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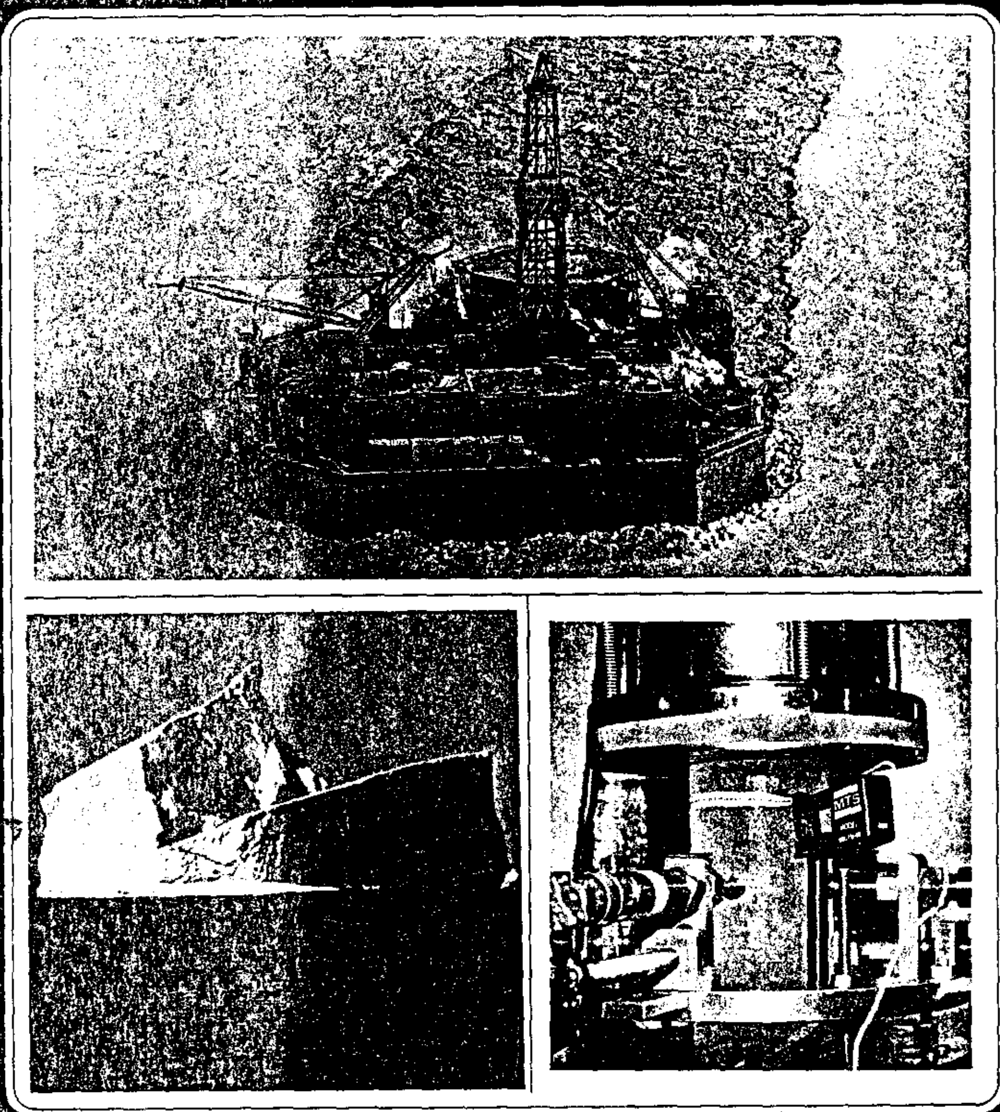

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A BOUNDARY INTEGRATION EQUATION METHOD WITHOUT MATRICES

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ABSTRACT

The Complex Variable Boundary Element Method or CVBEM provides for the exact solution of the two-dimensional Laplace equation which can be used as an approximation of a slow-moving, freezing front through wet soils. In this paper, a new approach for developing CVBEM approximation functions is presented. Using the ℓ^2 norm, the CVBEM approximation is developed which is the best approximation in an equivalent vector subspace. Because orthogonal functions are used, matrix solutions are eliminated; thus, considerably reducing the computational effort and requirements.

INTRODUCTION

The use of boundary integral equation methods (BIEM) for soil-water freezing front movement problems has received some recent attention in the literature. The current thrust in BIEM modeling is twofold: use of complex variable boundary elements or CVBEM and use of real variable boundary integral equations (e.g., Hromadka and Guymon, 1982). Both methods are similar in that a boundary integral is solved by numerical integration resulting in a square, fully-populated matrix of an order equal to the number of nodes placed on the problem boundary.

In this paper, the CVBEM is reanalyzed with the objectives of eliminating the need for a square matrix solution. The new method to be presented is based on generalized Fourier series theory, and satisfies the problem boundary conditions in a least-squares (L^2) sense. The resulting model is identical in capability as the previous CVBEM phase-change model (Hromadka, 1982), but provides the significant improvements of (1) satisfying boundary conditions in a L^2 norm, and (2) eliminates the matrix generation requirements.

