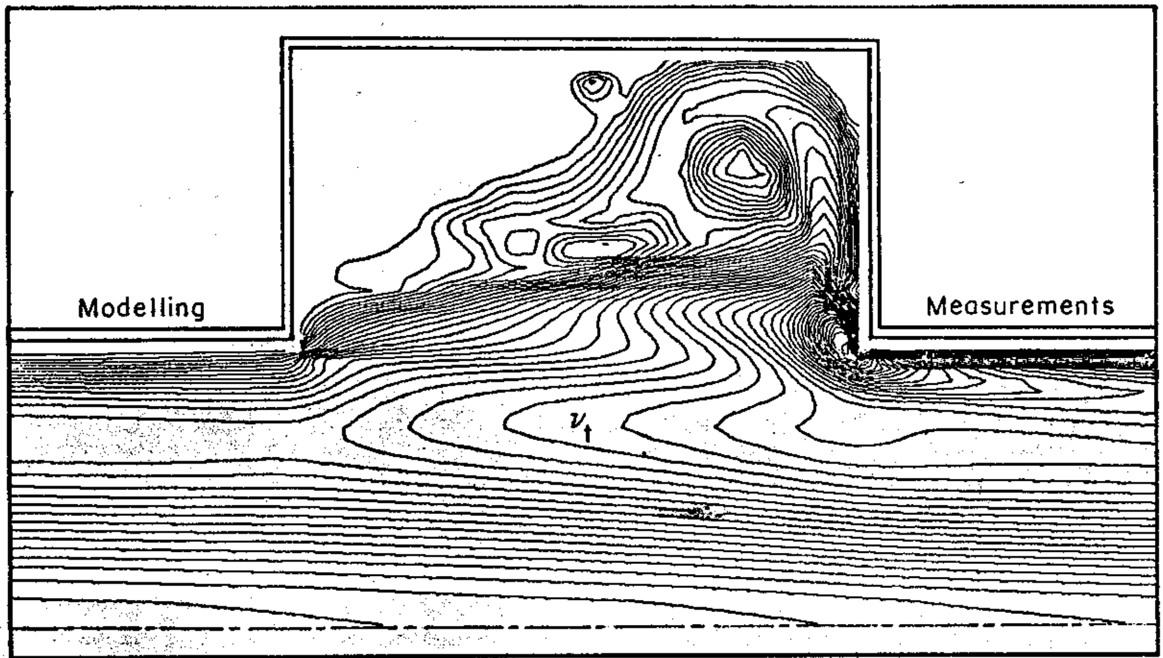
Proceedings

International Symposium On Refined Flow Modelling And Turbulence Measurements



16-18 September 1985

The University of Iowa
Iowa City, Iowa
USA

Sponsored by

Iowa Institute of Hydraulic Research

with the cooperation of

International Association for Hydraulic Research
American Society of Civil Engineers
American Society of Mechanical Engineers
U.S. National Science Foundation

VOLUME II

Modeling Complex Two-Dimensional Flows by the Complex-Variable Boundary-Element Method

by

Chintu Lai
Hydrologist
U.S. Geological Survey, WRD, NR
Reston, Virginia 22092

Ted V. Hromadka, II[1/]
Hydrologist
U.S. Geological Survey, WRD,
Laguna Niguel, California

SUMMARY

The Complex-Variable Boundary-Element Method (CVBEM) has emerged as a powerful, accurate and effective numerical technique for simulating two-dimensional potential flows. Its applicability to basic flow problems has been demonstrated in the recent literature; however, the CVBEM model can also be extended to a host of more complex two-dimensional problems in either the horizontal or vertical plane, for both surface water and ground water hydrology. Two examples are presented to illustrate the adaptability and applicability of the CVBEM to complex flow situations; these are flow over a spillway and advective-contaminant transport in ground water. These problems encompass either an iterative solution procedure or the superposition of a number of flow components.

1. Introduction

In computational fluid mechanics and hydraulics, numerical methods are constantly under development and refinement for the effective modeling of various fluid-flow problems. These methods can be classified into two categories according to the mode of numerical discretization--the domain approach and the boundary approach. Although the former approach is the dominant one at present, the latter has begun to attract the attention of modeling hydraulic engineers. Boundary techniques employ discretized boundary elements instead of discretized domain elements, as in domain methods, such as finite difference and finite element formulations. For physical processes to which the potential theory can be applied, the domain integration may be reduced to boundary integration through the theorems of Green, Stokes, Gauss or Cauchy. This results in considerable simplification of the modeling procedure.

