Convergence properties of the CVBEM: Development

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The Complex Variable Boundary Element Method or CVBEM is studied with respect to development of convergence properties. Using conformal mapping of the problem domain to the unit circle, convergence of the CVBEM is examined for Dirichlet and mixed boundary value problems. Convergence is examined with respect to both error bounds of the analytic functions involved, and with respect to the matrix systems developed for the CVBEM approximation technique.

Key Words: convergence, boundary value problems.

INTRODUCTION

The Complex Variable Boundary Element Method or CVBEM has been shown to be a useful numerical approach for the solution of two-dimensional potential problems.[1] In this paper, the CVBEM will be studied as to its convergence properties for both Dirichlet and mixed boundary value problems. These considerations have not been given rigorous attention elsewhere in the literature, and it is the main objective of this paper to remedy this need.

Because any analytic function can be recast by conformal mapping into an equivalent function on the unit circle (where known solutions exist), a rigorous convergence analysis can be developed with the use of only the well-known Poisson formula, and employment of the often-used $\| \|_\infty$ norms for the CVBEM matrices and vectors involved in the numerical technique.

Details of the CVBEM numerical approach are thoroughly presented in the cited reference, but are also briefly contained in the Appendix of this paper for the reader's convenience.

SOME UNIT CIRCLE GEOMETRIC PROPERTIES

Before investigating the convergence properties of the CVBEM on the unit circle, some preliminary results regarding the unit circle geometric relationships are useful.

Theorem A

Let the unit circle have m evenly spaced nodal points z_j such as shown in Fig. 1. In the figure, constant boundary elements are used where collocation points are defined at mid-element. Let collocation point $\bar{z}_1$ have co-ordinates $\bar{z}_1 = 1 + 0i$. Then the central angles θ between line segments $(z_{j+1}, \bar{z}_1)$ and $(z_j, \bar{z}_1)$ are all equal to π/m.

Proof

From Fig. 2, the circle is subdivided into m sectors with central angles $\alpha = 2\pi/m$. Let $d_3 = |z_3 - \bar{z}_1|$. Then the isosceles triangles of points $(z_3, z_0, \bar{z}_1)$ and $(z_2, z_0, \bar{z}_1)$ imply $\theta = \alpha/2 = \pi/m$.

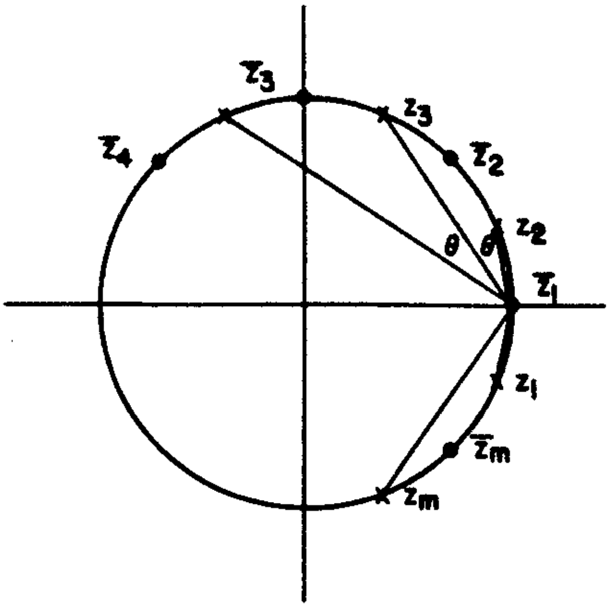

Figure 1

Theorem B

Consider the unit circle C with point $\bar{z}_1$ (see Fig. 3) defined at $1 + 0i$ and z an arbitrary point on C such that $0 \leqslant \theta \leqslant \pi$. Then the distance $d = |z - \bar{z}_1|$ satisfies $d \geqslant 2\theta/\pi$ for $0 \leqslant \theta \leqslant \pi$.

Proof

For the unit circle, $d^2 = (\cos\theta - 1)^2 + \sin^2\theta$ for $0 \leqslant \theta \leqslant \pi$. Let $f = (\cos\theta - 1)^2 + \sin^2\theta - (4/\pi^2)\theta^2$ then $f \geqslant 0$ for $0 \leqslant \theta \leqslant \pi$, with $f = 0$ at $\theta = 0$ and π.

Theorem C

Let the unit circle C have m evenly placed points such as shown in Fig. 4. Let point z' approach point $\bar{z}_1$ as shown. Then

$$\lim_{z' \to \bar{z}_1} \theta = \pi(m + 1)/m$$

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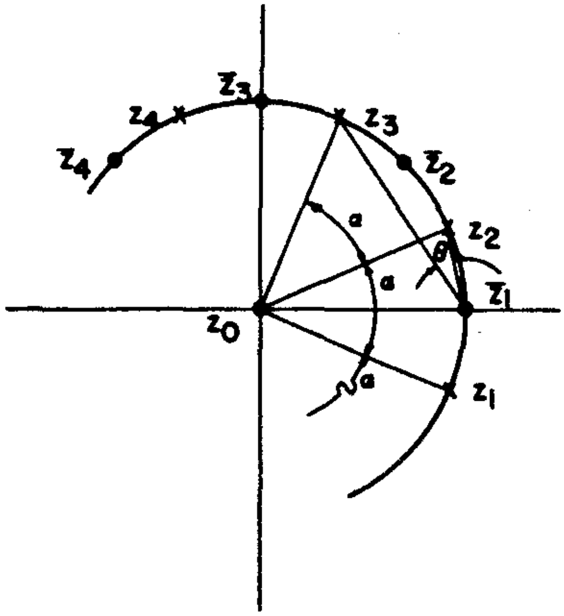