MODEL DEVELOPMENT

The CVBEM approximation function for linear (straight-line interpolation) basis functions results in the complex function (Hromadka, 1983, 1984)

$$\hat{\omega}(z) = \sum_{j=1}^{m} c_j (z - z_j) \operatorname{Ln} (z - z_j) \qquad (1)$$

where the c_j are complex constants $c_j = a_j + ib_j$; z_j are nodal points ($j = 1,2,\cdots,m$) defined on the problem boundary Γ (simple closed contour); and $\operatorname{Ln}(z - z_j)$ is the principal value complex logarithm function with branch cuts specified to intersect Γ only at z_j (see Fig. 1). Then $\hat{\omega}(z)$ is analytic over $\Omega \cup \Gamma - \{z_j\}$, and uniformly continuous over $\Omega \cup \Gamma$. Here, Ω is a simply connected domain enclosed by Γ. In fact, $\hat{\omega}(z)$ is analytic over the entire complex plane less the branch cuts. The c_j are calculated in the CVBEM technique by collocating to the boundary condition values known at the nodal points (Hromadka, 1984).

The c_j of Eq. (1) are calculated in the L_2 norm sense by finding the best choice of c_j to minimize the mean-square error in matching the boundary condition values continuously along Γ. Notation is used for the known and unknown function values along Γ,

$$\left.\begin{aligned} \omega(\zeta) &= \Delta\xi_k(\zeta) + \Delta\xi_u(\zeta) \\ \hat{\omega}(\zeta) &= \Delta\hat{\xi}_k(\zeta) + \Delta\hat{\xi}_u(\zeta) \end{aligned}\right\} \quad \zeta \in \Gamma \qquad (2)$$

where $\omega(z)$ is the solution to the boundary value problem over $\Omega \cup C$; $\hat{\omega}(z)$ is the CVBEM approximation over $\Omega \cup C$; Δ is a descriptor function such that $\Delta = 1, i$ depending whether the associated ξ_x or $\hat{\xi}_x$ function is the real or imaginary term; and ζ is notation for the case of $z \in \Gamma$. Therefore, the objective is to compute the c_j which, for a given nodal distribution on Γ, minimize

$$I = \|\xi_k - \hat{\xi}_k\|_2^2 = \int_\Gamma (\xi_k - \hat{\xi}_k)^2 \, d\Gamma \qquad (3)$$

ORTHOGONAL CVBEM FUNCTIONS AND THE BEST APPROXIMATION

The CVBEM approximation function of (1) can be written as

$$\hat{\omega}(z) = \sum_{j=1}^{m} c_j f_j \tag{4}$$

where $f_j = (z - z_j)\, \mathrm{Ln}\,(z - z_j)$. The Gram-Schmidt procedure can be used to orthogonalize the f_j such that

$$\hat{\omega}(z) = \sum_{j=1}^{m} \gamma_j g_j \tag{5}$$

where γ_j are complex constants and

$$(g_j, g_k) = \int_{\Gamma} g_j g_k \, d\Gamma = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{6}$$

In (6), (g_j, g_k) is notation for the inner-product.

The boundary conditions on Γ are given by ξ_k where $\phi(\zeta)$ is known continuously on contour Γ_ϕ and $\psi(\zeta)$ is known continuously on Γ_ψ where $\Gamma_\phi + \Gamma_\psi = \Gamma$ and $\Gamma_\phi \cap \Gamma_\psi$ only at nodal points. The Γ_ϕ and Γ_ψ can be composed of a finite number of contours. Then the γ_j are computed which minimize

$$I = \int_{\Gamma_\phi} (\phi(\zeta) - \mathrm{Re}\, \Sigma \gamma_j g_j)^2 d\Gamma + \int_{\Gamma_\psi} (\psi(\zeta) - \mathrm{Im}\, \Sigma \gamma_j g_j)^2 d\Gamma \tag{7}$$

Because the g_j are orthogonal, the γ_j are directly computed by

$$\gamma_j = (\xi_k, g_j)/(g_j, g_j) \tag{8}$$

Then the best approximation (in the L_2 norm) is given by

$$\hat{\omega}(z) = \sum_{j=1}^{m} (\xi_k, g_j) g_j/(g_j, g_j) \tag{9}$$

The c_j are then computed by back-substitution of the $\gamma_j g_j$ functions into the $c_j f_j$ functions. It is noted that by this approach, the c_j are computed directly without the use of a matrix system generation or matrix solution. This is important due to boundary integral methods resulting in the solution of fully populated, square matrix systems.

ORTHOGONAL VECTOR SYSTEMS AND THE BEST APPROXIMATION

Let F_j be linearly independent vectors of dimension n, for $j = 1,2,\cdots m$. Then the Gram-Schmidt procedure can be used to construct orthogonal vectors G_j of dimension n such that the dot product gives

$$G_j \cdot G_k = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{10}$$

Let B be a vector of dimension n. Then the best approximation of B in the subspace spanned by the G_j is given by the vector A where

$$A = \sum_{j=1}^{m} \eta_j G_j \tag{11}$$

where

$$\eta_j = (B \cdot G_j)/(G_j \cdot G_j) \tag{12}$$

The corresponding approximation to B with respect to the original F_j vectors is

$$A = \sum_{j=1}^{m} C_j F_j \tag{13}$$

where the C_j are computed by back-substitution of $\eta_j G_j$ into the respective F_j components.

