Best approximation of two-dimensional potential problems using the CVBEM

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The Complex Variable Boundary Element Method or CVBEM provides for the exact solution of the two-dimensional Laplace equation. In this paper, a new approach for developing CVBEM approximation functions is presented. Using the l_2 norm, the CVBEM approximation is developed which is the best approximation in an equivalent vector subspace. Because orthogonal functions are used, matrix solutions are eliminated; thus, considerably reducing the computational effort and memory requirements.

Key Words: analytic functions, boundary element methods, potential problems, least-squares

INTRODUCTION

The CVBEM approximation function for linear (straight-line interpolation) basis functions results in the complex function (Hromadka, 1983, 1984)

$$\hat{\omega}(z) = \sum_{j=1}^{m} c_j(z - z_j)\ln(z - z_j) \tag{1}$$

where the c_j are complex constants $c_j = a_j + ib_j$; z_j are nodal points $(j = 1, 2, \ldots, m)$ defined on the problem boundary Γ (simple closed contour); and $\ln(z - z_j)$ is the principal value complex logarithm function with branch cuts specified to intersect Γ only at z_j. Then $\hat{\omega}(z)$ is analytic over $\Omega \cup \Gamma - \{z_j\}$, and uniformly continuous over $\Omega \cup \Gamma$. Here, Ω is a simply connected domain enclosed by Γ. In fact, $\hat{\omega}(z)$ is analytic over the entire complex plane less the branch cuts. The c_j are calculated in the CVBEM technique by collocating to the boundary condition values known at the nodal points.[1]

In this paper, the c_j are calculated in the L_2 norm sense by finding the best choice of c_j to minimize the mean-square error in matching the boundary condition values continuously along Γ. Notation is used for the known and unknown function values along Γ,

$$\left.\begin{aligned} \omega(\zeta) &= \Delta\xi_k(\zeta) + \Delta\xi_u(\zeta) \\ \hat{\omega}(\zeta) &= \Delta\hat{\xi}_k(\zeta) + \Delta\hat{\xi}_u(\zeta) \end{aligned}\right\} \zeta \in \Gamma \tag{2}$$

where $\omega(z)$ is the solution to the boundary value problem over $\Omega \cup C$; $\hat{\omega}(z)$ is the CVBEM approximation over $\Omega \cup C$; Δ is a descriptor function such that $\Delta = 1, i$ depending whether the associated ξ_x or $\hat{\xi}_x$ function is the real or imaginary term; and ζ is notation for the case of $z \in \Gamma$. Then in this paper the objective is to compute the c_j which, for a given nodal distribution on Γ, minimize

$$I = \| \xi_k - \hat{\xi}_k \|_2^2 = \int_\Gamma (\xi_k - \hat{\xi}_k)^2 \, d\Gamma \tag{3}$$

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ORTHOGONAL CVBEM FUNCTIONS AND THE BEST APPROXIMATION

The CVBEM approximation function of (1) can be written as

$$\hat{\omega}(z) = \sum_{j=1}^{m} c_j f_j \tag{4}$$

where $f_j = (z - z_j)\ln(z - z_j)$. The Gram-Schmidt procedure can be used to orthogonalize the f_j such that

$$\hat{\omega}(z) = \sum_{j=1}^{m} \gamma_j g_j \tag{5}$$

where γ_j are complex constants and

$$(g_j, g_k) = \int_\Gamma g_j g_k \, d\Gamma = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{6}$$

In (6), (g_j, g_k) is notation for the inner-product.

The boundary conditions on Γ are given by ξ_k where $\phi(\zeta)$ is known continuously on contour Γ_ϕ and $\psi(\zeta)$ is

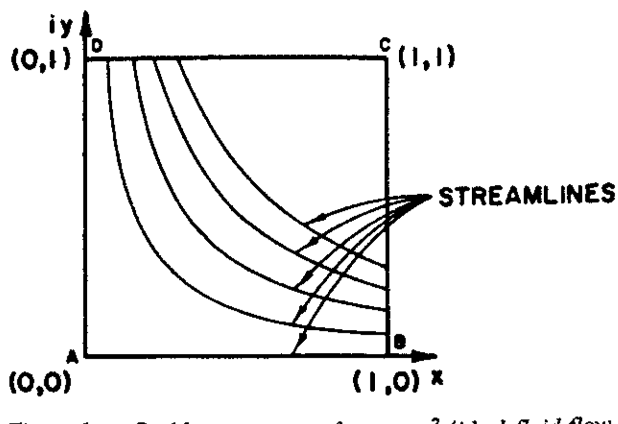

Figure 1a. Problem geometry for $\omega = z^2$ (ideal fluid flow around a corner)