The majority of boundary element formulations (e.g., 1,10) have dealt with a real-variable integration along the boundary of a real domain. The Complex Variable Boundary Element Method (CVBEM) on the other hand, handles the complex-variable integration along the boundary of a complex domain. The method is based on a function referred to as the H_n-Approximative Function, which is derived from the Cauchy integral formula, and is formulated for computer solution of the Laplace equation with appropriate boundary conditions. If the given flow is two-dimensional (2-D), transforming an areal integration to a one-dimensional (1-D) integration by the CVBEM is simpler and more efficient than a real-variable scheme based on Green's theorem. Furthermore, both harmonic functions, potential (ϕ) and stream (ψ) functions, are

1/ Presently with Williamson and Schmid, 17782 Park Blvd., Irvine, CA simultaneously available in the CVBEM model output. This also means that normal derivatives of the solution variables, which are required in the real-variable integration, are not needed.

In this study, two relatively complex flow problems are selected for simulation involving the use of the CVBEM. The first problem is a flow over a low spillway of a given crest and bucket profile, thus representing an open-channel flow which is 2-D in the vertical sense. This problem involves determination of an unknown free surface, for which numerical iteration is needed. The second problem treats advective contaminant transport in porous media resulting from recharging and pumping through wells, thus presenting an example of ground-water flow which is 2-D in the horizontal direction. This type of problem involves solutions of the Poisson equation, the analytic functions of sink and source, and boundary conditions which create a background flow.

2. A Review of Complex-Variable Boundary-Element Method (CVBEM)

The development of the CVBEM has been presented in several articles (2,3,5,6,7). A very brief review of the method is given here, which leads to the formulation of a computer algorithm that can be extended to simulate relatively complex 2-D flows in hydraulics.

In a complex region Ω, enclosed by a boundary Γ, the Cauchy integral formula,

$$\omega(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta)\, d\zeta}{\zeta - z} \tag{1}$$

relates a single-valued function ω at any interior point z to the integration of the function on the closed boundary. For computer solution, the continuum boundary of Eq. 1 can be discretized into a number of straight-line segments Γ_j, called "boundary element," (see Fig. 1). The exact function ω can be replaced by a continuous global trial function $G_n(z)$, resulting in the approximative function $\hat{\omega}(z)$,

$$\hat{\omega}(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\Gamma_j} \frac{G_n(\zeta)\, d\zeta}{\zeta - z} \tag{2}$$

in which, G_n is a complex polynomial of degree n, and m is the number of boundary elements. G_n is defined on each boundary element, and joined together at each (vertex) node to form a continuous global function. If n=1, G_n represents a linearly-varying (within each element) trial function, which will be the case considered throughout this study.

The function $\hat{\omega}(z)$ defined by Eq. 2, integrated in the usual positive sense, is referred to as the H_n Approximative Function. If $\{z_j;\ j = 1,2,\ldots m\}$ represent a set of nodal points on Γ, three types of nodal function values can be defined at these nodal points:--namely, exact (or continuum) nodal values, $\omega(z_j)$; approximative function (or discrete-form) nodal values, $\hat{\omega}(z_j)$; and specified nodal values, $\bar{\omega}(z_j)$. The nodal value of each type consists of the real part ϕ and the imaginary part ψ, which may also be identified with the corresponding type-notation, e.g. $\hat{\omega}(z_j) = \hat{\phi}(z_j) + i\hat{\psi}(z_j)$.

