Convergence properties of the CVBEM: Development

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The Complex Variable Boundary Element Method or CVBEM is studied with respect to development of convergence properties. Using conformal mapping of the problem domain to the unit circle, convergence of the CVBEM is examined for Dirichlet and mixed boundary value problems. Convergence is examined with respect to both error bounds of the analytic functions involved, and with respect to the matrix systems developed for the CVBEM approximation technique.

Key Words: convergence, boundary value problems.

INTRODUCTION

The Complex Variable Boundary Element Method or CVBEM has been shown to be a useful numerical approach for the solution of two-dimensional potential problems. In this paper, the CVBEM will be studied as to its convergence properties for both Dirichlet and mixed boundary value problems. These considerations have not been given rigorous attention elsewhere in the literature, and it is the main objective of this paper to remedy this need.

Because any analytic function can be recast by conformal mapping into an equivalent function on the unit circle (where known solutions exist), a rigorous convergence analysis can be developed with the use of only the well-known Poisson formula, and employment of the often-used \( \| \cdot \|_\infty \) norms for the CVBEM matrices and vectors involved in the numerical technique.

Details of the CVBEM numerical approach are thoroughly presented in the cited reference, but are also briefly contained in the Appendix of this paper for the reader's convenience.

SOME UNIT CIRCLE GEOMETRIC PROPERTIES

Before investigating the convergence properties of the CVBEM on the unit circle, some preliminary results regarding the unit circle geometric relationships are useful.

**Theorem A**

Let the unit circle have \( m \) evenly spaced nodal points \( z_j \) such as shown in Fig. 1. In the figure, constant boundary elements are used where collocation points are defined at mid-element. Let collocation point \( z_1 \) have co-ordinates \( z_1 = 1 + 0i \). Then the central angles \( \theta \) between line segments \( (z_{j+1}, z_j) \) and \( (z_j, z_1) \) are all equal to \( \pi/m \).

**Proof**

From Fig. 2, the circle is subdivided into \( m \) sectors with central angles \( \alpha = 2\pi/m \). Let \( d_2 = |z_2 - z_1| \). Then the isosceles triangles of points \( (z_2, z_0, z_1) \) and \( (z_2, z_0, z_3) \) imply \( \theta = \alpha/2 = \pi/m \).

**Theorem B**

Consider the unit circle \( C \) with point \( z_1 \) (see Fig. 3) defined at \( 1 + 0i \) and \( z \) an arbitrary point on \( C \) such that \( 0 \leq \theta \leq \pi \). Then the distance \( d = |z - z_1| \) satisfies \( d \geq 2\theta/\pi \) for \( 0 \leq \theta \leq \pi \).

**Proof**

For the unit circle, \( d^2 = (\cos \theta - 1)^2 + \sin^2 \theta \) for \( 0 \leq \theta \leq \pi \). Let \( f = (\cos \theta - 1)^2 + \sin^2 \theta - (4/m^2) \theta^2 \) then \( f \geq 0 \) for \( 0 \leq \theta \leq \pi \), with \( f = 0 \) at \( \theta = 0 \) and \( \pi \).

**Theorem C**

Let the unit circle \( C \) have \( m \) evenly spaced points such as shown in Fig. 4. Let point \( z' \) approach point \( z_1 \) as shown. Then

\[
\lim_{z' \to z_1} \theta = \pi (m + 1)/m
\]
Theorem D

Let $C = \{z : |z| = 1\}$ and $\Omega = \{z : |z| < 1\}$. Discretize $C$ into boundary elements using the nodal placement shown in Fig. 4. Let $\omega = \phi + i\psi$ be analytic over $\Omega$ and continuous over $C$. Let $\{\hat{\omega}_n(z)\}$ be a sequence of CVBEM approximation functions such that each $\hat{\omega}_n(z)$ is analytic over $\Omega$ and continuous over $C$. Let $\phi$ be known on $C$ (i.e., Dirichlet problem). Define $e_n(z) = \omega(z) - \hat{\omega}_n(z) = e\phi_n + i\psi_n$. For each boundary element $[z_j, z_{j+1}]$, let $e\phi_n$ be bounded by $|e\phi_n| \leq M_1$ for $z \in [z_j, z_{j+1}]$. Then $e\phi_n \to 0$ on $C$ and $\omega_n(z) - \omega(z) \to 0$ over $\Omega \cup C$.