REPRESENTATION OF THE CVBEM APPROXIMATION FUNCTION BY A DIMENSION mn VECTOR SPACE

Let Γ be discretized into m boundary element Γ_j, $j = 1,2,\cdots,m$. On each element, define n interior evaluation points (usually evenly spaced), resulting in a total of mn points t_i on Γ. For each function f_j (see Eq. (4)), develop the vector F_j of dimension mn by

$$F_j = \{f_j(t_i);\ i = 1,2,\cdots,mn\} \tag{14}$$

In (14), the coordinates of t_i are consistent for each vector F_j, $j = 1,2,\cdots,m$, such that points $(t_1, t_2, \cdots, t_n)$ occur in boundary element Γ_1. The resulting vectors F_j form the basis of a subspace F_{mn} where each vector $F \epsilon F_{mn}$ is given by

$$F = \sum_{j=1}^{m} \eta_j F_j \tag{15}$$

Similarly the boundary condition values defined on Γ can be represented by the vector B where

$$B = \{\xi_k(t_i);\ i = 1,2,\cdots,mn\} \tag{16}$$

The best approximation of the vector B (in the ℓ_2 norm analogy of the L_2 norm) by a vector $A \epsilon F_{mn}$ is given directly by (11) and (12). The corresponding estimate of the best approximation $\hat{\omega}(z)$ is given by

$$\hat{\omega}(z) = \sum_{j=1}^{m} \eta_j g_j \tag{17}$$

Thus in the above, the best approximation for $\hat{\omega}(z)$ is estimated by using the best approximation from a vector space spanned by the vectors G_j. Appendix A presents more details and theory of this CVBEM technique, including the proofs of several of the above statements.

IMPLEMENTATION

A FORTRAN computer program was prepared which developed the best approximation in a vector space (of dimension mn) in order to estimate the c_j coefficients of Eq. (1). The basic steps used in the program are as follows:

1. Data entry of nodal point (m) coordinates and boundary values
2. Number of evaluation points entered (n)
3. Develop dimension mn vectors F_j, $j = 1,2,\cdots,m$
4. Develop dimension mn vector B of boundary values
5. Develop orthogonal vectors G_j, $j = 1,2,\cdots,m$
6. Compute vector coefficients η_j
7. Back substitute G_j vectors into F_j and compute the coefficients C_j; $j = 1,2,\cdots,m$

8. Define $c_j = C_j$ to determine the CVBEM approximation function, $\hat{\omega}(z)$.

It is noted that the $c_j = \alpha_j + i\beta_j$. Thus the above program steps involve two vectors for each C_j. That is from (1),

$$\hat{\omega}(z) = \sum_{j=1}^{m} \alpha_j[(z - z_j) \ln (z - z_j)] + \sum_{j=1}^{m} \beta_j[i(z - z_j) \ln (z - z_j)] \tag{18}$$

Hence the f_j vectors corresponding to the c_j have two separate components which are used, respectively, with the α_j and β_j. Consequently, for m nodes there are 2m coefficients to be computed.

COMPUTATIONAL EFFICIENCY

The use of the new CVBEM technique appears to reduce computational effort by approximately 20-percent in comparison to the referenced CVBEM model of Hromadka (1982). Current work is ongoing to evaluate the performance of this new technique in development of approximative boundaries (Hromadka, 1984) and incorporation into general purpose frost heave models.

CONCLUSIONS

In this paper, a new approach for developing CVBEM approximation functions is presented. Using the ℓ_2 norm, the CVBEM approximation is developed which is the best approximation in an equivalent vector subspace. Because orthogonal functions are used, matrix solutions are eliminated; considerably reducing the computational effort and memory requirements. Current research is going to include additional model features such as frost heave, and soil consolidation.

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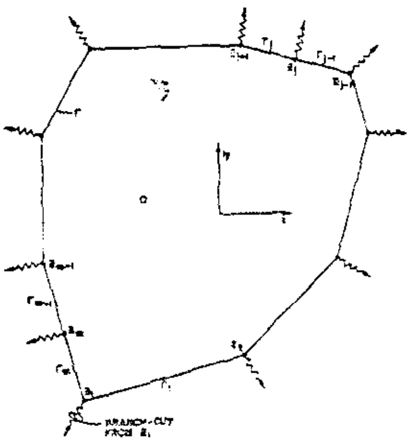

Fig. 1 Nodal Branch-Cut Placement for the CVBEM

APPENDIX A

Notation

For $\omega \in W_\Omega$, the following notation is used:

Ω: convex, simply connected domain

Γ: *simple closed contour forming the boundary* Ω

ζ, z: $\zeta \in \Gamma$, $z \in \Omega$; $\zeta = Re^{i\theta}$ for $0 \leq \theta < 2\pi$

$\phi = \text{Re}\ \omega$

$\psi = \text{Im}\ \omega$

$|\omega| = (\phi^2 + \psi^2)^{1/2}$

$d\mu = |d\zeta|$

$||\omega||p = (\int_\Gamma |\omega(\zeta)|^p d\mu)^{1/p}$

$\{z_j\}$: nodal points defined on Γ

Γ_j: boundary element (line segment) connecting z_j, z_{j+1}

B: $B = U\Gamma_j$; Ω_B = domain enclosed by B

Γ_ϕ, Γ_ψ: $\Gamma_\phi \cup \Gamma_\psi = \Gamma$ and $\Gamma_\phi \cap \Gamma_\psi$ at two points of Γ