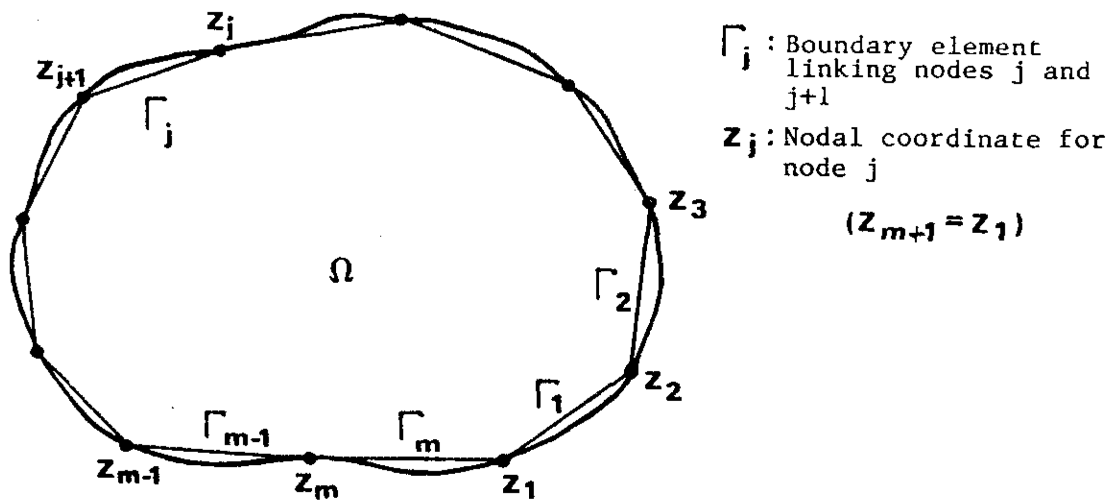


Fig. 1. CVBEM Boundary Discretization

In many engineering boundary problems, only one of the two "specified" nodal values $\bar{\Phi}$ and $\bar{\Psi}$ is actually given or known at each node, and the issue becomes that of finding the other. [An expanded definition for $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$ will be used here; that is, if at least one of the conjugate function values, ϕ_j or ψ_j, is known, all three symbols ω, ϕ, ψ will be capped with the overbar; thus, $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$, even one of $\bar{\phi}_j$ and $\bar{\psi}_j$ is unknown.] One commonly used approach is to let the interior point z in Eq. 2 approach each boundary node z_j in turn, and thus generating a system of m complex-variable equations, each having the form

$$\hat{\omega}(z_j) = \hat{\phi}(z_j) + i\hat{\psi}(z_j)$$

$$= \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\Gamma_k}[\alpha_\phi(\bar{\phi}_k, \bar{\phi}_{k+1},\zeta) + i\alpha_\psi(\bar{\psi}_k\ \bar{\psi}_{k+1},\zeta)]\frac{d\zeta}{\zeta - z_j} \qquad (3)$$

in which α_ϕ and α_ψ are both functions of real variables. [It can be proven that Cauchy principal values exist.]

Equating the real and imaginary parts on both sides of Eq. 3 (after integration), results in 2m equations. Because one half of the nodal variables ($\bar{\phi}_j$, $\bar{\psi}_j$) are known, the remaining unknowns (m in number) can be found by using only one half of the 2m equations.

3. Formulation of CVBEM Algorithm for Flow Modeling

By denoting the known part of $\bar{\omega}(z)$ as $\bar{\xi}_k$, $\bar{\Phi}(z)$ or $\bar{\Psi}(z)$, and the unknown part as $\bar{\xi}_u$, $\bar{\Phi}(z)$ or $\bar{\Psi}(z)$, and by using the symbol Δ to indicate 1 for $\phi(z)$ and i for $\psi(z)$, $\bar{\omega}(z_j)$ may be expressed as $\bar{\omega}(z_j) = \Delta\bar{\xi}_k(z_j) + \Delta\bar{\xi}_u(z_j)$. Furthermore, in Eq. 3, the part in $\hat{\omega}$ corresponding to $\bar{\xi}_k$, that is, $\hat{\Phi}$ or $\hat{\Psi}$, will be denoted as $\hat{\xi}_k$, and the other part, which corresponds to $\bar{\xi}_u$, as $\hat{\xi}_u$. Then the corresponding expression should be $\hat{\omega}(z_j) = \Delta\hat{\xi}_k + \Delta\hat{\xi}_u$. With the foregoing notation, the 2m equations formulated in the preceding section may be grouped into two sets of m equations as follows (6).

$$\text{Case I} \quad [\hat{\xi}_k] = C_1 [\bar{\xi}_k] + C_2 [\bar{\xi}_u] \tag{4}$$

$$\text{Case II} \quad [\hat{\xi}_u] = C_3 [\bar{\xi}_k] + C_4 [\bar{\xi}_u] \tag{5}$$

in which C_1, C_2, C_3 and C_4 are each m x m matrices composed of real numbers.

The m unknowns, ξ_u, can then be solved either from the first system, Case I, by letting $\hat{\xi}_k = \xi_k$ (i.e. collocating the knowns explicitly), or from the second system, Case II, by letting $\hat{\xi}_u = \xi_u$ (i.e. collocating the unknowns implicitly).