Proof

From the Poisson formula, $e\phi_n$ known on $C$ determines $e\psi_n$ to within a constant by

$$e\psi_n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} e\phi_n W(\theta) \, d\theta$$

where

$$W(\theta) = \frac{\sin \theta}{1 - \cos \theta}$$

(see Fig. 5). Thus a bound for $|e\psi_n|$ is determined by

$$|e\psi_n| \leq \frac{2}{2\pi} \int_{-\pi}^{\pi} e\phi_n W(\theta) \, d\theta + \frac{2}{2\pi} \int_{0}^{\delta} M_1 W(\theta) \, d\theta$$

where $\delta = eM_1$ and $|e\phi_n| \leq e$. Solving,
Proof
By assumption, \( \omega(z) \) is analytic over \( C' \). Then for \( |z'| = R' < R \), the Cauchy formula gives

\[
2\pi i \omega(z^-) = \oint_{C} \frac{\omega(\xi)}{\xi - z^-} \, d\xi
\]

Consider

\[
I = \left| \frac{\omega(\xi)}{\xi - z^-} \right|_C - \left| \frac{\omega(\xi)}{\xi - z^-} \right|_{C'}
\]

\[
= \left| \int_{\theta = 0}^{2\pi} \left[ \frac{\omega(e^{i\theta}) e^{i\theta}}{e^{i\theta} - z^-} - \frac{\omega(Re^{i\theta}) e^{i\theta}}{Re^{i\theta} - z^-} \right] \, d\theta \right|
\]

\[
= \left| \int_{\theta = 0}^{2\pi} \frac{(\omega(e^{i\theta}) - \omega(Re^{i\theta})) Re^{i\theta}}{(e^{i\theta} - z^-)(Re^{i\theta} - z^-)} \, d\theta \right|
\]

\[
\leq 2\pi \left| (e^{i\theta} - z^-)(Re^{i\theta} - z^-) \right|
\]

\[
\omega(z) \text{ is uniformly continuous over } \Omega \cup C. \text{ Hence for every } \epsilon > 0 \text{ there exists } R_z < 1 \text{ such that } |\omega(e^{i\theta}) - \omega(Re^{i\theta})| < \epsilon \text{ for } R_z < R < 1. \text{ For } |\omega(z)| \text{ uniformly bounded on } \Omega \cup C \text{ by } M_x,
\]

\[
I \leq 2\pi \frac{\left| e^{i\theta} - z^- \right| + \left| Re^{i\theta} - z^- \right|}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
\]

\[
\leq 2\pi \left| \frac{e^{i\theta} - z^-}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|} \right|
\]

\[
\leq \frac{2\pi}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
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\]

\[
I \leq 2\pi \frac{\left| e^{i\theta} - z^- \right| + \left| Re^{i\theta} - z^- \right|}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
\]

\[
\leq 2\pi \left| \frac{e^{i\theta} - z^-}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|} \right|
\]

\[
\leq \frac{2\pi}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
\]

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\]

\[
I \leq 2\pi \frac{\left| e^{i\theta} - z^- \right| + \left| Re^{i\theta} - z^- \right|}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
\]

\[
\leq 2\pi \left| \frac{e^{i\theta} - z^-}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|} \right|
\]

\[
\leq \frac{2\pi}{|e^{i\theta} - z^-||Re^{i\theta} - z^-|}
\]

Thus for every \( \epsilon > 0 \),

\[
\lim_{R \to 1} I \leq 4\pi \epsilon(1 - \epsilon)^2
\]

DISCUSSION ON THE CONVERGENCE PROPERTIES OF THE CVBEM MATRIX SYSTEM

Let \( \omega(z) \) be analytic in \( \Omega \) : \( |z| < 1 \) and continuous on \( C : |z| = 1 \). Define \( M \) as the length of the boundary of \( C \), and \( M \) as the length of the boundary of \( C_j \). Define a global trial function \( G_m(z) \) on \( C \) by
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\[ G_m(\xi) = \sum_{j=1}^{m} N_j(\xi) \omega(z_j) \]

where

\[ N_j(\xi) = \begin{cases} (z - z_{j-1})(z_j - z_{j-1}), & \xi \in C_{j-1} \\ (z_{j-1} - \xi)(z_{j+1} - z_j), & \xi \in C_j \\ 0, & \text{otherwise} \end{cases} \]

For \( \xi \in C \), define \( f_m(\xi) \) by

\[ \omega(\xi) = G_m(\xi) + f_m(\xi) \]

where necessarily \( f_m(\xi) \) is continuous on \( C \) and \( f_m(z_j) = 0, j = 1, 2, \ldots, m \). For every \( m \), let \( C_j \) be uniformly bounded on the interior of each \( C_j \) by \( |f_m(\xi)| \leq M_2, \xi \in C_j \) (i.e. \( \omega(\xi) \leq M_2 \) for \( \xi \in C_j \)).