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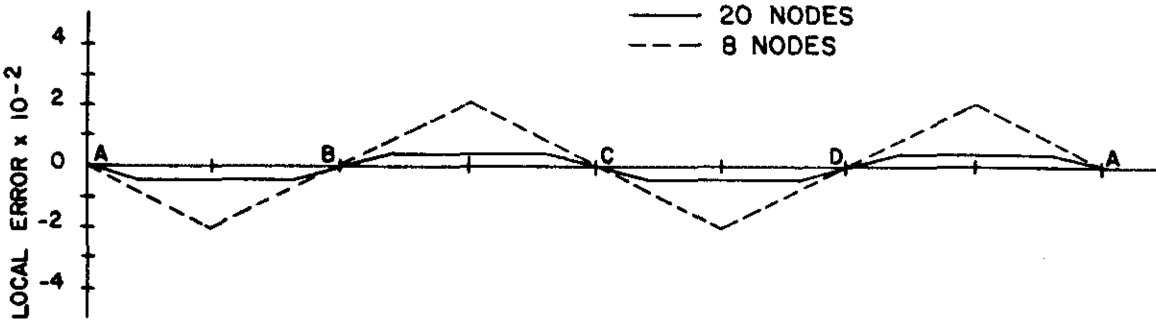

Figure 1b. Local error in matching boundary conditions

known continuously on Γ_ψ where $\Gamma_\phi + \Gamma_\psi = \Gamma$ and $\Gamma_\phi \cap \Gamma_\psi$ only at nodal points. The Γ_ϕ and Γ_ψ can be composed of a finite number of contours. Then the γ_j are computed which minimize

$$I = \int_{\Gamma_\phi} (\phi(\zeta) - \mathrm{Re}\, \Sigma\gamma_j g_j)^2\, d\Gamma + \int_{\Gamma_\psi} (\psi(\zeta) - \mathrm{Im}\, \Sigma\gamma_j g_j)^2\, d\Gamma \tag{7}$$

Because the g_j are orthogonal, the γ_j are directly computed by

$$\gamma_j = (\xi_k, g_j)/(g_j, g_j) \tag{8}$$

Then the best approximation (in the L_2 norm) is given by

$$\hat{\omega}(z) = \sum_{j=1}^{m} (\xi_k, g_j)\, g_j/(g_j, g_j) \tag{9}$$

The c_j are then computed by back-substitution of the $\gamma_j g_j$ functions into the $c_j f_j$ functions. It is noted that by this approach, the c_j are computed directly without the use of a matrix system generation or matrix solution. This is important due to boundary integral methods resulting in the solution of fully populated, square matrix systems.

ORTHOGONAL VECTOR SYSTEMS AND THE BEST APPROXIMATION

Let $\mathbf{F}_j$ be linearly independent vectors of dimension n, for $j = 1, 2, \ldots, m$. Then the Gram-Schmidt procedure can be used to construct orthogonal vectors $\mathbf{G}_j$ of dimension n such that the dot product gives

$$\mathbf{G}_j \cdot \mathbf{G}_k = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{10}$$

Let $\mathbf{B}$ be a vector of dimension n. Then the best approximation of $\mathbf{B}$ in the subspace spanned by the $\mathbf{G}_j$ is given by the vector $\mathbf{A}$ where

$$\mathbf{A} = \sum_{j=1}^{m} \eta_j \mathbf{G}_j \tag{11}$$

where

$$\eta_j = (\mathbf{B} \cdot \mathbf{G}_j)/(\mathbf{G}_j \cdot \mathbf{G}_j) \tag{12}$$