In this study, the Case II approach has been used. After all ξ's are found, the values of $\bar{\omega}(z_j) = \bar{\phi}(z_j) + i\bar{\psi}(z_j)$, $j=1,2,\ldots m$, are used to compute the H_1 approximative function, $\hat{\omega}(z_j)$, from which error estimates are made. The process is then repeated for numerical improvement.

4. Flow Over a Low Spillway

Two-dimensional ideal-fluid flow (incompressible and inviscid) can usually be represented by complex potential and thus satisfies Laplace equations,

$$\nabla^2\phi = 0, \qquad \nabla^2\psi = 0 \tag{6}$$

Therefore, if appropriate boundary conditions are given, that is, if enough ξ_k's can be specified along the boundary, the remaining ξ_u's can be computed (by Case I or II) and the flow problem may be solved.

Flow problems become complicated and difficult to solve if the boundary geometry is not defined. Flows with a free surface are typical examples of the class of problems in which determination of the boundary becomes a part of the problem solution. A steady ideal-fluid flow over a low spillway is selected here to illustrate the feasibility of extending the basic CVBEM solution to complex flow problems for which a close-form solution does not exist. A similar type of problem, aside from the classical flow-net approach (11), has been addressed by various computational methods in recent years;--e.g. the Finite Difference Method (FDM) by Southwell and Vaisey (12), the Finite Element Method (FEM) by Ikegawa and Washizu (8) and the real-variable Boundary Element Method (BEM) by Liggett (9).

A steady ideal-fluid flow over a low spillway with the crest and bucket profile as depicted in Fig. 2 is considered. The problem is to determine the free-surface profile together with the families of equi-potential and stream lines, i.e. a flow net, which satisfy assumed free-surface conditions.[2/]

The solution to this type of problem entails some form of iteration. An approximate free-surface is first assumed and two vertical lines are taken sufficiently far upstream and downstream where a uniform flow is assured. Using this flow domain, bounded by surface and bottom profiles and two end-verticals, the CVBEM is applied and ϕ's are

2/The problem is derived from one of the flow-net-problem assignments, given to the students of Intermediate Mechanics of Fluids 59:103 by Dr. Hunter Rouse at the University of Iowa. The phrase "the flow-net solution" in the text, when comparing solution results of different approaches, refers to that of the first writer when he constructed such a flow net in the aforementioned course.