$(\omega,\omega) = \int_{\Gamma_\phi} \phi^2 d\mu + \int_{\Gamma_\psi} \psi^2 d\mu$

$||\omega|| = (\omega,\omega)^{1/2}$

L = length of Γ

$\bar{\omega}(\zeta) = \lim_{\delta \to 1} \omega(\delta\zeta)$, $0 \leq \delta < 1$. Similarly, $\bar{\omega}(\zeta) = \bar{\phi}(\zeta) + i\bar{\psi}(\zeta)$

z_c = centroid of Ω oriented such that $z_c = 0 + 0i$

δ = a coordinate reduction factor, $0 \leq \delta < 1$

$\Gamma_\delta = \{\delta\zeta,\ \zeta \in \Gamma\}$

$\Omega_\delta = \{\delta z,\ z \in \Omega\}$

$\bar{\Omega} = \Omega \cup \Gamma$

$\bar{\Omega}_\delta = \Omega_\delta \cup \Gamma_\delta$

ℓ = *minimum boundary element length*

DEFINITION OF W_Ω

Let Ω be a simply connected convex domain enclosed by the simple-closed contour Γ, where the centroid z_c of $\bar{\Omega} = \Omega \cup \Gamma$ is located at $z_c = 0 + 0i$. Then W_Ω is the set of complex valued *functions which are analytic over* Ω and satisfy

(1) $|\omega(\zeta)| \in L^2(\Gamma)$

(2) $\lim_{\delta \to 1} \omega(\delta\zeta) = \omega(\zeta)$ ae

DEFINITION OF W_Ω^A

W_Ω^A is the set of all $\omega \in W_\Omega$ such that ω is analytic over Γ.

Theorem

$W_\Omega^A \subseteq W_\Omega$.

Proof

Let $\omega \in W_\Omega^A$. Then $\omega \in W_\Omega$.

DEFINITION OF W_Ω^L

Let the boundary Γ of Ω be piecewise linear. Then W_Ω is noted as W_Ω^L.

Theorem

Let $\omega \in W_\Omega$. Then $\omega(\delta\zeta) \to \omega(\zeta)$ in $L^2(\Gamma)$.

Proof

Consider just $\phi(z)$ of $\omega(z) = \phi(z) + i\psi(z)$ and suppose $\phi(z) \geq 0$ for discussion purposes. Define sets E_N and R_N by

$$E_N = \{\zeta \in \Gamma: \ \phi(\zeta) \leq N\}$$

$$R_N = \Gamma - E_N$$

Define functions $\phi_N(z)$ and $\phi_N^*(z)$ both elements of W_Ω such that the boundary values on Γ are given by two Dirichlet problems

$$\phi_N(\zeta) = \begin{cases} \bar{\phi}(\zeta), & \zeta \in E_N \\ N, & \zeta \in R_N \end{cases}$$

$$\phi_N^*(\zeta) = \begin{cases} 0, & \zeta \in E_N \\ \bar{\phi}(\zeta) - N, & \zeta \in R_N \end{cases}$$

Then $\phi_N(\zeta) + \phi_N^*(\zeta)$ on Γ and $\phi_N(z) + \phi_N^*(z) = \bar{\phi}(z)$ over Ω.

It is noted that $\phi(\zeta) \in L^2(\Gamma)$ implies

$$A = \int_\Gamma [\phi(\zeta)]^2 d\mu = \int_\Gamma [\bar{\phi}(\zeta)]^2 d\mu \geq \int_{R_N} [\phi(\zeta)]^2 d\mu \geq N^2 m R_N$$

and thus

$$mR_N < A/N^2$$

Also as $N \to \infty$, $mR_N \to 0$ and

$$\int_{R_N} [\bar{\phi}(\zeta)]^2 d\mu \to 0 \quad \text{as} \quad N \to \infty.$$

Because $\phi_N(\delta\zeta)$ is dominated by the constant N for all $0 \leq \delta \leq 1$, the Lebesgue Dominated Convergence Theorem applies and $\phi_N(\delta\zeta) \to \phi_N(\delta\zeta)$ in $L^2(\Gamma)$, for all N.

$$\text{Let } R_\infty = \lim_{N\to\infty} R_\infty = \bigcap_{N=1}^{\infty} R_N.$$

Let $mR_\infty = 0$, let $\zeta_0 \in R_\infty$. Then because $\phi(z)$ is harmonic over Ω it cannot have a maximum (or minimum) in Ω unless $\phi(z)$ is constant over Ω. Thus $\sup[\phi(z)]$ must occur for $z \in \Gamma$; that is, $z \in R_\infty$. Then $\phi(\delta\zeta_0) \leq \bar{\phi}(\zeta_0)$ for all $\zeta_0 \in R_\infty$ and $\phi(\delta\zeta_0)$ is dominated by $\bar{\phi}(\zeta_0)$ over R_∞. Therefore, $\phi(\delta\zeta) \to \phi(\zeta)$ in $L^2(\Gamma)$. Similarly, $\psi(\delta\zeta) \to \psi(\zeta)$ in $L^2(\Gamma)$, and $\omega(\delta\zeta) \to \omega(\zeta)$ in $L^2(\Gamma)$.