Define the \( m \) nodes on \( C \) by \( z_j = e^{\theta j}, j = 1, 2, \ldots, m \) and interior points \( z'_j \) by \( z'_j = Re^{\theta j}, R < 1 \). For \( \omega(\xi) \) uniformly continuous over \( \Omega \cup C \),

\[ \lim_{z_k \to z_k} \omega(z_k) = \omega(z_k) \]

where from the previous theorem,

\[ \omega(z_k) = \frac{1}{2\pi i} \int_C \frac{\omega(\xi) \, d\xi}{\xi - z_k} = \frac{1}{2\pi i} \int_C \frac{[G_m(\xi) + f_m(\xi)] \, d\xi}{\xi - z_k} \]

\[ = \frac{1}{2\pi i} \int_C \frac{G_m(\xi) \, d\xi}{\xi - z_k} + \frac{1}{2\pi i} \int_C \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

\[ = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{C_j} \frac{G_m(\xi) \, d\xi}{\xi - z_k} + \frac{1}{2\pi i} \int_C \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

Hence for nodal co-ordinate \( z_k \in C \),

\[ \omega(z_k) = \lim_{z_k \to z_k} \omega(z_k) \]

\[ = \lim_{z_k \to z_k} \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{C_j} \frac{G_m(\xi) \, d\xi}{\xi - z_k} + \lim_{z_k \to z_k} \frac{1}{2\pi i} \int_C \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

For each element \( C_j, |f_m(\xi)| \leq M_2 |l - s|^2/2 \) for \( 0 \leq l \leq 1 \),

where \( l = 2\pi/\theta \), \( |l - s| \), \( \xi \in C_j \). Then for \( / \neq k \) nor \( k \),

\[ \lim_{z_k \to z_k} \omega(z_k) = \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\xi) \, d\xi}{\xi - z_k} + \frac{1}{2\pi i} \int_C \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

\[ \leq \frac{1}{2\pi} \int_0^\infty \frac{M_2 |s|^2}{2\pi \theta} \, ds \]

where \( d_j = \min |s - z_k|, \xi \in C_j \). Then

\[ \lim_{z_k \to z_k} \omega(z_k) = \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\xi) \, d\xi}{\xi - z_k} = M_2 \theta \frac{(m-1)}{24\pi d_j} \]

\[ \sum_{j=2}^{m-1} \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

where \( d_j \geq 2\pi\theta, M_2 \theta \leq \pi \). Noting that the \( \theta_j \) differ in value sequentially by \( 2\pi/\theta \),

\[ \lim_{z_k \to z_k} \sum_{j=2}^{m-1} \frac{1}{2\pi i} \int_{C_j} \frac{f_m(\xi) \, d\xi}{\xi - z_k} \leq \frac{M_2 \theta}{24} \left( \frac{m}{2\pi} \right) \]

\[ \times \left[ \frac{1}{2} \sum_{j=2}^{m-1} \frac{1}{2} \sum_{j=2}^{m-1} \frac{M_2 \theta}{24} \left( \frac{m}{2\pi} \right) \right] = M_2 \theta \left( \frac{ln m}{m^2} \right) \]

Similarly for \( C_{k-1} \) and \( C_k \), and for \( 2|\xi - z_k| \geq |\xi - z_k| = s \) (local co-ordinate):

\[ \lim_{z_k \to z_k} \frac{1}{2\pi i} \int_{C_{k-1} \cup C_k} \frac{f_m(\xi) \, d\xi}{\xi - z_k} \leq \frac{1}{2\pi} \int_0^{\infty} \frac{M_2 (s - s^2)}{(s/2)^2} \, ds \]

\[ = \frac{2M_2}{m^2} = M_2 \theta \left( \frac{1}{m^2} \right) \]

where again \( l = 2\pi/\theta \). Hence for any node \( z_k \in C \),

\[ F_k = \frac{1}{2\pi i} \int_C \frac{f_m(\xi) \, d\xi}{\xi - z_k} \]

\[ F_k = \lim_{z_k \to z_k} F_k \]

and

\[ \| F_k \| = M_2 \theta \left( \frac{ln m}{m^2} \right) \]