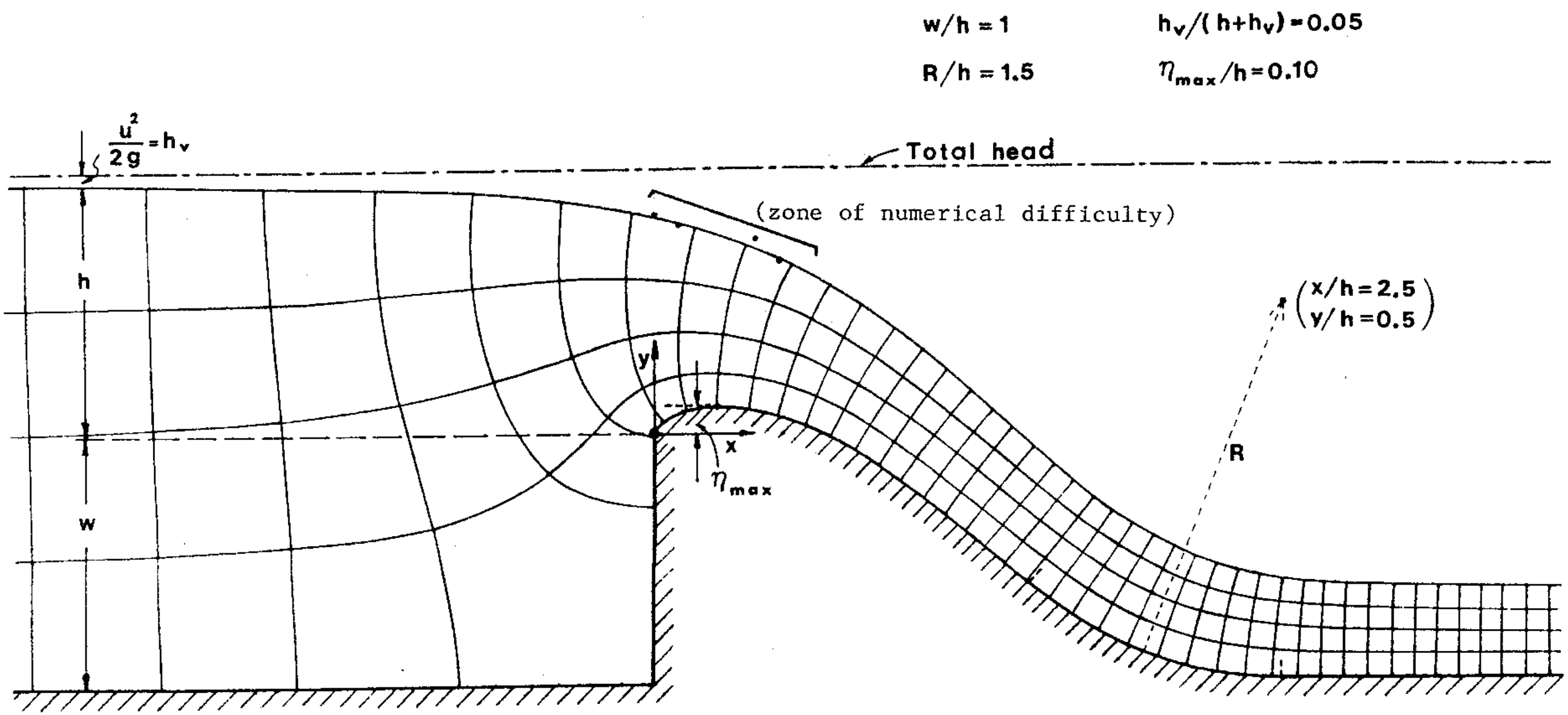


Fig. 2. Ideal-Fluid Flow Over a Low Spillway

evaluated along the closed boundary. The velocity head at each surface nodal point is computed from the corresponding spacings of ϕ-values along the surface. The total head is evaluated from the Bernoulli equation at the surface nodes and compared with the prescribed total head. Adjustments of the surface are then made to reduce the discrepancy between the prescribed and computed total heads at each node to a tolerable level.

Numerical instability has been experienced (also reported by Liggett, 9) in carrying out iteration procedures for achieving a satisfactory water-surface profile. After various trials, the following procedures have been found most satisfactory: (a) find the discrepancy, ε, between the given and the computed total head at each node; (b) set a base surface-correction value ε_1 to $\varepsilon_1 = 0.2\ \varepsilon$ (or $0.25\ \varepsilon$); (c) multiply ε_1 by a factor $f = \Delta s/\Delta s_0$ to obtain the correction value ε_2, i.e. $\varepsilon_2 = f\varepsilon_1$, in which Δs and Δs_0 are ϕ-spacings at the local and at the infinite upstream points, respectively; (d) modify the sign of ε_2 (i.e. the direction in which the surface node of interest should be moved) according to the specific energy curve (e.g., referred to as the E-y curve in Henderson, 4).

It was impossible to eliminate the numerical oscillation completely even with the above procedure. Some points with the largest discrepancies are shown in Fig. 2, together with the surface profile obtained from the flow-net solution. These points all occurred near the critical velocity point (near the spillway crest). The deviation of these nodal points also affected the location and spacing of flow-net lines near these points. Discrepancies at other nodal points are too small to plot in the figure.

Those who previously performed numerical calculation of flow over a spillway reported a zone of poor agreement in their numerical results. For example, Liggett (9) indicated that in the zone immediately upstream from the spillway the computed points did not agree well with the experimental profile. A similar difficulty has been experienced with the CVBEM solution, except in this case the zone of difficulty occurred over the spillway crest where the flow is critical. In addition, the surface profile that was compared with the CVBEM solution was from the classical flow-net solution 2/ instead of an experimental profile.

Because the CVBEM deals directly with the complex potential, both ϕ and ψ values are directly available in the model output; a clear advantage for flow-net construction. Other valuable flow information can, in turn, be provided from the flow net. Algorithm based on the CVBEM are generally regarded as more efficient than those based on a Green's function or real-variable formulation (3).

5. Advective Contaminant Transport

Using superposition principles the basic CVBEM model can be expanded for or be adapted to modeling compound 2-D ground-water flow and transport problems. In other words, by joint use of the CVBEM, analytic functions, and the Poisson equation, different forms of flow such as background flows, sources and sinks, precipitation or seepage, and other flows introduced by the boundary conditions, can be combined together for studies of contaminant transport in ground water.