Figure 2

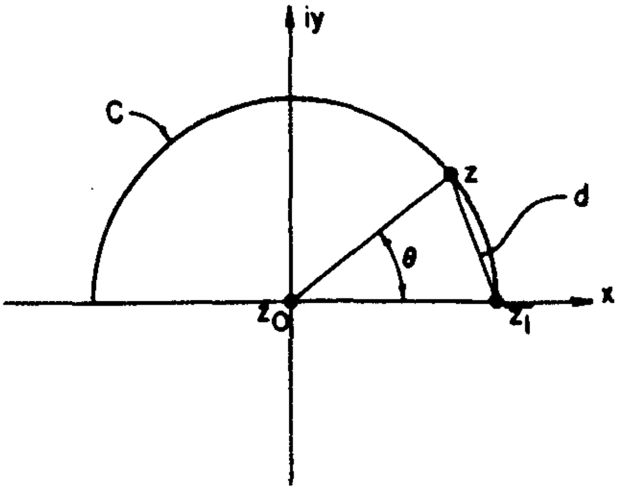


Figure 3

Proof

The m nodes on C result in $(m-1)$ sectors which extend from point $\bar z_1$ to each of the nodes z_2 through z_1 (counterclockwise direction). Thus the figure angle θ is calculated as

$$\theta = 2\pi - (m-1)\left(\frac{\pi}{m}\right) = \pi(m+1)/m$$

where each of the $(m-1)$ sectors has a central angle of π/m from Theorem A.

CONVERGENCE OF THE CVBEM FOR DIRICHLET PROBLEMS

In the following, the CVBEM is studied as to convergence of the approximation function $\hat\omega(z)$ to the exact solution of the boundary value problem $\omega(z)$. Because the unit circle is employed, $\omega(z)$ is given by the well-known Poisson formula. Theorem D considers the convergence of $\hat\omega(z)\to\omega(z)$ in the maximum norm sense of $|\omega(z)-\hat\omega(z)|\to 0$. The Discussion following Theorem E considers $\hat\omega(z)\to\omega(z)$ as the element size diminishes uniformly to zero, using norms of the matrices and vectors used in the CVBEM solution approach. It is noted that this Discussion applies to both the Dirichlet and mixed boundary value problems.

Theorem D

Let $C=\{z:|z|=1\}$ and $\Omega=\{z:|z|<1\}$. Discretize C into boundary elements using the nodal placement shown in Fig. 4. Let $\omega=\phi+i\psi$ be analytic over Ω and continuous over C. Let $\{\hat\omega_n(z)\}$ be a sequence of CVBEM approximation functions such that each $\hat\omega_n(z)$ is analytic over Ω and continuous over C. Let ϕ be known on C (i.e. Dirichlet problem). Define $e_n(z)=\omega(z)-\hat\omega_n(z)=e\phi_n+ie\psi_n$. For each boundary element $[z_j,z_{j+1}]$, let $e'\phi_n$ be bounded by $|e'\phi_n|\leqslant M_1$ for $\zeta\in(z_j,z_{j+1})$. Then $|e\phi_n|\to 0$ on $C\to|\omega_n(z)-\omega(z)|\to 0$ over $\Omega\cup C$.

Proof

From the Poisson formula, $e\phi_n$ known on C determines $e\psi_n$ to within a constant by

$$e\psi_n = -\frac{1}{2\pi}\int_{-\pi}^{\pi} e\phi_n W(\theta)\,\mathrm{d}\theta$$

where

$$W(\theta)=\frac{\sin\theta}{1-\cos\theta}$$

(see Fig. 5). Thus a bound for $|e\psi_n|$ is determined by

$$|e\psi_n|\leqslant\frac{2}{2\pi}\int_{\delta}^{\pi}\epsilon W(\theta)\,\mathrm{d}\theta+\frac{2}{2\pi}\int_{0}^{\delta}M_1\theta W(\theta)\,\mathrm{d}\theta$$