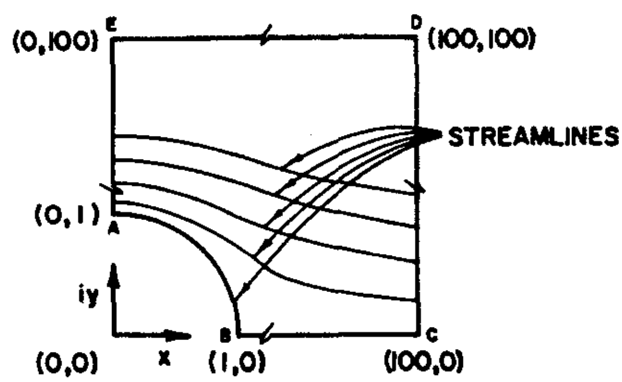

Figure 2a. Problem geometry for $\omega = z + z^{-1}$ (ideal fluid flow over a cylinder)

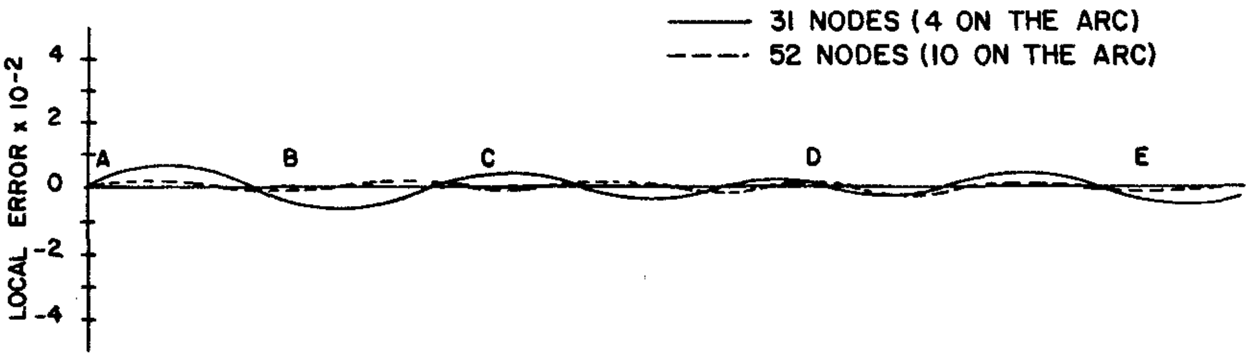

Figure 2b. Local error in matching boundary conditions

The corresponding approximation to **B** with respect to the original $\mathbf{F}_j$ vectors is

$$\mathbf{A} = \sum_{j=1}^{m} \mathbf{C}_j \mathbf{F}_j \tag{13}$$

where the $\mathbf{C}_j$ are computed by back-substitution of $\eta_j \mathbf{G}_j$ into the respective $\mathbf{F}_j$ components.

REPRESENTATION OF THE CVBEM APPROXIMATION FUNCTION BY A DIMENSION *mn* VECTOR SPACE

Let Γ be discretized into m boundary elements Γ_j, $j = 1, 2, \ldots, m$. On each element, define n interior evaluation points (usually evenly spaced), resulting in a total of mn points t_i on Γ. For each function f_j (see equation (4)), develop the vector $\mathbf{F}_j$ of dimension mn by

$$\mathbf{F}_j = \{f_j(t_i);\ i = 1, 2, \ldots, mn\} \tag{14}$$

In (14), the co-ordinates of t_i are consistent for each vector $\mathbf{F}_j$, $j = 1, 2, \ldots, m$, such that points $(t_1, t_2, \ldots, t_n)$ occur in boundary element Γ_1. The resulting vectors $\mathbf{F}_j$ form the basis of a subspace F_{mn} where each vector $\mathbf{F} \in F_{mn}$ is given by

$$\mathbf{F} = \sum_{j=1}^{m} \eta_j \mathbf{F}_j \tag{15}$$

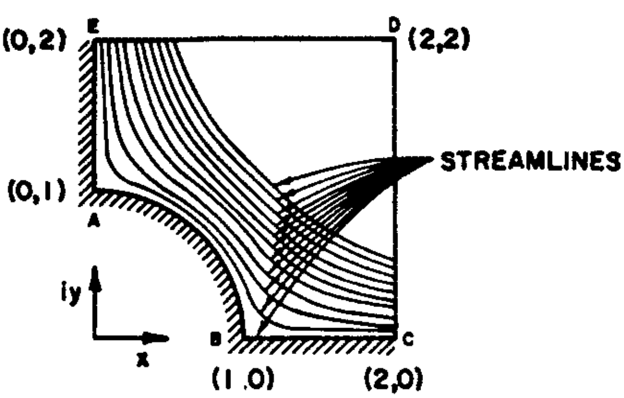

Figure 3a. Problem geometry for $\omega = z^2 + z^{-2}$ (ideal fluid flow around a cylindrical corner)