Theorem

Let $\omega \in W_\Omega$. Then for $z \in \Omega$,

$$\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta) d\zeta}{\zeta - z}$$

Proof

Let $\Gamma_\delta = \{\delta\zeta, \ \zeta \in \Gamma \text{ and } 0<\delta<1\}$.

Then ω is analytic over Γ_δ and the domain enclosed by Γ_δ noted as Ω_δ. Choose Γ_δ such that $z \in \Omega_\delta$. Then

$$\omega(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{\omega(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\delta\zeta) d(\delta\zeta)}{\delta\zeta - z}$$

Consider

$$\chi = \left| \omega(z) - \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta) d\zeta}{\zeta - z} \right| = \frac{1}{2\pi} \left| \int_\Gamma \frac{\omega(\delta\zeta)\delta d\zeta}{\delta\zeta - z} - \int_\Gamma \frac{\omega(\zeta) d\zeta}{\zeta - z} \right|$$

$$= \frac{1}{2\pi} \left| \int_\Gamma \frac{(\delta\zeta[\omega(\delta\zeta) - \omega(\zeta)] - z[\delta\omega(\delta\zeta) - \omega(\zeta)]) d\zeta}{(\delta\zeta - z)(\zeta - z)} \right|$$

Letting $d = \min|\zeta - z|$ for $\zeta \in \Gamma$, then as $\delta \to 1$

$$\lim_{\delta\to 1} \chi \leq \frac{1}{2\pi d} \int_\Gamma |\omega(\delta\zeta) - \omega(\zeta)| d\mu = \frac{1}{2\pi d} ||\omega(\delta\zeta) - \omega(\zeta)||_1$$

But $|\omega(\delta\zeta)| \to |\omega(\zeta)|$ in L^2 implies L^1 convergence, and $\lim_{\delta\to 1} \chi = 0$.

Thus

$$\omega(z) = \frac{1}{2\pi i} \int \frac{\omega(\zeta) d\zeta}{\zeta - z}, \quad z \in \Omega.$$

Theorem

Let $\{f_j\} = \{1, z, (z - z_1) \, Ln_1(z - z_1), \cdots, (z - z_j) \, Ln_j(z - z_j), \cdots\}$ where $z_j \in \Gamma$ and the $\{z_j\}$ are separate nodal points of Γ. Then $\{f_j\}$ are linearly independent functions on Γ.

Proof

Suppose the first (n+2) of the functions are linearly independent. That is,

$$\hat{\omega}_n = C_{n+1} + C_{n+2} \, z + \sum_{j=1}^{n} C_j (z - z_j) \, Ln_j(z - z_j)$$

and the next function $(z - z_k) \, Ln_k(z - z_k)$ is not linearly independent. Then for some complex constant C_k,

$$C_k (z - z_k) \, Ln_k(z - z_k) = \hat{\omega}_n(z) \text{ for all } z \in \Gamma.$$

Taking the second derivative with respect to z gives

$$\frac{\kappa}{(z-z_k)} = \sum_{j=1}^{m} \frac{C_j}{(z-z_j)}, \text{ for } z \in \Gamma \text{ and } z = z_1, z_2, \cdots, z_k$$

Rearranging terms, linear dependence for node z_k sets

$$C_k \prod_{j=1}^{n} (z - z_m) = (z - z_k) \sum_{j=1}^{n} C_j \prod_{\substack{m=1 \\ m\neq j}}^{n} (z - z_i)$$

Hence, as $z \to z_m$ a contradiction arises for every m. Thus, the $\{f_i\}$ are linearly independent on Γ by induction (the case of $n = 1$ is trivial).

Notation

The symbol $||\omega||$ for $\omega \in W_\Omega$ is notation for

$$||\omega|| = \left[\int_{\Gamma_\phi} (\mathrm{Re}\ \omega)^2\, d\mu + \int_{\Gamma_\psi} (\mathrm{Im}\ \omega)^2\, d\mu\right]^{\frac{1}{2}}$$

The symbol $||\omega||p$ for $\omega \in W_\Omega$ is notation for

$$||\omega||p = \left[\int_\Gamma |\omega(\zeta)|^p\, d\mu\right]^{\frac{1}{p}}, \ p \geq 1$$

of importance is the case of $p = 2$:

$$||\omega||_2 = \left[\int_\Gamma |\omega(\zeta)|^2 d\mu\right]^{\frac{1}{2}} = \left[\int_\Gamma \left\{[\mathrm{Re}\ \omega]^2 + [\mathrm{Im}\ \omega]^2\right\} d\mu\right]^{\frac{1}{2}}$$

Because sets of Lebesque measure zero have no effect on integration, almost-everywhere (ae) equality on Γ indicates the same class of element. Thus for $\omega \in W_\Omega$, $[\omega] = \{\omega \in W_\Omega\colon \omega(\zeta)$ are equal ae for $\zeta \in \Gamma\}$. For example, $[0] = \{\omega \in W_\Omega\colon \omega(\zeta) = 0$ ae, $\zeta \in \Gamma\}$. When understood, the notation "[ ]" will be dropped when there is no confusion.