where $\delta=\epsilon/M_1$ and $|e\phi_n|\leqslant\epsilon$. Solving,

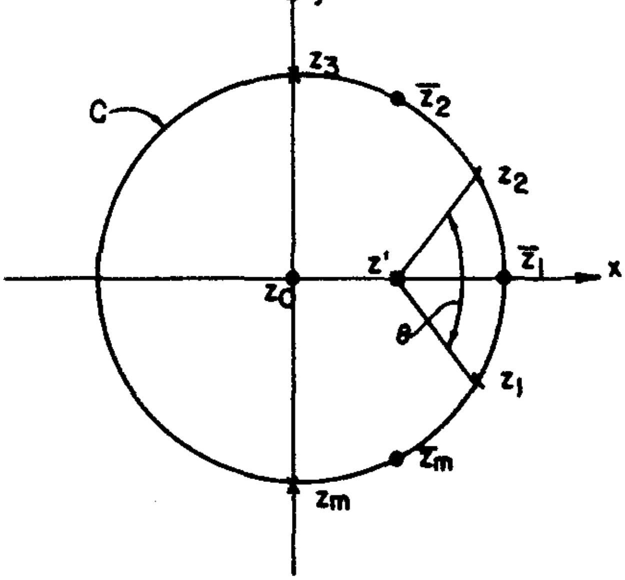


Figure 4

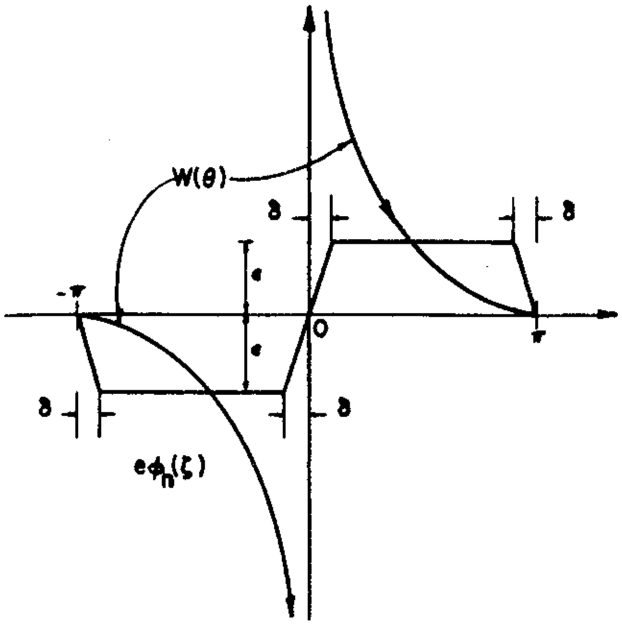


Figure 5

$$\frac{1}{\pi}\int_{\delta}^{\pi} \epsilon W(\theta)\,\mathrm{d}\theta = \frac{\epsilon}{\pi}\int_{\delta}^{\pi}\frac{\mathrm{d}(1-\cos\theta)}{1-\cos\theta} = \frac{\epsilon}{\pi}\ln(1-\cos\theta)\Big|_{\delta}^{\pi}$$

$$= \frac{\epsilon}{\pi}\,[\ln 2 - \ln(1-\cos\delta)]$$

For δ small, $(1-\cos\delta)\sim\delta^2/2$ and

$$\frac{\epsilon}{\pi}\,[\ln 2 - \ln(1-\cos\delta)] \sim \frac{2\epsilon}{\pi}\,[1-\ln\epsilon+\ln M_1]$$

The other integral is solved for δ small by

$$\frac{M_1}{\pi}\int_0^{\delta}\theta W(\theta)\,\mathrm{d}\theta \sim \frac{M_1}{\pi}\int_0^{\delta}\frac{\theta^2\,\mathrm{d}\theta}{\theta^2/2} = \frac{2M_1}{\pi}\delta = \frac{2\epsilon}{\pi}$$

Thus

$$|e\psi_n| \leqslant \frac{2\epsilon}{\pi}\,[2-\ln\epsilon+\ln M_1]$$

and for $\epsilon = \max|e\phi_n|$ on C, $\epsilon\to 0 \Rightarrow |e\psi_n|\to 0$. Thus,

$$|\omega(z)-\omega_n(z)| = |e\phi_n + ie\psi_n| \leqslant |e\phi_n| + |e\psi_n|$$

$$\leqslant \epsilon + \frac{2\epsilon}{\pi}\,[2-\ln\epsilon+\ln M_1]$$

and

$$\epsilon\to 0 \Rightarrow |\omega(z)-\omega_n(z)|\to 0$$

Theorem E

Let $\omega(z)$ be analytic over $\Omega:\{z:|z|<1\}$ and continuous over $C:\{z:|z|=1\}$. Define boundaries C and C^- by $C=e^{i\theta}$, $C^-=Re^{i\theta}$ for $0\leqslant\theta<2\pi$, $0<R<1$. Then for $|z^-|=R^-<R$,

$$\int_C \frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} = \lim_{R\to 1}\int_{C^-}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} = 2\pi i\,\omega(z^-)$$

Proof

By assumption, $\omega(z)$ is analytic over C^-. Then for $|z^-|=R^-<R$, the Cauchy formula gives

$$2\pi i\,\omega(z^-) = \int_{C^-}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-}$$

Consider

$$I = \left|\int \frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} - \int_{C^-}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-}\right|$$

$$= \left| i\int_{\theta=0}^{2\pi}\left[\frac{\omega(e^{i\theta})\,e^{i\theta}\,\mathrm{d}\theta}{e^{i\theta}-z^-} - \frac{\omega(Re^{i\theta})\,Re^{i\theta}\,\mathrm{d}\theta}{Re^{i\theta}-z^-}\right]\right|$$

$$= \left| i\int_0^{2\pi}\frac{(\omega(e^{i\theta})-\omega(Re^{i\theta}))\,Re^{i2\theta} + z^-e^{i\theta}(R\omega(Re^{i\theta})-\omega(e^{i\theta}))}{(e^{i\theta}-z^-)\,(Re^{i\theta}-z^-)}\Bigg]\mathrm{d}\theta\right|$$

$$\leqslant \int_0^{2\pi}\frac{(R\,|\omega(e^{i\theta})-\omega(Re^{i\theta})||e^{i2\theta}| + |z^-||e^{i\theta}||R| \times|\omega(Re^{i\theta})-\omega(e^{i\theta})| + |\omega(e^{i\theta})||R-1||z^-||e^{i\theta}|)}{|e^{i\theta}-z^-||Re^{i\theta}-z^-|}\,|\mathrm{d}\theta|$$