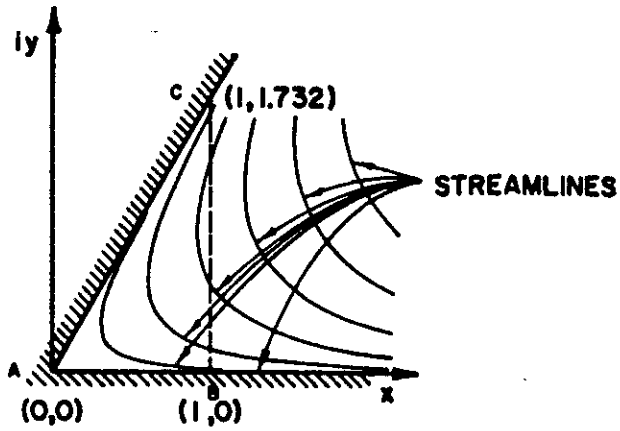

Figure 4a. Problem geometry for $\omega = z^3$ (ideal flow around an angular region)

Similarly the boundry condition values defined on Γ can be represented by the vector **B** where

$$\mathbf{B} = \{\xi_k(t_i);\ i = 1, 2, \ldots, mn\} \tag{16}$$

The best approximation of the vector **B** (in the l_2 norm analogy of the L_2 norm) by a vector $\mathbf{A} \in F_{mn}$ is given directly by (11) and (12). The corresponding estimate of the best approximation $\hat{\omega}(z)$ is given by

$$\hat{\omega}(z) = \sum_{j=1}^{m} \eta_j g_j \tag{17}$$

Thus in the above, the best approximation for $\hat{\omega}(z)$ is estimated by using the best approximation from a vector space spanned by the vectors $\mathbf{G}_j$. Appendix A of this paper presents an example approximation problem which demonstrates the above discussions.

IMPLEMENTATION

A FORTRAN computer program was prepared which developed the best approximation in a vector space (of dimension mn) in order to estimate the c_j coefficients of equation (1). The basic steps used in the program are as follows:

1. Data entry of nodal point (m) co-ordinates and boundary values

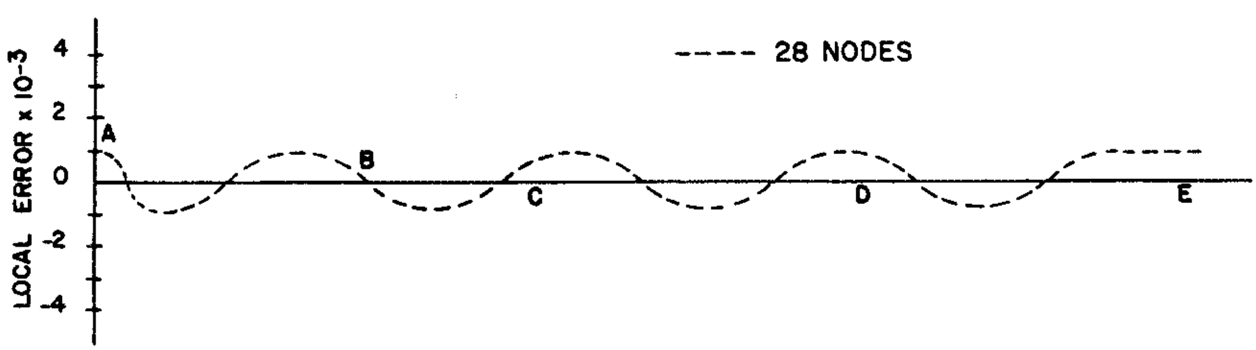

Figure 3b. Local error in matching boundary conditions

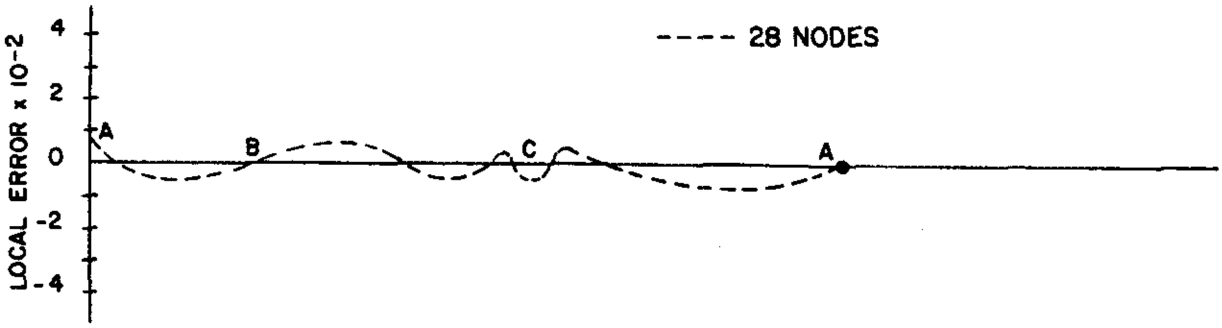

Figure 4b. Local error in matching boundary conditions

2. Number of evaluation points entered (n)
3. Develop dimension mn vectors $\mathbf{F}_j, j = 1, 2, \ldots, m$
4. Develop dimension mn vector $\mathbf{B}$ of boundary values
5. Develop orthogonal vectors $\mathbf{G}_j, j = 1, 2, \ldots, m$
6. Compute vector coefficients $\boldsymbol{\eta}_j$
7. Back substitute $\mathbf{G}_j$ vectors into $\mathbf{F}_j$ vectors and compute the coefficients $\mathbf{C}_j; j = 1, 2, \ldots, m$
8. Define $c_j = \mathbf{C}_j$ to determine the CVBEM approximation function, $\hat{\omega}(z)$.

It is noted that the $c_j = \alpha_j + i\beta_j$. Thus the above program steps involve two vectors for each $\mathbf{C}_j$.

That is from (1),

$$\hat{\omega}(z) = \sum_{j=1}^{m} \alpha_j [(z - z_j) \ln(z - z_j)] + \sum_{j=1}^{m} \beta_j [i(z - z_j) \ln(z - z_j)] \quad (18)$$

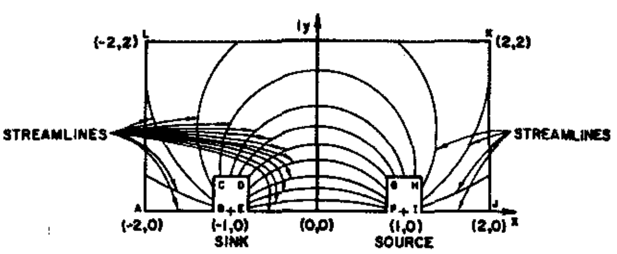

Figure 5a. Problem geometry for $\omega = \ln(z + 1/z - 1)$ (heat source and sink of equal strength)

Hence the f_j vectors corresponding to the c_j have two separate components which are used, respectively, with the α_j and β_j. Consequently, for m nodes there are $2m$ coefficients to be computed.

APPLICATIONS

In the following, figures are provided which plot flow nets for heat transport and groundwater flow problems. The error in matching the boundary condition values are also included. Because $\hat{\omega}(z)$ is analytic over Ω, the Laplace equation is solved exactly over Ω. Hence, the maximum error of approximation must occur on the boundary, Γ.

CONCLUSIONS

In this paper, a new approach for developing CVBEM approximation functions is presented. Using the l_2 norm, the CVBEM approximation is developed which is the best approximation in an equivalent vector subspace. Because orthogonal functions are used, matrix solutions are eliminated; considerably reducing the computational effort and memory requirements.