Theorem ($||\omega|| = 0$ implies $\omega = [0] \in W_\Omega$)

Given $\omega \in W_\Omega$. Let $\omega = \phi + i\psi$ where $\phi = 0$ ae on Γ_ϕ and $\psi = 0$ ae on Γ_ψ. Then $||\omega|| = 0$, and $\omega = [0] \in W_\Omega$.

Proof

Let $f(z) = u + iv$ be a conformal mapping of Ω onto G where Γ_ϕ is split into two equal parts C_1 and C_2, and $C_3 = \Gamma_\psi$, and G is the region $0<u<1$, $v>0$ so that: the image of C_1 is the ray $u = 0$, $v>0$; the image of C_2 is the ray of $u = 1$, $v>0$; and the image of C_3 is the segment $0<u<1$ of the u-axis (the Riemann Mapping Theorem guarantees the existence of $f(z)$).

The boundary condition of $\psi = 0$ on C_3 is equivalent of $\partial\phi/\partial n = 0$ on C_3. Therefore, in G the problem is to find $\phi^*(u,v)$ such that $\phi^*(0,v) = 0$ for $v>0$ and $\phi^*(1,v)=0$ for $v>0$, and $\partial\phi/\partial n^* = \phi_v^*(u,0) = 0$. The solution is $\phi^*(u,v) = 0$. Thus, $\phi(x,y) = 0$. By continuity of $\phi(x,y)$ over $\Omega, \psi(x,y) = 0$. Thus, $||\omega|| = 0$ implies $\omega = [0] \in W_\Omega$.

Theorem (relationship of $||\omega||$ to $||\omega||_2$)

Let $\omega \in W_\Omega$. Then $||\omega||_2^2 = ||\omega||^2 + ||i\omega||^2$

Proof

Let $\omega = \phi + i\psi$.

Then
$$||\omega||_2^2 = \int_\Gamma |\omega(\zeta)|^2 d\mu$$
$$= \int_\Gamma (\phi^2 + \psi^2) d\mu$$
$$= \int_{\Gamma_\phi} \phi^2 d\mu + \int_{\Gamma_\psi} \phi^2 d\mu + \int_{\Gamma_\phi} \psi^2 d\mu + \int_{\Gamma_\psi} \psi^2 d\mu$$
$$= ||\omega||^2 + \int_{\Gamma_\phi} (-\psi)^2 d\mu + \int_{\Gamma_\psi} \phi^2 d\mu$$
$$= ||\omega||^2 + ||i\omega||^2$$

Theorem

Let $\omega \in W_\Omega$. Then $||\omega|| = 0 \Rightarrow ||\omega||_2 = 0$.

Proof

$||\omega|| = 0$ implies $\omega = [0]$. And $i\omega = [0]$.

Thus, $||\omega||_2 = \sqrt{||\omega||^2 + ||i\omega||^2} = 0$.

Theorem

Let $\omega \in W_\Omega$. Then $||\omega|| = 0$ iff $||\omega||_2 = 0$

Proof

Follows by previous theorems.

Theorem

Let $\omega \in W_\Omega$. Then $||\omega|| \leq ||\omega||_2$.

Proof

Let $\omega = \phi + i\psi$.

Then

$$||\omega||_2^2 = \int_\Gamma |\omega(\zeta)|^2 d\mu = \int_\Gamma \phi^2 d\mu + \int_\Gamma \psi^2 d\mu$$
$$= ||\omega||^2 + ||i\omega||^2$$

Because $||i\omega||^2 \geq 0$, then $||\omega||^2 \leq ||\omega||_2^2$.

Theorem (inner-product on W_Ω)

Let (α,β) be defined for $\{\alpha,\beta\} \in W_\Omega$ by

$$(\alpha,\beta) = \int_{\Gamma_\phi} \mathrm{Re}\alpha\ \mathrm{Re}\beta d\mu + \int_{\Gamma_\psi} \mathrm{Im}\alpha\ \mathrm{Im}\beta d\mu$$

Then (α,β) is an inner-product on W_Ω.

Proof

(i) $(\alpha,\alpha) = \int_{\Gamma_\phi} (\mathrm{Re}\alpha)^2 d\mu + \int_{\Gamma_\psi} (\mathrm{Im}\alpha)^2 d\mu \geq 0$

(ii) $(\alpha,\alpha) = 0$ implies $\alpha = [0] \in W_\Omega$:
$(\alpha,\alpha) = 0$ implies $\mathrm{Re}\alpha = 0$ ae on Γ_ϕ
and $\mathrm{Im}\alpha = 0$ ae on Γ_ψ. Thus, α $[0] \in W_\Omega$.

(iii) $(k\alpha,\beta) = \int_{\Gamma_\phi} Rek\alpha \; Re\beta d\mu + \int_{\Gamma_\psi} Imk\alpha \; Im\beta d\mu = k(\alpha,\beta)$

(iv) $(\alpha,\beta) = (\beta,\alpha)$

(v) $(\alpha+\beta,\gamma) = (\alpha,\gamma) + (\beta,\gamma)$

Theorem

W_Ω is an inner-product space.