$\omega(z)$ is uniformly continuous over $\Omega\cup C$. Hence for every $\epsilon>0$ there exists an $R_\epsilon<1$ such that $|\omega(e^{i\theta})-\omega(Re^{i\theta})|<\epsilon$ for $R_\epsilon<R<1$. For $|\omega(z)|$ uniformly bounded on $\Omega\cup C$ by M,

$$I\leqslant\int_0^{2\pi}\frac{[R\epsilon + R^-R\epsilon + R^-M(1-R)]\,\mathrm{d}\theta}{(1-R^-)\,(R-R^-)}$$

$$\leqslant\int_0^{2\pi}\frac{[2\epsilon + M(1-R)]\,\mathrm{d}\theta}{(R-R^-)^2}$$

$$\leqslant 2\pi\,[2\epsilon + M(1-R)]/(R-R^-)^2$$

Then

$$\lim_{R\to 1} I \leqslant 4\pi\epsilon/(1-R^-)^2$$

Thus for every $\epsilon>0$,

$$\lim_{R\to 1}\left|\int_C\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} - 2\pi i\,\omega(z^-)\right| < \epsilon$$

Hence,

$$\lim_{R\to 1}\int_{C^-}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} = \int_C\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z^-} = 2\pi i\,\omega(z^-)$$

DISCUSSION ON THE CONVERGENCE PROPERTIES OF THE CVBEM MATRIX SYSTEM

Let $\omega(z)$ be analytic in $\Omega:\{z:|z|<1\}$ and continuous on $C:\{z:|z|=1\}$. Define m equilength boundary elements C_j (arcs) with nodal points located on the endpoints of C_j. Define a global trial function $G_m(\zeta)$ on C by

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\,\omega(z_j)$$

where

$$N_j(\zeta) = \begin{cases} (\zeta - z_{j-1})/(z_j - z_{j-1}), & \zeta \in C_{j-1} \\ (z_{j+1} - \zeta)/(z_{j+1} - z_j), & \zeta \in C_j \\ 0, \text{ otherwise} \end{cases}$$

For $\zeta \in C$, define $f_m(\zeta)$ by

$$\omega(\zeta) = G_m(\zeta) + f_m(\zeta)$$

where necessarily $f_m(\zeta)$ is continuous on C and $f_m(z_j) = 0$, $j = 1, 2, \ldots, m$. For every m, let $|f_m''(\zeta)|$ be uniformly bounded on the interior of each C_j by $|f_m''(\zeta)| \leqslant M_2$, $\zeta \in C_j - \{z_j, z_{j+1}\}$ (i.e. $|\omega''(\zeta)| \leqslant M_2$ for $\zeta \in C_j - \{z_j, z_{j+1}\}$). Define the m nodes on C by $z_j = e^{i\theta j} = e^{i2\pi(j-1)/m}$, $j = 1, 2, \ldots, m$ and interior points z_j^- by $z_j^- = Re^{i\theta j}, R < 1$. For $\omega(z)$ uniformly continuous over $\Omega \cup C$,

$$\lim_{z_k^- \to z_k} \omega(z_k^-) = \omega(z_k)$$

where from the previous theorem,

$$\begin{aligned} \omega(z_k^-) &= \frac{1}{2\pi i} \int_C \frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \\ &= \frac{1}{2\pi i} \int_C \frac{[G_m(\zeta) + f_m(\zeta)]\,\mathrm{d}\zeta}{\zeta - z_k^-} \\ &= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{C_j} \frac{G_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} + \frac{1}{2\pi i} \int_C \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \end{aligned}$$

Hence for nodal co-ordinate $z_k \in C$,

$$\begin{aligned} \omega(z_k) &= \lim_{z_k^- \to z_k} \omega(z_k^-) \\ &= \lim_{z_k^- \to z_k} \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{C_j} \frac{G_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} + \lim_{z_k^- \to z_k} \frac{1}{2\pi i} \int_C \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \end{aligned}$$

For each element C_j, $|f_m(s)| \leqslant M_2(ls - s^2)/2$ for $0 \leqslant s \leqslant l$, where s is a linear local co-ordinate system defined on C_j, and $l = 2\pi/m$. Then for $j \neq k$ nor $(k-1)$,

$$\lim_{z_k^- \to z_k} \left| \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \right| \leqslant \frac{1}{2\pi} \int_0^l \frac{M_2(ls - s^2)\,\mathrm{d}s}{2d_j}$$

where $d_j = \min|\zeta - z_k|, \zeta \in C_j$. Then

$$\lim_{z_k^- \to z_k} \left| \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \right| \leqslant \frac{M_2 l^3}{24\pi d_j}, \quad j = k, (k-1)$$

