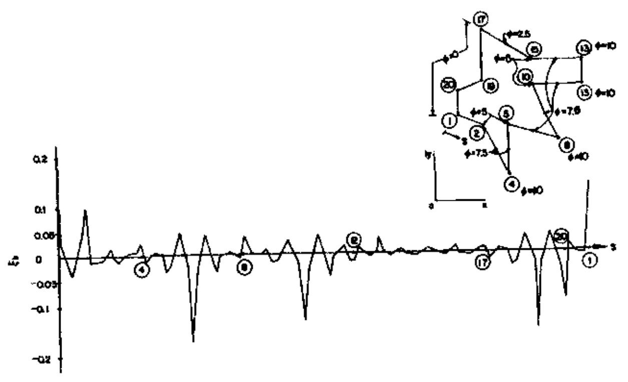

Figure 3. Error plot for irregular two-dimensional cross-section

matching boundary values for both the known and unknown boundary conditions. Figure 2 shows the CVBEM computed flow net.

Example 2. Figure 3 shows an irregular two-dimensional cross-section with boundary conditions. The purpose of this example is to show how the approximate boundary is used to evaluate computational error for an irregular section problem. Figure 4 shows a very good match between the exact and approximate boundaries.

Example 3. A long and shallow unconfined aquifer (see Fig. 5) is used to compare the results between the dual formulation technique[5] and the proposed CVBEM technique. The mean deviation between the exact and approximate boundary is about 0.001^{m} and 0.2^{m} for the water table and impervious boundary, respectively. The 0.2^{m} deviation is based on the 10^{-4} magnitude difference between the exact and approximate boundary. If this magnitude increases to 10^{-2}, the approximate boundary can be considered to coincide with the exact boundary. The equipotential lines shown on Fig. 6 approximate those shown on Fig. 8(b) in Frind *et al.*'s[5] paper. The stream lines are not orthogonal to the equipotential lines because of the different scales in x- and y-directions.