Expressing the above in an equation form, a potential function

$$F(z) = \hat{\omega}(z) + \sum_{k=1}^{n} \frac{-Q_k}{2\pi} \ln (z-z_k) + \phi_p (z), \quad z\varepsilon\Omega\cup\Gamma \qquad (7)$$

has been developed which satisfies the Laplace equation in domain Ω. (6) Here, Q_k is the discharge from well k (of n) located at z_k [i.e. a sink (-); (+) for a source], ϕ_p is a particular solution for the given Poisson equation, and $\hat{\omega}(z)$ is a CVBEM approximator representing the background flow field. The boundary conditions needed to develop $\hat{\omega}(z)$ can be obtained by subtracting the effects of the source-sink and the particular solution terms (the 2nd and the 3rd term on the right-hand side of Eq. 7) at the boundary from the actual boundary conditions.

Some applications of the aforementioned expanded CVBEM model to ground-water advective contaminant transport are illustrated below. Diffusion-dispersion effects are not considered and the hydraulic conductivity is assumed constant. Each application has the same nodal point placement as shown in Fig. 3.

(a) Figure 3 shows three completely penetrating ground-water wells, each with discharge 50 m^3/hr, located at the coordinates (500,500), (300, -300), and (-500, -500), respectively, in a homogeneous isotropic aquifer of thickness 10 m. Contaminated water is being recharged at an injection well located at the coordinates (-300, 300) with a rate of 50m^3/hr. Negligible background ground-water flow is assumed. Figure 3 depicts the contaminant front at 0.5, 2, and 4 years. It takes 4.32 years for the contaminated water to reach the middle well (300,-300), and about 5.58 years to the other two discharge wells.

(b) An injection well and a discharge well are located at (-300,0) and (300,0), respectively. A differential potential of 2 m is assumed across the western and eastern boundaries, which contributes west-to-east background flow. The potential along the northern and southern boundaries is linearly distributed. Figure 4 portrays the contaminant front at 0.5, 1, 1.5, and 1.98 years. The contaminated water takes about 2 years to reach the discharge well.

(c) Consider a steady flow pattern produced by a single pumping well operating at 50 m^3/hr at (0,0), near a landfill with an equi-potential boundary ϕ = 2m along x = -1000. From the computed results, it takes 8.96 years for the contaminated water to reach the production well. Two additional wells have subsequently been installed at (-500, 250) and (-500, -250) with strength equal to 10 m^3/hr to retard the contaminant front. Figures 5 and 6 describe the front movement and flow (streamline) pattern for these two case studies. The unsymmetrical flow pattern revealed in Fig. 6 is probably due to the relatively coarse nodal- and time-intervals used.

An interesting and useful feature in development of the CVBEM model demonstrated in this problem is its simplicity and flexibility for modular expansion. To the basic CVBEM model, an arbitrary number of sinks or/and sources may be added, or a Poisson equation may be attached. It is also feasible to combine two or more CVBEM solutions--such as a large nonhomogeneous region subdived into a few homogeneous subregions, each with an appropriate CVBEM solution,--by a proper treatment of interface boundary conditions.

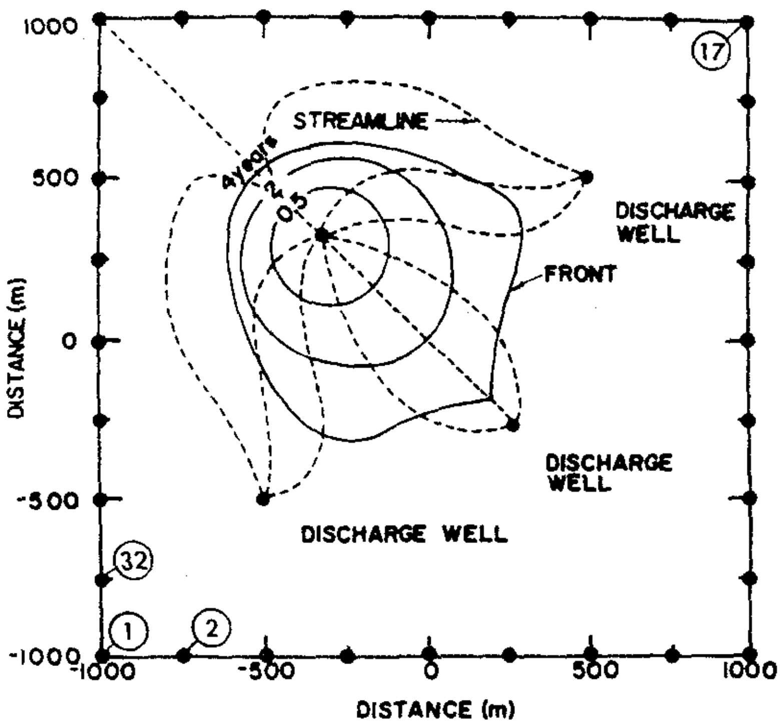


Fig. 3. Contaminant Transport for Example (a)