Summing terms for $j = 2$ through $(m-1)$ gives

$$\begin{aligned} \lim_{z_k^- \to z_k} \sum_{j=2}^{m-1} \left| \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \right| &\leqslant \frac{M_2 l^3}{24\pi} \sum_{j=2}^{(m-1)} \frac{1}{d_j} \\ &\leqslant \frac{2M_2 l^3}{48} \sum_{j=2}^{(m+1)/2} \frac{1}{\theta_j} \end{aligned}$$

where $d_j \geqslant 2\theta_j/\pi$, for $0 \leqslant \theta_j \leqslant \pi$. Noting that the θ_j differ in value sequentially by $2\pi/m$,

$$\begin{aligned} \lim_{z_k^- \to z_k} \sum_{j=2}^{(m-1)} \left| \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \right| &< \left(\frac{M_2 l^3}{24} \right) \left(\frac{m}{2\pi} \right) \\ &\times \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{2}{(m+1)} \right) = M_2 \theta \left(\frac{\ln m}{m^2} \right) \end{aligned}$$

Similarly for C_{k-1} and C_k, and for $2|\zeta - z_k^-| \geqslant |\zeta - z_k| = s$ (local co-ordinate):

$$\begin{aligned} \lim_{z_k^- \to z_k} \left| \frac{1}{2\pi i} \int_{C_{k-1} + C_k} \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \right| &\leqslant \frac{1}{2\pi} \int_0^l \frac{M_2(ls - s^2)\,\mathrm{d}s}{(s/2)} \\ &= \frac{l^2 M_2}{2\pi} = M_2 \theta \left(\frac{1}{m^2} \right) \end{aligned}$$

where again $l = 2\pi/m$. Hence for any node $z_k \in C$,

$$F_k^- \equiv \frac{1}{2\pi i} \int_C \frac{f_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-}, \quad F_k \equiv \lim_{z_k^- \to z_k} F_k^-$$

and

$$\| F_k \| = M_2 \theta \left(\frac{\ln m}{m^2} \right)$$

Now consider

$$\begin{aligned} &\lim_{z_k^- \to z_k} \frac{1}{2\pi i} \int_C \frac{G_m(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \\ &= \lim_{z_k^- \to z_k} \frac{1}{2\pi i} \sum_{j=1}^{m} \int_C \frac{N_j(\zeta)\,\omega(z_j)\,\mathrm{d}\zeta}{\zeta - z_k^-} \\ &= \sum_{j=1}^{m} \omega(z_j) \lim_{z_k^- \to z_k} \frac{1}{2\pi i} \int_C \frac{N_j(\zeta)\,\mathrm{d}\zeta}{\zeta - z_k^-} \\ &\equiv \sum_{j=1}^{m} \omega(z_j)\,\eta_{jk} \end{aligned}$$

where the η_{jk} are complex constants $\eta_{jk} = \alpha_{jk} + i\beta_{jk}$.

Combining the above results, for any node $z_k \in C$, the uniform continuity of $\omega(z)$ over $\Omega \cup C$ gives:

$$\omega(z_k) = \lim_{z_k^- \to z_k} \omega(z_k^-) = \sum_{j=1}^{m} \eta_{jk}\omega(z_j) + F_k$$

In engineering problems, usually only one of the nodal values ϕ_j or ψ_j is known for each node j. The CVBEM develops estimates for the unknown nodal values (designated in vector notation by $\boldsymbol{\xi}_u$) as a function of the known nodal values (designated by $\boldsymbol{\xi}_k$) by setting the unknowns equal to the appropriate real or imaginary part of the CVBEM approximation function, $\hat{\omega}(z)$. That is if ϕ_j is unknown, the CVBEM solves for

$$\hat{\phi}_k = Re\,\hat{\omega}(z_k)$$

and for ψ_j unknown,

$$\hat{\psi}_k = Im\, \hat{\omega}(z_k)$$

where for $z_k \in C$,

$$\hat{\omega}(z_k) = \lim_{z_k^- \to z_k} \hat{\omega}(z_k^-)$$

Then letting $\hat{\omega}(z_j) = \hat{\phi}_j + i\hat{\psi}_j$, the CVBEM solves for an unknown $\hat{\phi}_k$ by

$$\hat{\phi}_k = \sum_{j=1}^{m} (\alpha_{jk}\hat{\phi}_j + \beta_{jk}\hat{\psi}_j)$$

or for an unknown $\hat{\psi}_k$ value using

$$\hat{\psi}_k = \sum_{j=1}^{m} (\alpha_{jk}\hat{\psi}_j + \beta_{jk}\hat{\phi}_j)$$

This solution procedure is written in matrix form by

$$\hat{\boldsymbol{\xi}}_u = \mathbf{N}_u\hat{\boldsymbol{\xi}}_u + \mathbf{N}_k\boldsymbol{\xi}_k$$

where $\mathbf{N}_u$ and $\mathbf{N}_k$ are matrices composed of the above α_{jk} or β_{jk} coefficients which correspond to the unknown and known nodal values, respectively. (It is noted that $\boldsymbol{\xi}_k$ does not have the hat designation due to $\boldsymbol{\xi}_k$ being known values and the basis functions $N_j(\zeta)$ being exact for the $\boldsymbol{\xi}_k$. Additionally, integration contributions computed by the CVBEM from the known boundary conditions are assumed to be exact due to a proper choice of the basis functions.)