APPENDIX A

In this appendix, an example problem is worked to illustrate the procedure outlined in the main text of the paper. Let $\xi_k(\zeta) = \sin 30\zeta$ for $0 \leqslant \zeta \leqslant 3$, where ζ is a co-ordinate along Γ. Let $f_1 = \zeta$, and $f_2 = \zeta^3$ be approximation functions with $w = c_1 f_1 + c_2 f_2 = c_1 \zeta + c_2 \zeta^3$. For evaluation points, use $t_i = (0, 1, 2, 3)$. Hence vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are given by

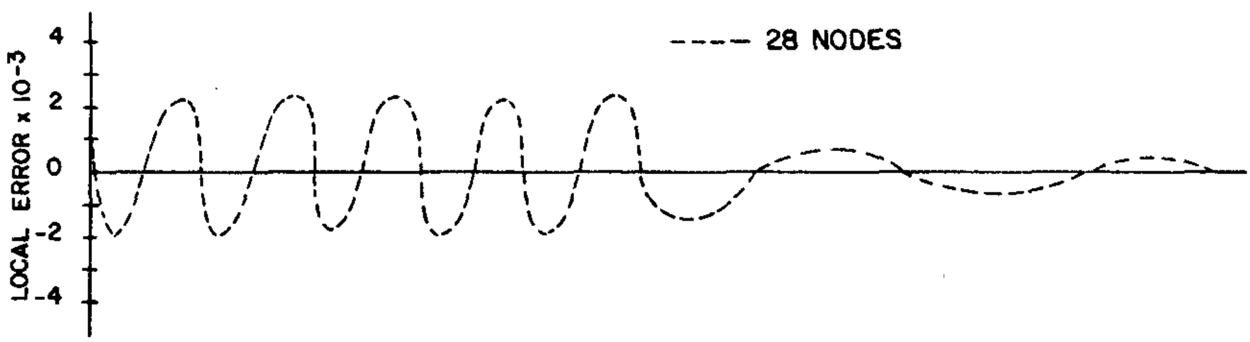

Figure 5b. Local error in matching boundary conditions

$\mathbf{F}_1 = (0, 1, 2, 3)$, $\mathbf{F}_2 = (0, 1, 8, 27)$. Similarly, $\mathbf{B} = (0, 0.5, 0.866, 1)$. Orthogonal vectors $\mathbf{G}_1$ and $\mathbf{G}_2$ are developed by $\mathbf{G}_1 = \mathbf{F}_1$, and $\mathbf{G}_2 = \mathbf{F}_2 - (\mathbf{F}_2 \cdot \mathbf{G}_1)\mathbf{G}_1/(\mathbf{G}_1 \cdot \mathbf{G}_1) = \mathbf{F}_2 - 7\mathbf{G}_1 = (0, -6, -6, 6)$. It is readily seen that $\mathbf{G}_1 \cdot \mathbf{G}_2 = 0$ and are therefore orthogonal vectors. From (12), $\eta_1 = (\mathbf{B} \cdot \mathbf{G}_1)/(\mathbf{G}_1 \cdot \mathbf{G}) = 0.373714$. Similarly, $\eta_2 = (\mathbf{B} \cdot \mathbf{G}_2)/(\mathbf{G}_2 \cdot \mathbf{G}_2) = -0.020333$. Thus the best approximation $\mathbf{A}$ is given by $\mathbf{A} = \eta_1 \mathbf{G}_1 + \eta_2 \mathbf{G}_2 = 0.373714\mathbf{G}_1 - 0.020333\mathbf{G}_2$. Back substituting the $\mathbf{G}_j$ into $\mathbf{F}_j$ vectors, $\mathbf{A} = \mathbf{C}_1\mathbf{F}_1 + \mathbf{C}_2\mathbf{F}_2 = 0.516045\mathbf{F}_1 - 0.020333\mathbf{F}_2$. Letting the $c_j = \mathbf{C}_j$, the approximation for $\xi_k(\zeta)$ is given by $\sin 30\zeta \sim 0.516045\zeta - 0.020333\zeta^3$. In comparison the vector (0, 0.5, 0.866, 1) is approximated as (0, 0, 495712, 0.869426, 0.999144). Similarly, $\| \sin 30\zeta - \Sigma c_j f_j \|_\infty \simeq 0.006$.

REFERENCES

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