Now consider

\[ \lim_{z_k \to z_k} \frac{1}{2\pi i} \int_C \frac{G_m(\xi) \, d\xi}{\xi - z_k} \]

\[ = \lim_{z_k \to z_k} \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{N_j(\xi) \, d\xi}{\xi - z_k} \]

\[ = \sum_{j=1}^{m} \omega(z_j) \eta_{jk} \]

where the \( \eta_j \) are complex constants \( \eta_{jk} = \alpha_{jk} + i\beta_{jk} \).

Combining the above results, for any node \( z_k \in C \), the uniform continuity of \( \omega(z) \) over \( \Omega \cup C \) gives:

\[ \omega(z_k) = \lim \omega(z_k) = \sum_{j=1}^{m} \eta_{jk} \omega(z_j) + F_k \]

In engineering problems, usually only one of the nodal values \( \phi_j \) or \( \psi_j \) is known for each node. The CVBEM develops estimates for the unknown nodal values (designated in vector notation by \( \xi_j \)) as a function of the known nodal values (designated by \( \xi_k \)) by setting the unknowns equal to the appropriate real or imaginary part of the CVBEM approximation function, \( \omega(z) \). That is if \( \phi_j \) is unknown, the CVBEM solves for

\[ \hat{\phi}_k = Re \omega(z_k) \]

and for \( \psi_j \) unknown,
\[
\hat{\psi}_k = \text{Im} \hat{\omega}(z_k)
\]

where for \( z_k \in C \),

\[
\hat{\omega}(z_k) = \lim_{z_k \to z_k} \hat{\omega}(z_k)
\]

Then letting \( \hat{\omega}(z_j) = \hat{\phi}_j + i\hat{\psi}_j \), the CVBEM solves for an unknown \( \hat{\phi}_k \) by

\[
\hat{\phi}_k = \sum_{j=1}^{m} (\alpha_{jk} \hat{\phi}_j + \beta_{jk} \hat{\psi}_j)
\]

or for an unknown \( \hat{\psi}_k \) value using

\[
\hat{\psi}_k = \sum_{j=1}^{m} (\alpha_{jk} \hat{\psi}_j + \beta_{jk} \hat{\phi}_j)
\]

This solution procedure is written in matrix form by

\[
\hat{\xi}_u = N_u \hat{\xi}_u + N_k \hat{\xi}_k
\]

where \( N_u \) and \( N_k \) are matrices composed of the above \( \alpha_{jk} \) or \( \beta_{jk} \) coefficients which correspond to the unknown and known nodal values, respectively. (It is noted that \( \hat{\xi}_k \) does not have the hat designation due to \( \hat{\xi}_k \) being known values and the basis functions \( N_j(\xi) \) being exact for the \( \hat{\xi}_k \). Additionally, integration contributions computed by the CVBEM from the known boundary conditions are assumed to be exact due to proper choice of the basis functions.)

Similar to the CVBEM matrix system, \( \omega(z_k) \) values can be written as \( \omega(z_k) = \phi_k + i\psi_k \) where

\[
\phi_k = \lim_{z_k \to z_k} Re \frac{1}{2\pi i} \int_C \frac{G_m(\xi) \, d\xi}{\xi - z_k} + Re \lim_{z_k \to z_k} F_k
\]

\[
\psi_k = \lim_{z_k \to z_k} Im \frac{1}{2\pi i} \int_C \frac{G_m(\xi) \, d\xi}{\xi - z_k} + Im \lim_{z_k \to z_k} F_k
\]

In comparison to the CVBEM matrix system,

\[
\hat{\xi}_u = N_u \hat{\xi}_u + N_k \hat{\xi}_k + F_k
\]

Thus the error of approximation is given by

\[
(\hat{\xi}_u - \hat{\xi}_u) = (1 - N_u)^{-1} F_k
\]

Hence

\[
\| \hat{\xi}_u - \hat{\xi}_u \| \leq \| (1 - N_u)^{-1} \| \| F_k \|
\]

For the unit circle and evenly spaced nodes, empirical evidence (see Figs. 6 and 7) indicates

\[
\| (1 - N_u)^{-1} \| = \theta(m)
\]