Similar to the CVBEM matrix system, $\omega(z_k)$ values can be written as $\omega(z_k) = \phi_k + i\psi_k$ where

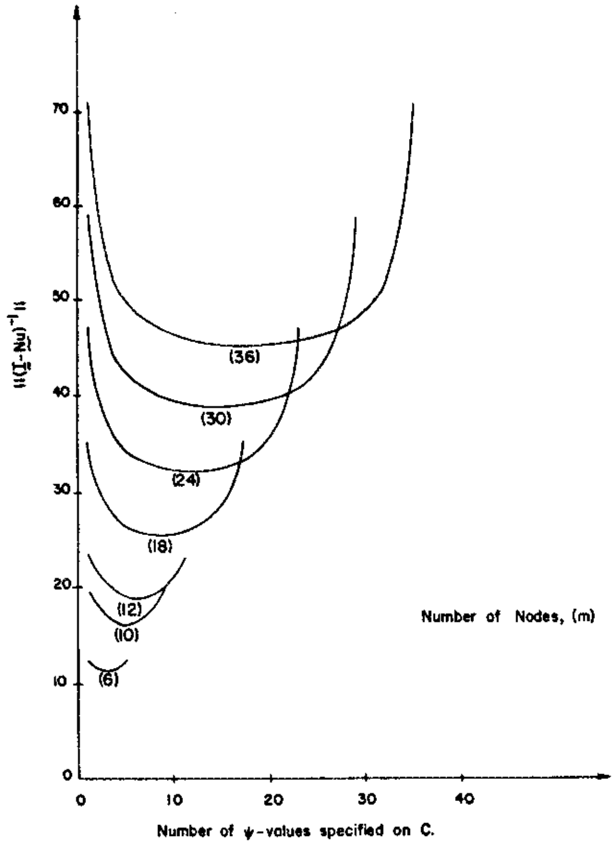
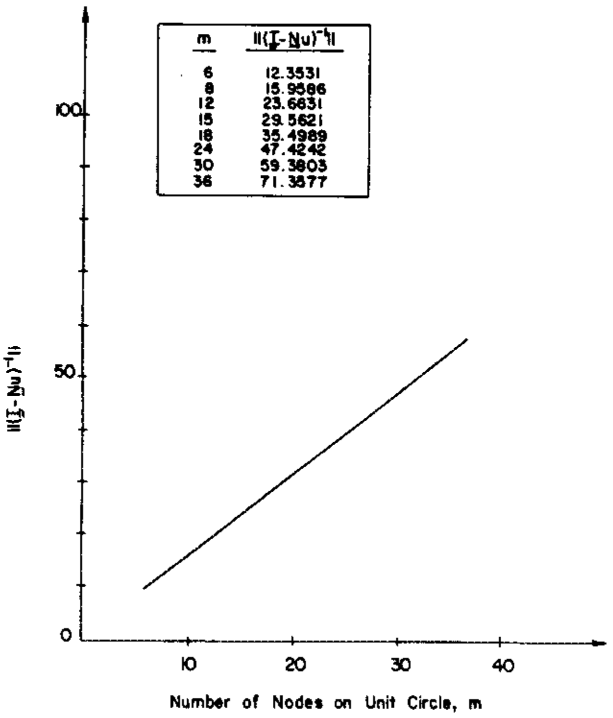


Figure 6



m	$\|(\mathbf{I}-\mathbf{N}_u)^{-1}\|$
6	12.3531
8	15.9586
12	23.6631
15	29.5621
18	35.4989
24	47.4242
30	59.3803
36	71.3877

Figure 7

$$\phi_k = \lim_{z_k^- \to z_k} Re \frac{1}{2\pi i} \int_C \frac{G_m(\zeta)\, d\zeta}{\zeta - z_k^-} + Re \lim_{z_k^- \to z_k} F_k^-$$

$$\psi_k = \lim_{z_k^- \to z_k} Im \frac{1}{2\pi i} \int_C \frac{G_m(\zeta)\, d\zeta}{\zeta - z_k^-} + Im \lim_{z_k^- \to z_k} F_k^-$$

In comparison to the CVBEM matrix system,

$$\boldsymbol{\xi}_u = \mathbf{N}_u\boldsymbol{\xi}_u + \mathbf{N}_k\boldsymbol{\xi}_k + F_k$$

Thus the error of approximation is given by

$$(\boldsymbol{\xi}_u - \hat{\boldsymbol{\xi}}_u) = (\mathbf{I} - \mathbf{N}_u)^{-1}\mathbf{F}_k$$

Hence

$$\|\boldsymbol{\xi}_u - \hat{\boldsymbol{\xi}}_u\| \leqslant \|(\mathbf{I} - \mathbf{N}_u)^{-1}\|\,\|\mathbf{F}_k\|$$

For the unit circle and evenly spaced nodes, empirical evidence (see Figs. 6 and 7) indicates

$$\|(\mathbf{I} - \mathbf{N}_u)^{-1}\| = \theta(m)$$

Thus

$$\|\boldsymbol{\xi}_u - \hat{\boldsymbol{\xi}}_u\| = \theta(m)\,\theta\left(\frac{\ln m}{m^2}\right) = \theta^2\left(\frac{\ln m}{m}\right)$$

and

$$\lim_{m \to \infty} \|\boldsymbol{\xi}_u - \hat{\boldsymbol{\xi}}_u\| = 0$$

(The matrix norm $\|(\mathbf{I} - \mathbf{N}_u)^{-1}\|$ is computed in Figs. 6 and 7 for several values of m. The nodes are evenly spaced on C in this analysis. For each value of m, the number of specified ψ-values are increased sequentially from 1 through m and $\|(\mathbf{I} - \mathbf{N}_u)^{-1}\|$ is computed for each case. All specified ψ-values on C are located contiguously. Figure 7 summarizes the results shown in Fig. 6.)