Theorem

The function $||\omega|| = (\omega,\omega)^{1/2}$ *is a norm for* $\omega \in W_\Omega$.

Theorem ($\hat{\omega}_m$ can approximate an analytic function on W_Ω^L)

Let Γ be the union of a finite number of straight line segments such that Γ is a simple closed contour forming the boundary of the simply connected domain Ω. Let ω be analytic over $\Omega \cup \Gamma$. Then for any $\varepsilon > 0$ there exists a CVBEM approximator $\hat{\omega}_m$ such that $||\omega - \hat{\omega}_m|| < \varepsilon$.

Proof

Define m nodal points $\{z_j\}$ on Γ such that a node is located at each angle point of Γ, and the length of each boundary element ℓ_j satisfies max $\{\ell_j\} \le 2$ min $\{\ell_j\}$. Using a linear global trial function $G_m(\zeta)$, define $e_m(\zeta) = \omega(\zeta) - G_m(\zeta)$ for $\zeta \in \Gamma$. Also define $||e_m|| = \max|\omega(\zeta) - G_m(\zeta)|$. Then $|e_m'(\zeta)| \le |\omega'(\zeta)| + ReG_m'(\zeta)| + ImG_m'(\zeta)| \le 3M_1$ where $M_1 = \max|\omega'(\zeta)|$. For $M_2 = \max|\omega''(\zeta)|$, For $M_2 = \max|\omega''(\zeta)|$, $||e_m|| \le M_2\ell^2$ where $\ell = \min\{\ell_j\}$.

Let point $z_0 \in$ *boundary element* Γ_k *with* $z_0 \ne z_k$ *nor* z_{k+1} (the end point nodes of Γ_k).

Define χ by

$$\chi = |\omega(z_0) - \hat{\omega}_m(z_0)| = \frac{P}{2\pi}\left|\int_\Gamma \frac{[\omega(\zeta) - G_m(\zeta)]d\zeta}{\zeta - z_0}\right|$$

where P is notation for the Cauchy Principal Value of the integral, or limit as $z \to z_0$ sectorally for $z \in \Omega$. Because $\omega(z)$ is analytic at z_0, there exists a domain $\mathcal{D}(z_0)$ such that $\mathcal{D}(z_0) = \{z: |z - z_0| < R, R > 0\}$ and $\omega(z)$ is analytic in $\mathcal{D}(z_0)$ and can be expanded as a Taylor series. Choose m sufficiently large such that $\Gamma_k \subset \mathcal{D}(z_0)$. Then

$$\chi \le \frac{1}{2\pi}\int_{\Gamma-\Gamma_k} \frac{|e_m(\zeta)|d\mu}{|\zeta - z_0|} + \frac{P}{2\pi}\left|\int_{\Gamma_k} \frac{e_m(\zeta)d\zeta}{\zeta - z_0}\right|$$

$$\le \frac{3L||e_m||}{2\pi\ell} + \frac{P}{2\pi}\left|\int_{\Gamma_k} \frac{[e_m(\zeta) - e_m(z_0)]d\zeta}{\zeta - z_0}\right|$$

$$+ \frac{|e_m(z_0)|P}{2\pi}\left|\int_{\Gamma_k} \frac{d\zeta}{\zeta - z_0}\right|$$

But

$$|e_m(\zeta) - e_m(z_0)| \le |Re[e_m(\zeta) - e_m(z_0)]| + |Im[e_m(\zeta) - e_m(z_0)]|$$

$$\le 2|e_m'(\zeta)||\zeta - z_0| + 2|e_m'(\zeta)||\zeta - z_0|$$

$$\le 12M_1|\zeta - z_0|$$

That is, $e_m(\zeta)$ satisfies a Lipshitz condition on Γ_k with an exponent of 1. Also

$$\frac{P}{2\pi}\left|\int_{\Gamma_k} \frac{d\zeta}{\zeta - z_0}\right| = \frac{P}{2\pi}\left|Ln\frac{z_{k+1} - z_0}{z_k - z_0}\right| \le \frac{1}{2\pi}\left|Ln\frac{2\ell}{\frac{\ell}{3}} + \pi\right| < 1$$

Thus

$$\chi \le \frac{3L}{2\pi\ell}||e_m|| + ||e_m|| + \frac{P}{2\pi}\int_{\Gamma_k} \frac{12M_1|\zeta - z_0|d\mu}{|\zeta - z_0|}$$

$$< \frac{L}{\ell}||e_m|| + ||e_m|| + 12M_1\ell$$

But $||e_m|| \le M_2\ell^2$. Then $\chi = \theta(\ell) = \theta\left(\frac{1}{m}\right)$

and $\lim_{\substack{m\to\infty \\ \ell\to 0}} \chi = 0.$

Choose m sufficiently large such that the above development is valid *and also* $|\omega(\zeta) - \hat{\omega}_m(\zeta)| < \varepsilon/\sqrt{L}$ *where* $\varepsilon > 0$. Then

$$||\omega - \hat{\omega}_m||_2 = \left[\int_\Gamma |\omega - \hat{\omega}_m|^2 d\mu\right]^{1/2} < \left\{\int_\Gamma (\varepsilon^2/L)d\mu\right\}^{1/2} = \varepsilon$$

and for any $\varepsilon > 0$, *there exists a* $\hat{\omega}_m$ *such that*

$$||\omega - \hat{\omega}_m|| \le ||\omega - \hat{\omega}_m||_2 < \varepsilon.$$

(A similar argument holds for $z_0 \in \{z_j\}$.)