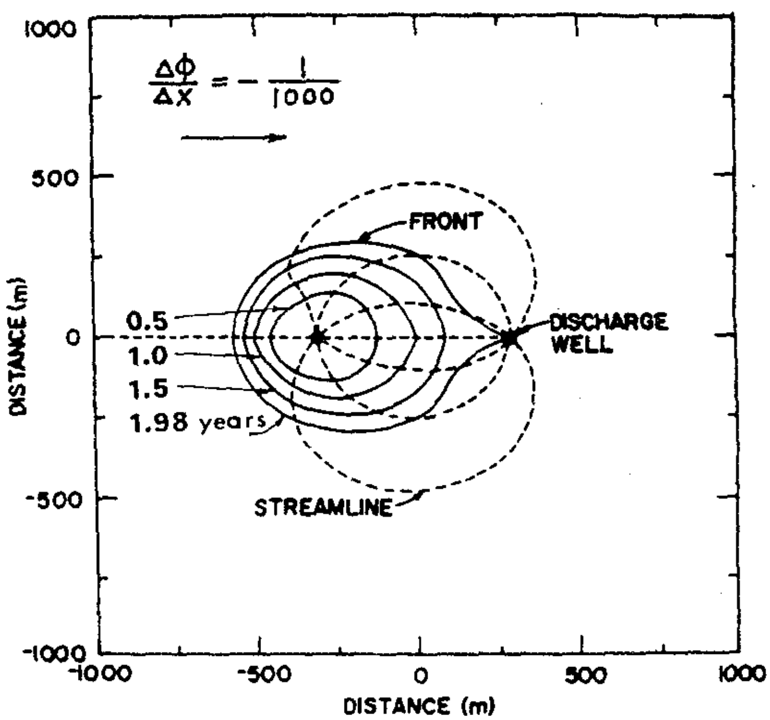


Fig. 4. Contaminant Transport for Example (b)