Thus

\[
\| \hat{\xi}_u - \hat{\xi}_u \| = \theta(m) \psi \left( \frac{\ln m}{m^2} \right) = \theta \left( \frac{\ln m}{m^2} \right)
\]

and

\[
\lim_{m \to \infty} \| \hat{\xi}_u - \hat{\xi}_u \| = 0
\]

(The matrix norm \( \| (1 - N_u)^{-1} \| \) is computed in Figs. 6 and 7 for several values of \( m \). The nodes are evenly spaced on \( C \) in this analysis. For each value of \( m \), the number of specified \( \psi \)-values is increased sequentially from 1 through \( m \) and \( \| (1 - N_u)^{-1} \| \) is computed for each case. All specified \( \psi \)-values on \( C \) are located contiguously. Figure 7 summarizes the results shown in Fig. 6.)
CONVERGENCE OF THE CVBEM FOR MIXED BOUNDARY VALUE PROBLEMS.

The remaining theorems further address the convergence performance of the CVBEM for the common case of mixed boundary conditions; that is, values of \( \phi \) or \( \psi \), or gradients of \( \phi \) or \( \psi \) on arcs of the unit circle \( C \). The previous discussion addresses convergence by consideration of the CVBEM matrix systems.

Theorem \( F \) assumes that \( \omega(z) \) is analytic on \( C \) and therefore the error functions \( e(z), e'(z) \) and \( e''(z) \) are uniformly bounded over the unit circles and its interior. The discussion following Theorem \( F \) recasts the mixed boundary value problem into a simpler Dirichlet problem which dominates the original problem’s error function.

**Theorem \( F \)**

Let \( \omega(z) \) be analytic over \( \Omega \cup C \) where \( \Omega = \{ z : |z| < 1 \} \)
and \( C = \{ z : |z| = 1 \} \). Let \( \{ \omega_n(z) \} \) be a sequence of functions analytic over \( \Omega \cup C \) such that for each \( m \) the functions \( \omega(z), \omega_n(z) \) and \( \omega_m(z) \) are uniformly bounded in magnitude by some \( M^2 \in \mathbb{R} \). Define \( e_m(z) = \omega(z) - \omega_n(z) \) where \( e_m(z) = \phi_m(z) + i\psi_m(z) \). Subdivide \( C \) into two arcs \( C_\phi \) and \( C_\psi \) such that \( C_\phi + C_\psi = C \) and \( \phi_m \) is known on \( C_\phi \) and \( \psi_m \) is known on \( C_\psi \). For each \( m \), let

\[
E_m = \max \{ |e_m(z)|, z \in C_\phi; \max \{ |\partial e_m(z)/\partial s|, z \in C_\psi \}
\]

where \((a, s)\) are normal and tangential co-ordinates on \( C \).

Then

\[
E_m \to 0 \Rightarrow \omega_n(z) \to \omega(z) \quad \text{for} \quad z \in \Omega \cup C
\]

**Proof**

From the hypothesis, \( e_m(z) \) is analytic over \( \Omega \cup C \). Thus each function \( e_m(z), e'_m(z), \) and \( e''_m(z) \) is analytic and uniformly continuous over \( \Omega \cup C \) for every \( m \). There exists an \( M^2 \in \mathbb{R} \) which uniformly bounds (in magnitude) \( \omega(z), \omega_n(z), \) and \( \omega_m(z) \) over \( \Omega \cup C \).

Let \( M = M^1 + M^2 \). Then for every \( m \), each \( e_m(z), e'_m(z), \) and \( e''_m(z) \) is uniformly bounded by \( M \).

By the Cauchy-Riemann equations,

\[
\frac{\partial \psi_m(z)}{\partial s} = \frac{\partial \phi_m(z)}{\partial n}
\]

thus

\[
\left| \frac{\partial \phi_m(z)}{\partial n} \right| \leq E_m \quad \text{for} \quad z \in C_\psi
\]

Then Green's theorem gives

\[
I_m = \iint_{\Omega} \left[ \left( \frac{\partial \phi_m}{\partial x} \right)^2 + \left( \frac{\partial \phi_m}{\partial y} \right)^2 \right] \, d\Omega
\]

\[
= \int_C \phi_m \frac{\partial \phi_m}{\partial n} \, dC + \int_{\Omega} \nabla^2 \phi_m \, d\Omega
\]

where

\[
\nabla^2 \phi_m = \frac{\partial^2 \phi_m}{\partial x^2} + \frac{\partial^2 \phi_m}{\partial y^2} = 0 \quad \text{over} \quad \Omega \cup C
\]