Theorem

Let $\omega \in W_\Omega$. Let $\{z_j\}$ be a CVBEM nodal partition of Γ. Define as the union of the boundary elements,

$$B = \bigcup_{j=1}^{m} \Gamma_j$$

Then $B \subset \Omega \cup \Gamma$.

Proof

Because $\Omega \cup \Gamma$ is convex, each CVBEM boundary element is a straight line segment connecting to points in $\Omega \cup \Gamma$, and hence is contained in $\Omega \cup \Gamma$.

Theorem

Let $\omega \in W_\Omega^A$. Approximate Γ by m nodes $\{z_j\}$ to develop B. For every $\varepsilon > 0$ there is a B such that for $\zeta \in \Gamma$ and $\zeta_B \in B$, where $\zeta = \zeta(\theta)$ and $\zeta_B = \zeta_B(\theta)$, $0 \le \theta < 2\pi$ and $||\omega[\zeta(\theta)] - \omega[\zeta_B(\theta)]|| < \varepsilon$.

Proof

$\omega \in W_\Omega^A$ implies ω is analytic over $\Omega \cup \Gamma$ and also B. Hence ω is uniformly continuous over $\Omega \cup \Gamma$. As $m\to\infty$, $B \to \Gamma$ and $\max|\zeta(\theta) - \zeta_B(\theta)| \to 0$. Choose m such that *max* $|\omega[\zeta(\theta)] - \omega[\zeta_B(\theta)]| < \varepsilon/\sqrt{L}$. *Then*

$$||\omega - \omega_B|| \le ||\omega - \omega_B||_2 \le \left[\int_\Gamma |\omega[\zeta(\theta)] - \omega[\zeta_B(\theta)]|^2 d\mu\right]^{1/2}$$

$$\le \left[\int_\Gamma (\varepsilon^2/L)d\mu\right]^{1/2} = \varepsilon$$

Theorem

Let $\omega \varepsilon W_{\Omega}$. Then for every $\varepsilon > 0$ there exists a CVBEM approximation $\hat{\omega}_m$ and associated boundary B such that

$$||\omega(\zeta) - \hat{\omega}_m(\zeta_B)|| < \varepsilon$$

where $\zeta \varepsilon \Gamma$ and $\zeta_B \varepsilon B$.

Proof

Approximate $\omega(\zeta)$ by $\omega(\delta\zeta)$ such that $||\omega(\zeta) - \omega(\delta\zeta)|| < \varepsilon/2$ and $0<\delta<1$. Then $\omega(\delta\zeta)$ is analytic over $\Omega \cup \Gamma$ and $\omega(\delta\zeta) \varepsilon W_{\Omega}^{A}$. Approximate $\omega(\delta\zeta)$ by a $\hat{\omega}_m$ such that $||\omega(\delta\zeta) - \hat{\omega}(\zeta_B)|| < \varepsilon/2$ where $\zeta_B \varepsilon B$ and $\zeta \varepsilon \Gamma$. Then

$$||\omega(\zeta) - \hat{\omega}_m(\zeta_B)|| \leq ||\omega(\zeta) - \omega(\delta\zeta)|| + ||\omega(\delta\zeta) - \hat{\omega}_m(\zeta_B)|| < \varepsilon$$

Theorem

Let $\omega \varepsilon W_{\Omega}$ where $\Gamma = B$ for some finite number of nodes. (That is, Γ is a union of a finite number of line segments.) Then

$$||\omega(\zeta) - \hat{\omega}_m(\zeta_B)|| = ||\omega(\zeta) - \hat{\omega}_m(\zeta)||$$

Proof

For the assumed $\Gamma = B$, $\zeta_B \varepsilon \Gamma$.

Theorem

Let $\omega \varepsilon W_{\Omega}^{L}$ (where Γ is a piecewise linear contour). Then the set of functions $\{f_j\}$ forms a basis for W_{Ω}.

Proof

Approximate $\omega(\zeta)$ by $\omega(\delta\zeta)$ for some $0<\delta<1$. Approximate $\omega(\delta\zeta) \varepsilon W_{\Omega}^{A}$ by a CVBEM $\hat{\omega}_m(\zeta)$. Then

$$||\omega(\zeta) - \hat{\omega}_m(\zeta)|| \leq ||\omega(\zeta) - \omega(\delta\zeta)|| + ||\omega(\delta\zeta) - \hat{\omega}_m(\zeta)||$$

Engineering Problems

In practical engineering problems, we are involved with W_{Ω}^{L} spaces. Additionally, the boundary values of $\omega \varepsilon W_{\Omega}^{L}$ are piecewise continuous in the first derivative, and continuous with respect to both the potential and stream functions over Γ.

From previous work, the $\{f_j\}$ forms a basis for W_{Ω}^{L}. Thus the defined norm $||\omega||$ for $\omega \varepsilon W_{\Omega}^{L}$ provides an immediate best approximation using a generalized Fourier series expansion of ω using $\{f_j\}$.