CONVERGENCE OF THE CVBEM FOR MIXED BOUNDARY VALUE PROBLEMS.

The remaining theorems further address the convergence performance of the CVBEM for the common case of mixed boundary conditions; that is, values of ϕ, or ψ, or gradients of ϕ or ψ on arcs of the unit circle C. The previous Discussion addresses convergence by consideration of the CVBEM matrix systems.

Theorem F assumes that $\omega(z)$ is analytic on C and therefore the error functions $e(z)$, $e'(z)$ and $e''(z)$ are uniformly bounded over the unit circles and its interior. The Discussion following Theorem F recasts the mixed boundary value problem into a simpler Dirichlet problem which dominates the original problem's error function.

Theorem F

Let $\omega(z)$ be analytic over $\Omega \cup C$ where $\Omega = \{z : |z| < 1\}$ and $C = \{z : |z| = 1\}$. Let $\langle \hat{\omega}_n(z) \rangle$ be a sequence of functions analytic over $\Omega \cup C$ such that for each m the functions $\hat{\omega}(z)$, $\hat{\omega}'_m(z)$ and $\omega''_m(z)$ are uniformly bounded in magnitude by some $M^1 \in \mathbb{R}$. Define $e_m(z) = \omega(z) - \hat{\omega}_m(z)$ where $e_m(z) = e\phi_m(z) + ie\psi_m(z)$. Subdivide C into two arcs $C\phi$ and $C\psi$ such that $C\phi + C\psi = C$ and ϕ_m is known on $C\phi$ and ψ_m is known on $C\psi$. For each m, let

$$E_m = \max \quad |e\phi_m(z)|, z \in C\phi; \left| \frac{\partial e\psi_m(z)}{\partial s} \right|, z \in C\psi$$

where (n, s) are normal and tangential co-ordinates on C. Then

$$E_m \to 0 \Rightarrow \hat{\omega}_n(z) \to \omega(z) \text{ for } z \in \Omega \cup C$$

Proof

From the hypothesis, $e_m(z)$ is analytic over $\Omega \cup C$. Thus each function $e_m(z)$, $e'_m(z)$, and $e''_m(z)$ is analytic and uniformly continuous over $\Omega \cup C$ for every m. There exists an $M^2 \in \mathbb{R}$ which uniformly bounds (in magnitude) $\omega(z)$, $\omega'(z)$, and $\omega''(z)$ over $\Omega \cup C$. Let $M = M^1 + M^2$. Then for every m, each $e_m(z)$, $e'_m(z)$, and $e''_m(z)$ is uniformly bounded by M.

By the Cauchy-Riemann equations,

$$\frac{\partial e\psi_m(z)}{\partial s} = \frac{\partial e\phi_m(z)}{\partial n}$$

thus

$$\left| \frac{\partial e\phi_m(z)}{\partial n} \right| \leqslant E_m \quad \text{for } z \in C\psi$$

Then Green's theorem gives

$$I_m = \int_\Omega \left[\left(\frac{\partial e\phi_m}{\partial x} \right)^2 + \left(\frac{\partial e\phi_m}{\partial y} \right)^2 \right] d\Omega$$

$$= \int_C e\phi_m \frac{\partial e\phi_m}{\partial n} \, dC + \int_\Omega e\phi_m \nabla^2 e\phi_m \, d\Omega \tag{1}$$

where

$$\nabla^2 e\phi_m = \frac{\partial^2 e\phi_m}{\partial x^2} + \frac{\partial^2 e\phi_m}{\partial y^2} = 0 \quad \text{over } \Omega \cup C$$

Thus bounds on I_m are calculated by

$$|I_m| = I_m = \left| \int_C e\phi_m \frac{\partial e\phi_m}{\partial n} \, dC \right|$$

$$\leqslant \left| \int_{C_\phi} e\phi_m \frac{\partial e\phi_m}{\partial n} \, dC \right| + \left| \int_{C_\psi} \frac{\partial e\psi_m}{\partial s} e\phi_m \, dC \right|$$

$$\leqslant 2\pi E_m M \tag{2}$$

Thus, $I_m \to 0$ as $E_m \to 0$.

Equivalently,

$$\left(\frac{\partial e\phi_m}{\partial x} \right)^2 + \left(\frac{\partial e\phi_m}{\partial y} \right)^2 \to 0 \quad \text{as } E_m \to 0 \tag{3}$$

But

$$|e'_m(z)|^2 = \left(\frac{\partial e\phi_m(z)}{\partial x} \right)^2 + \left(\frac{\partial e\psi_m(z)}{\partial x} \right)^2$$

$$= \left(\frac{\partial e\phi_m(z)}{\partial x} \right)^2 + \left(\frac{\partial e\phi_m(z)}{\partial y} \right)^2 \tag{4}$$

From (1), (2) and (4),

$$I_m = \int_\Omega |e'_m(z)|^2 \, d\Omega \leqslant 2\pi M E_m \tag{5}$$

Because each $e''_m(z)$ is uniformly bounded in magnitude by M and each $e'_m(z)$ is uniformly continuous over $\Omega \cup C$, then $|e'_m(z)| \to 0$ as $E_m \to 0$. Thus as $E_m \to 0$, $e_m(z)$ approaches a constant function over $\Omega \cup C$. By continuity over $\Omega \cup C$, $e_m(z) \to 0$. Thus $|\omega(z) - \hat{\omega}_m(z)| \to 0$ as $E_m \to 0$.