Thus bounds on \( I_m \) are calculated by

\[
|I_m| \leq I_m = \left| \int_C \phi_m \frac{\partial \phi_m}{\partial n} \, dC \right| + \left| \int_{C_\phi} \frac{\partial \phi_m}{\partial s} \, dC \right| + \left| \int_{C_\psi} \frac{\partial \psi_m}{\partial s} \, dC \right| \leq 2\pi M E_m M
\]

Thus, \( I_m \to 0 \) as \( E_m \to 0 \).

Equivalently,

\[
\left( \frac{\partial \phi_m}{\partial x} \right)^2 + \left( \frac{\partial \phi_m}{\partial y} \right)^2 \to 0 \quad \text{as} \quad E_m \to 0
\]

But

\[
|e_m'(z)|^2 = \left( \frac{\partial \phi_m(z)}{\partial x} \right)^2 + \left( \frac{\partial \psi_m(z)}{\partial x} \right)^2
\]

\[
= \left( \frac{\partial \phi_m(z)}{\partial x} \right)^2 + \left( \frac{\partial \psi_m(z)}{\partial y} \right)^2
\]

From (1), (2) and (4),

\[
I_m = \int_{\Omega} |e_m'(z)|^2 \, d\Omega \leq 2\pi M E_m
\]

Because each \( e''_m(z) \) is uniformly bounded in magnitude by \( M \) and each \( e'_m(z) \) is uniformly continuous over \( \Omega \cup C \), then \( |e'_m(z)| \to 0 \) as \( E_m \to 0 \). Thus as \( E_m \to 0 \), \( e_m(z) \) approaches a constant function over \( \Omega \cup C \). By continuity over \( \Omega \cup C, e_m(z) \to 0 \). Thus \( |\omega(z) - \omega_m(z)| \to 0 \) as \( E_m \to 0 \).

**DISCUSSION**

Although \( \omega(z) \) is assumed to be analytic over \( \Omega \cup C \) in Theorem \( F \), the discussion following Theorem \( E \) only used the assumption that \( |\omega(z)| \) is bounded on \( C \) and \( \omega_m(z) \) is analytic over \( \Omega \). In the following, the mixed boundary value problem is reinvestigated by recasting the original problem into a Dirichlet problem where convergence of the Dirichlet problem implies convergence of the mixed boundary value problem.

The mixed boundary value problem results in an error function \( e(z) = \omega(z) - \omega(z) \) for \( z \in C \) where \( e(z) = e_\phi(z) + ie_\psi(z) \) and \( e_\phi(z) \) and \( e_\psi(z) \) is known on \( C_\phi \) and \( e_\psi(z) \) is known on \( C_\psi \).
$C_\psi$. For study purposes, suppose $e_\psi(\Omega)$ and $e_\phi(\Omega)$ are
known as shown in Fig. 8. From the figure, $C_\psi$ comprises
only a small portion of the total boundary, C. It is also seen
that $e \gg |e_\psi|$ and $e \gg |e_\phi|$ for the corresponding contours
$C_\phi$ and $C_\psi$. The goal is to determine the max $|e_\phi|$ over $\Omega \cup C$
given the bound $e$ of Fig. 8, as conclude that max $|e_\phi| \to 0$
as $e \to 0$.

Consider the mixed boundary value problem with the
$C_\phi$ and $C_\psi$ boundary values shown in Fig. 9. In the figure,
the goal is to make $e_\phi$ as large as possible. Appealing to an
analogy to steady state heat transport, it is immediately
seen that $e_\phi$ must be as large as possible over $C$ such as
shown in Fig. 9. Necessarily $e_\phi = e_\psi = 0$ at points $A$ and
$B$ due to both conjugate functions of $\omega$ known at these
points. Also, $e = 0M_1$.

A still 'warmer' situation would be to define $e_\psi/A = e_\psi/B = e$, and 'insulate' $C_\psi$ such that $e_\psi = e$ (see Fig. 10).
Then for this problem $|e_\psi| \leq 2e$ and from Theorem F,
$2e \to 0 \Rightarrow |e_\psi| \to 0$; hence, max $|e_\phi| \to 0$.

REFERENCES

1 Hromadka II, T. V. The Complex Variable Boundary Element
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