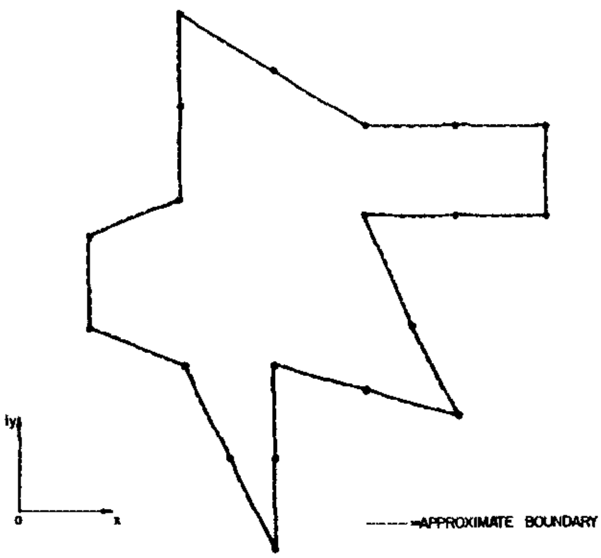

Figure 4. Approximate boundary for irregular two-dimensional cross-section

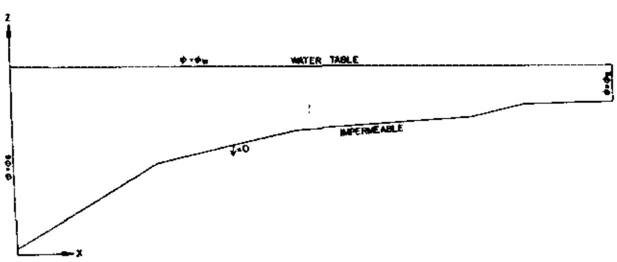

Figure 5. Boundary conditions for shallow unconfined aquifer

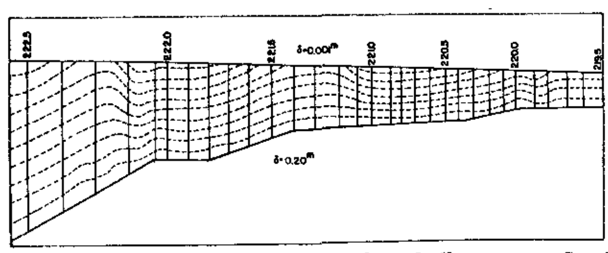

Figure 6. Computed flow net for shallow unconfined aquifer

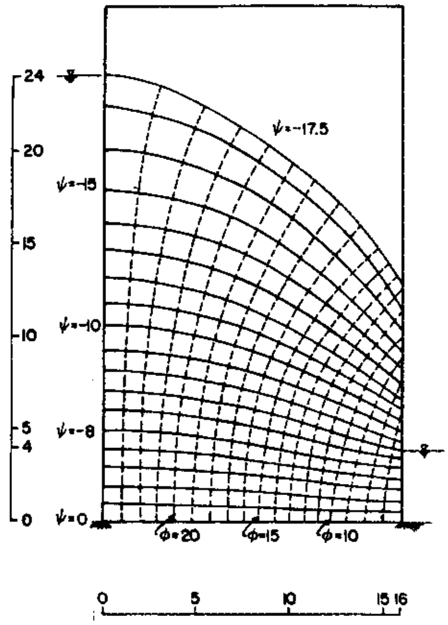

Figure 7. Computed flow net for soil-water flow through a homogeneous soil

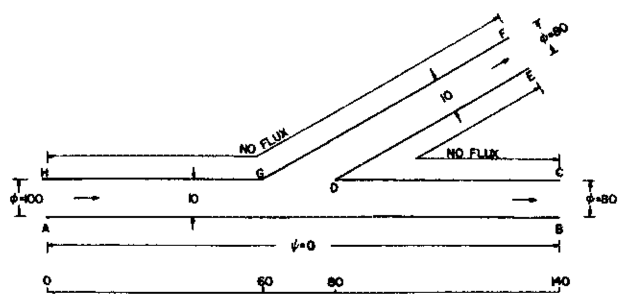

Figure 8. Boundary conditions for hydraulic bifurcation structure

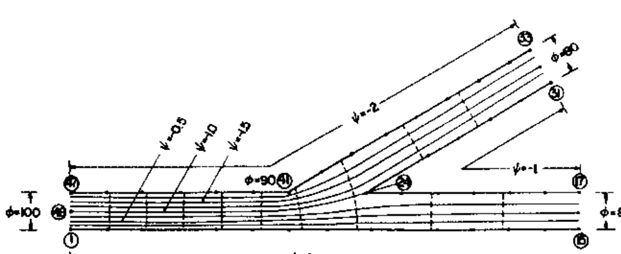

Figure 9. Computed flow net for hydraulic bifurcation structure

Example 4. Figure 7 shows streamlines and equipotential lines for soil-water flow through a homogeneous soil. The locations of the phreatic surface and the seepage face can be easily determined by the approximate boundary technique.

Example 5. Figure 8 shows a hydraulic bifurcation structure with boundary conditions specified at up- and downstream ends. First, the stream function is specified for the horizontal boundary (AB) and 'no-flux' boundary conditions are specified for the remaining boundaries. The computed results show a constant streamline along the boundary CDE and linear variation between boundaries AB, CDE, and FGH. Utilising the above results, constant streamline boundary conditions are specified for the problem. Figure 9 depicts the boundary conditions and the resulting flow net for the problem. The approximate boundary is considered to coincide with the geometry boundary with a 10^{-2} magnitude difference between the approximate and specified boundary conditions along the boundary.

CONCLUSIONS

The CVBEM can be used to develop approximations of two-dimensional flow-field problems. In this paper is advanced a major innovation in the use of boundary integral equation methods. By use of orthonormalised nodal point expansion functions generated from the CVBEM, the need for solving a square matrix system is entirely eliminated. Additionally, the boundary conditions are met in a minimised mean-squares fit sense. The resulting CVBEM computer code is fast, efficient, easy to use, and can be accommodated on 64K home computers which support FORTRAN.

Because the generalised Fourier series approach is applicable to real-variable expansions, the results developed for the CVBEM can be applied to real-variable boundary integral equation methods (BIEM) and boundary element methods (BEM).

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APPENDIX A

Derivation of CVBEM approximation function

Let $\omega \in W_\Omega^A$, and P_n be a nodal partition of Γ. Define a global trial function $G_m(\zeta)$ on Γ by

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\,\omega_j$$

where $\omega_j = \omega(z_j)$ and $\zeta \in \Gamma$. Develop the integral function $A(z)$ defined by

$$A(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{G_m(\zeta)\,d\zeta}{\zeta - z}, \quad z \in \Omega$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\Gamma_j} \frac{G_m(\zeta)\,d\zeta}{\zeta - z}$$