DISCUSSION

Although $\omega(z)$ is assumed to be analytic over $\Omega \cup C$ in Theorem F, the Discussion following Theorem E only used the assumption that $|\omega''(z)|$ is bounded on C and $\omega(z)$ is analytic over Ω. In the following, the mixed boundary value problem is reinvestigated by recasting the original problem into a Dirichlet problem where convergence of the Dirichlet problem implies convergence of the mixed boundary value problem.

The mixed boundary value problem results in an error function $e(\zeta) = \omega(\zeta) - \hat{\omega}(\zeta)$ for $\zeta \in C$ where $e(\zeta) = e_\phi(\zeta) + ie_\psi(\zeta)$ and $e_\phi(\zeta)$ is known on C_ϕ and $e_\psi(\zeta)$ is known on

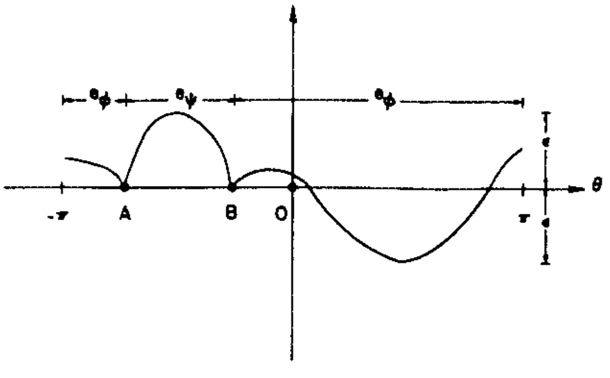


Figure 8

C_ψ. For study purposes, suppose $e_\phi(\zeta)$ and $e_\psi(\zeta)$ are known as shown in Fig. 8. From the figure, C_ψ comprises only a small portion of the total boundary, C. It is also seen that $\epsilon \geqslant |e_\phi|$ and $\epsilon \geqslant |e_\psi|$ for the corresponding contours C_ϕ and C_ψ. The goal is to determine the max $|e_\phi|$ over $\Omega \cup C$ given the bound ϵ of Fig. 8, as conclude that max $|e_\phi| \to 0$ as $\epsilon \to 0$.

Consider the mixed boundary value problem with the C_ϕ and C_ψ boundary values shown in Fig. 9. In the figure, the goal is to make e_ϕ as large as possible. Appealing to an analogy to steady state heat transport, it is immediately seen that e_ϕ must be as large as possible over C such as shown in Fig. 9. Necessarily $e_\phi = e_\psi = 0$ at points A and B due to both conjugate functions of ω known at these points. Also, $\epsilon = \delta M_1$.

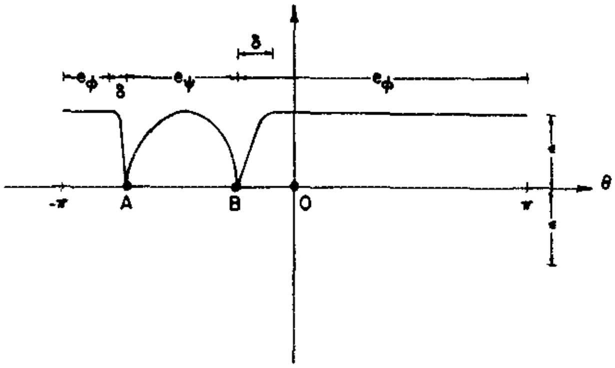


Figure 9

A still 'warmer' situation would be to define $e_\psi/_A = e_\psi/_B = \epsilon$, and 'insulate' C_ϕ such that $e_\psi = \epsilon$ (see Fig. 10). Then for this problem $|e_\psi| \leqslant 2\epsilon$ and from Theorem E, $2\epsilon \to 0 \Rightarrow |e_\psi| \to 0$; hence, max $|e_\phi| \to 0$.

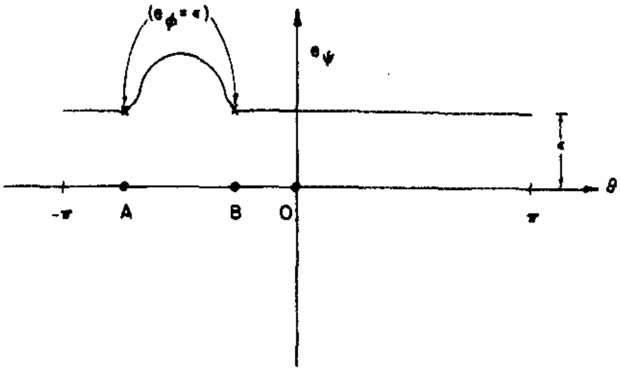


Figure 10

REFERENCES

1 Hromadka II, T. V. *The Complex Variable Boundary Element Method*, Springer-Verlag, 1984