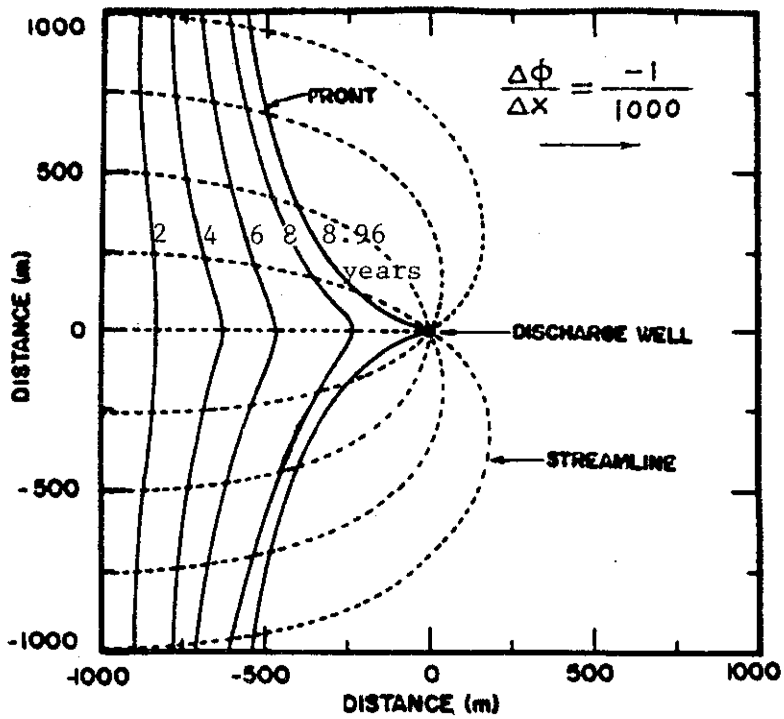


Fig. 5. Contaminant Transport for Example (c), Case 1

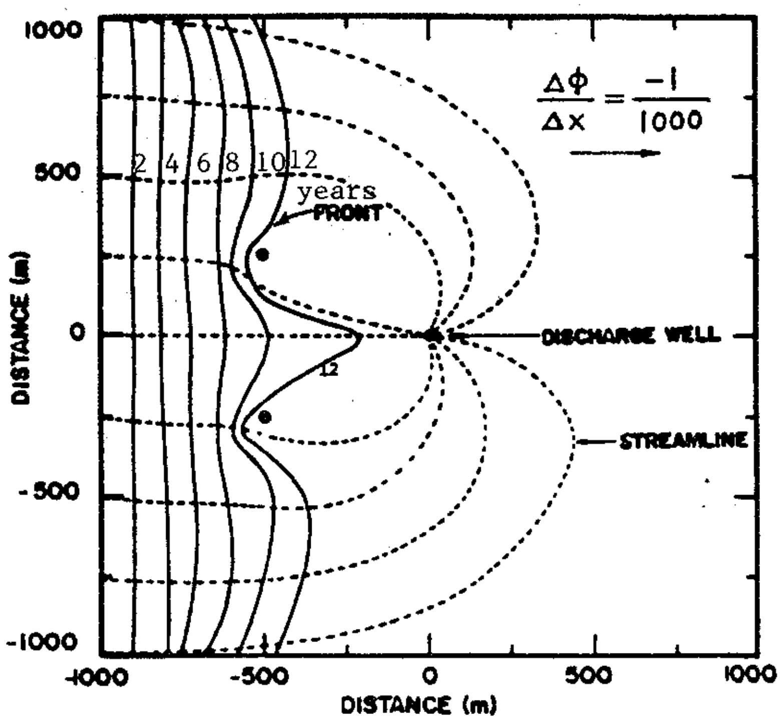


Fig. 6. Contaminant Transport for Example (c), Case 2

Although the Poisson equation was not used for those cases considered above, it can be easily programmed into the model if an appropriate condition and data are given and a suitable particular solution can be found (see Eq. 7).

7. Conclusions

The basic CVBEM (complex-variable boundary element method) model can be extended to simulate relatively complex two-dimensional potential flows in surface and ground water. "Two-dimensional" can be either in horizontal or vertical sense, and "complex flows" may include coupled flows requiring coupled, interactive, or feedback solution, and compound flows which consist of a number of different types of flows but can be solved by superposition. They may involve varied forms of application, such as determination of free surface profile or study of contaminant arrival time. Two illustrative examples given in this paper, jointly include all features mentioned above and serve to exhibit the capability, adaptability, and useful features of the CVBEM.

8. References

1. Brebbia, C.A., (1978), The Boundary Element Method for Engineers, Pentech Press, London.
2. Brevig, P.M., Greenhow, M., and Vinje, T., (1982), "Extreme Wave Forces on Submerged Wave Energy Devices," Applied Ocean Research, Vol. 4, No. 4.
3. Dold, J. W., and Peregrine, D.H., (1984), "Steep Unsteady Water Waves: An Efficient Computational Scheme," School of Mathematics, University of Bristol Internal Rep. AM-16-04.
4. Henderson, F.M., (1966), Open Channel Flow, McMillan Co.
5. Hromadka II, T.V., and Guyman, G.L., (1983), "The Complex Variable Boundary Element Method: Development," Internat'l Jour. Numerical Methods in Engineering, April.
6. Hromadka II, T.V., and Lai, C., The Complex Variable Boundary Element Method for Engineers and Scientists, (1986, Springer Verlag; in process).
7. Hunt, B., and Isaacs, L.T., (1981), "Integral Equation Formulation for Groundwater Flow," Jour. of the Hydraulics Div., ASCE, Vol. 107, No. HY10, pp. 1197-1209.
8. Ikegawa, M. and Washizu, K., (1973), Finite Element Method Applied to Analysis of Flow over a Spillway Crest, Interna'l Jour. for Numerical Methods in Engineering, Vol. 6, pp. 179-189.
9. Liggett, J.A., (1982), "The Boundary Element Method," in Engineering Applications of Computational Hydraulics, (M.B. Abbott and J.A. Cunge, ed.), Chap. 8, Vol. 1, Pitman Publ. Inc., Boston.
10. Liggett, J.A., and Liu, P. L-F., (1983), The Boundary Integral Equation Method for Porous Media Flow, George Allen and Unwin, London.
11. Rouse, H., Ed., (1950), Engineering Hydraulics, John Wiley & Sons, New York.
12. Southwell, R.V., and Vaisey, G., (1946), "Relaxation Methods Applied to Engineering Problems: XIII, Fluid Motions Characterized by Free Streamlines," Phil. Tran., Roy. Soc. London, Sec. A., 240, pp. 117-161.