On Γ_j,

$$G_m(\zeta) = \omega_j[(z_{j+1} - \zeta)/(z_{j+1} - z_j)] + \omega_{j+1}[(\zeta - z_j)/(z_{j+1} - z_j)]$$

and

$$\int_{\Gamma_j} \frac{G_m(\zeta)\,d\zeta}{\zeta - z} = \omega_{j+1}\left[1 + \left(\frac{z - z_j}{z_{j+1} - z_j}\right)(\ln(z_{j+1} - z) - \ln(z_j - z))\right] - \omega_j\left[1 + \left(\frac{z - z_{j+1}}{z_{j+1} - z_j}\right) \times (\ln(z_{j+1} - z) - \ln(z_j - z))\right]$$

Summing from $j = 1$ to m, (and noting $\omega_{m+1} = \omega_1$ and $z_{m+1} = z_1$)

$$\chi = \sum_{j=1}^{m} \int_{\Gamma_j} \frac{G_m(\zeta)\,d\zeta}{\zeta - z} = \sum_{j=1}^{m} (\omega_{j+1} - \omega_j) + \sum_{j=1}^{m} \frac{[(\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1})) \times (\ln(z_{j+1} - z) - \ln(z_j - z))]}{(z_{j+1} - z_j)}$$

where

$$\ln(z_{m+1} - z) = \ln(z_1 - z) + 2\pi i$$

Thus

$$\chi = \sum_{j=1}^{m-1} [\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1})] \times [\ln(z_{j+1} - z) - \ln(z_j - z)]/(z_{j+1} - z_j) + [\omega_1(z - z_m) - \omega_m(z - z_1)] \times [\ln(z_1 - z) + 2\pi i - \ln(z_m - z)]/(z_1 - z_m)$$

Combining terms with respect to the $\ln(z_j - z)$ functions gives

$$\chi = \sum_{j=1}^{m}\left(\frac{[\omega_j(z - z_{j-1}) - \omega_{j-1}(z - z_j)]}{(z_j - z_{j-1})} - \frac{[\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1})]}{(z_{j+1} - z_j)}\right)\ln(z_j - z) + 2\pi i\,\frac{[\omega_1(z - z_m) - \omega_m(z - z_1)]}{(z_1 - z_m)}$$

Thus if $P_j(\zeta)$ is the interpolation function on Γ_j given by

$$P_j(\zeta) = \begin{cases} N_j(\zeta)\,\omega_j + N_{j+1}(\zeta)\,\omega_{j+1}, & \zeta \in \Gamma_j \\ 0, & \text{otherwise} \end{cases}$$

then by substituting z into ζ of $P_j(\zeta)$

$$\chi = \sum_{j=1}^{m} (P_{j-1}(z) - P_j(z))\ln(z_j - z) + 2\pi i P_m(z)$$

where now $P_0(z) \equiv P_m(z)$. Finally, $P_j(z_j) = P_{j-1}(z_j)$ implies that

$$A(z) = P_m(z) + \frac{1}{2\pi i}\sum_{j=1}^{m}\left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})}\right] \times (z_j - z)\ln(z_j - z)$$

Complex constants k_j can be used to simplify the writing of $A(z)$ by

$$A(z) = P_m(z) + \sum_{j=1}^{m} k_j(z_j - z)\ln(z_j - z)$$

where

$$k_j = \frac{1}{2\pi i}\left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})}\right]$$

Noting

$\ln(z_j - z) = \ln(z - z_j) + \ln(-1) = \ln(z - z_j) + i\pi$, $A(z)$

can be rewritten as

$$A(z) = P_m(z) - \sum_{j=1}^{m} k_j(z - z_j)\ln(z - z_j) + i\pi \sum_{j=1}^{m} k_j(z_j - z)$$

In the above, $\ln(z - z_j)$ is measured with respect to point $z \in \Omega$ as the branch point. It is desirable to define branch points at each z_j with branch cuts lying exterior of Ω. This process introduces an additional angle term θ^j of the branch cut for each node such that

$$\ln(z - z_j) = \ln_j(z - z_j) + i\theta^j$$

where $\ln_j$ is notation of individual logarithm functions. Thus $A(z)$ is of the form

$$A(z) = R_1(z) + \sum_{j=1}^{m} c_j(z - z_j)\ln_j(z - z_j)$$

where $R_1(z)$ is a first degree complex polynomial

$$R_1(z) = P_m(z) + \sum_{j=1}^{m} i(\pi + \theta^j)\,k_j(z_j - z)$$

and each $c_j = -k_j$.

APPENDIX B

Convergence of CVBEM approximator

Let $\omega \in W_{\Omega}^{A}$ and $z \in \Omega$.

Let

$$A(z) = P_m(z) + \sum_{j=1}^{m} k_j(z_j - z)\ln(z_j - z)$$

where

$$k_j = \frac{1}{2\pi i}\left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})}\right]$$

and

$$P_m(z) = \omega_1 + \left[\frac{\omega_1 - \omega_m}{z_1 - z_m}\right](z - z_1)$$

Because $\omega(z)$ is analytic on $\bar{\Omega}$, l can be chosen small enough such that any three neighboring (in sequence) nodes z_{j-1}, z_j, z_{j+1} lie within the radius of convergence of the Taylor series about z_j. That is,

$$\omega_{j+1} = \omega_j + \omega_j'(z_{j+1} - z_j) + \omega_j''(z_{j+1} - z_j)^2/2! + \dots$$
$$\omega_{j-1} = \omega_j - \omega_j'(z_j - z_{j-1}) + \omega_j''(z_j - z_{j-1})^2/2! + \dots$$

Thus

$$(\omega_{j+1} - \omega_j)/(z_{j+1} - z_j) = \omega_j' + \omega_j''(z_{j+1} - z_j)/2! + \dots$$
$$(\omega_{j-1} - \omega_j)/(z_j - z_{j-1}) = -\omega_j' + \omega_j''(z_j - z_{j-1})/2! + \dots$$

and

$$k_j = \frac{1}{2\pi i}\left[\omega_j'' \frac{(z_{j+1} - z_{j-1})}{2} + r_j\right]$$

where r_j is the residual terms of the Taylor series such that $r_j \to 0$ in order 2 as $l \to 0$. Thus

$$\lim_{\substack{m\to\infty \\ l\to 0}} A(z) = \lim_{\substack{m\to\infty \\ l\to 0}} \left[P_m(z) + \sum_{j=1}^{m} \frac{1}{2\pi i} \omega_j'' \frac{(z_{j+1} - z_{j-1})}{2} \times (z_j - z)\ln(z_j - z) + \sum_{j=1}^{m} \frac{1}{2\pi i} r_j (z_j - z)\ln(z_j - z)\right]$$

Evaluating terms,

$$\lim_{\substack{m\to\infty \\ l\to 0}} P_m(z) = \lim_{l\to 0}\left[\omega_1 + \left(\frac{\omega_1 - \omega_m}{z_1 - z_m}\right)(z - z_1)\right]$$
$$= \omega_1 + \left.\frac{d\omega}{d\zeta}\right|_{z_1} (z - z_1)$$

and therefore

$$\lim_{\substack{m\to\infty \\ l\to\infty}} A(z) = \omega_1 + \left.\frac{d\omega}{d\zeta}\right|_{z_1} (z - z_1) + \frac{1}{2\pi i}\int_\Gamma \frac{d^2\omega}{d\zeta^2} \times (\zeta - z)\ln(\zeta - z)\, dz$$

Integrating by parts,

$$\int_\Gamma \frac{d^2\omega}{d\zeta^2}(\zeta - z)\ln(\zeta - z)\, dz$$
$$= (\zeta - z)\ln(\zeta - z)\left.\frac{d\omega}{d\zeta}\right|_\Gamma - \int_\Gamma \frac{d\omega}{d\zeta}(1 + \ln(\zeta - z))\, d\zeta$$

where

$$(\zeta - z)\ln(\zeta - z)\left.\frac{d\omega}{d\zeta}\right|_\Gamma = 2\pi i(z_1 - z)\left.\frac{d\omega}{d\zeta}\right|_{z_1}$$

Integrating by parts again the remaining integral term,

$$-\int_\Gamma \frac{d\omega}{d\zeta}(1 + \ln(\zeta - z))\, d\zeta = -[\omega \ln(\zeta - z)]\Big|_\Gamma + \int_\Gamma \frac{\omega\, d\zeta}{\zeta - z}$$

But

$$-[\omega \ln(\zeta - z)]\Big|_\Gamma = -2\pi i \omega_1$$

Thus dividing the necessary terms by $2\pi i$,

$$\lim_{\substack{m\to\infty \\ l\to 0}} A(z) = \left[\omega_1 + \left.\frac{d\omega}{d\zeta}\right|_{z_1} (z - z_1)\right] + \left[(z_1 - z)\left.\frac{d\omega}{d\zeta}\right|_{z_1} - \omega_1 + \frac{1}{2\pi i}\int_\Gamma \frac{\omega\, d\zeta}{\zeta - z}\right]$$
$$= \frac{1}{2\pi i}\int_\Gamma \frac{\omega\, d\zeta}{\zeta - z} = \omega(z)$$

Hence,

$$\lim_{\substack{m\to\infty \\ l\to 0}} A(z) = \omega(z)$$