Advances in Computational Methods for Modeling Groundwater Mound Evolution

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Abstract

In this work, we develop a numerical method for modeling the evolution in time of a groundwater mound on a rectangular domain. The global initial-boundary value problem is assumed to have specified Dirichlet boundary conditions. To model this phenomenon, the global problem is decomposed, a steady-state component and a transient component. These components are governed by the Laplace and diffusion partial differential equations, respectively. The Complex Variable Boundary Element Method (CVBEM) is used to develop an approximation of a steady-state solution known as a background flow regime. A linear combination of basis functions that are the product of a two-dimensional Fourier sine series and an exponential function is used to develop an approximation of the transient solution. The global approximation function is the sum of the CVBEM approximation function for the steady-state component and the Fourier series approximation function for the transient component of the global problem. To validate this method, we apply it to two sample problems. This work was presented at the 2019 American Institute for Professional Geologists conference and has subsequently been refined based on feedback there.

Description of Groundwater Test Problems

Groundwater mounding occurs when water infiltrates the subsurface at a rate faster than it dissipates below the normal water table, forming a bulge. As time passes, the excess water continues to infiltrate below the water table and the mound dissipates. Figure 1 depicts this dissipation process modelled by a sinusoidal function of two spatial variables.

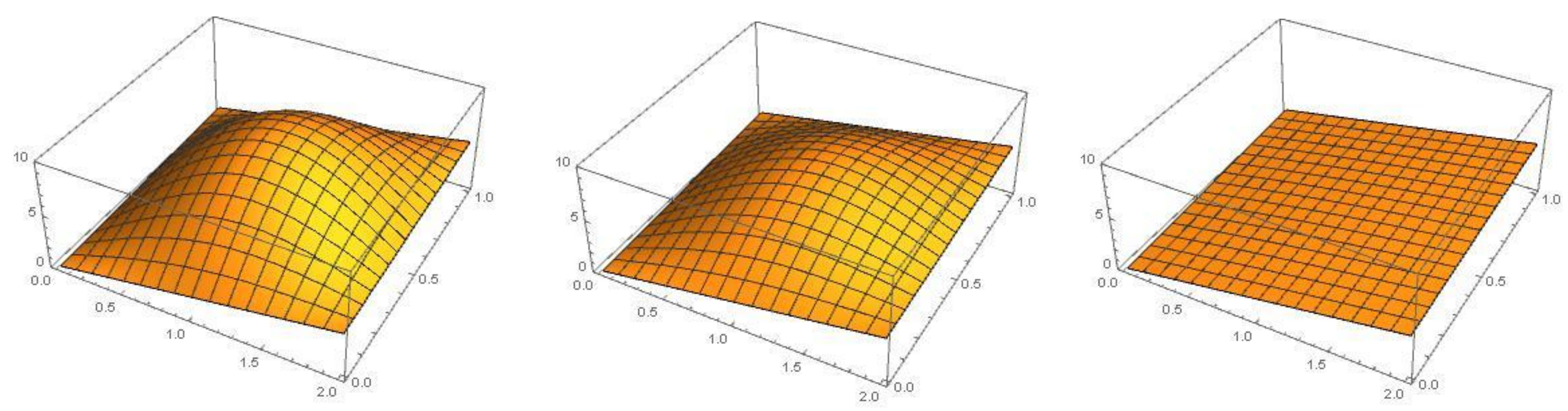


Figure 1: General evolution of a groundwater mounding phenomenon with background flow regime

In this investigation, groundwater flow is formulated into two problems where the same mounding effect occurs over different background flow regimes. These problems will act as tests for evaluating the computational method developed in this study.

Test Problem A models a background regime consisting of flow with a 90 bend. Problem A is formally stated as;

Governing PDE: $u_{xx} + u_{yy} = u_t$ on $\Omega = [0, 2] \times [0, 1]$

Boundary conditions: $u(x, y, t) = [z^2] = x^2 - y^2$ on Γ

Initial condition: $u(x, y, 0) = 100 \sin \frac{\pi x}{2} \sin(\pi y) + x^2 - y^2$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain. The analytic solution of the initial-boundary value problem is

$$u(x, y, t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2 (5/4) t} + x^2 - y^2.$$

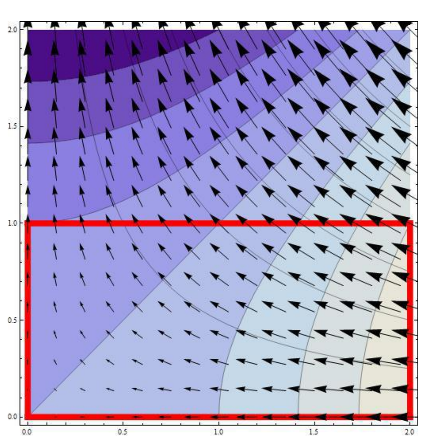


Figure 2: A Contour plot of the steady state surface corresponding to a 90° turn flow background regime.

Test Problem B models a background flow regime consisting of planar flow. Problem B is formally stated as;

Governing PDE: $u_{xx} + u_{yy} = u_t$ on $\Omega = [0, 2] \times [0, 1]$

Boundary conditions: $u(x, y, t) = 2x + y$ on Γ

Initial condition: $u(x, y, 0) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y) + 2x + y$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain.

As the mound dissipates with time, the global flow regime (i.e., background flow plus mound) is restored to just the background flow regime (the steady-state solution). The analytic solution of this initial-boundary value problem is

$$u(x, y, t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2 (5/4) t} + 2x + y.$$

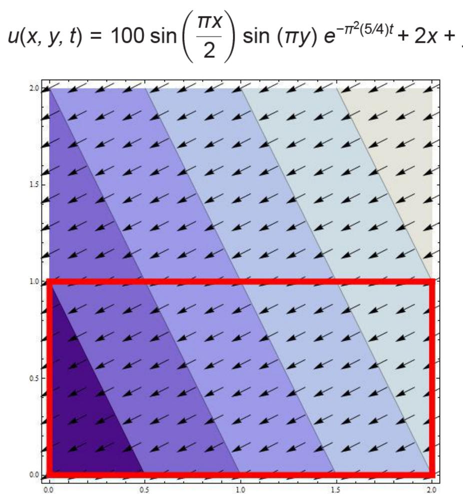


Figure 3: A Contour plot of the steady state surface corresponding to the planar flow background regime.

Numerical Method Development

To develop a numerical method for modelling groundwater mound evolution, the global problem is decomposed into two components; a steady-state component and a transient component. The steady state component is governed by the Laplace partial differential equation, $\Delta u_1 = 0$, and the diffusion partial differential equation, $\Delta u_2 = \frac{\partial u}{\partial t}$. The global solution is the sum, $u = u_1 + u_2$ of a steady-state and transient solutions.

The Complex Variable Boundary Element Method (CVBEM) is used to develop the approximate potential function description of the steady-state condition. The CVBEM is a linear combination of analytic complex variable basis functions of the form

$$\hat{w}(z) = \sum_{k=1}^{p} c_k g_k(z)$$

where c_k is the k^{th} complex coefficient, $g_k(z)$ is the k^{th} member of the family of basis functions being used in the approximation, and p is the number of basis functions being used in the approximation. To approximate a solution to the steady-state problem, i.e. determine suitable c_k coefficients, the CVBEM is applied to the boundary conditions of the global BVP. Once the coefficients are known, it is possible to approximate the potential function of the steady-state situation by applying the coefficients to the real part of the CVBEM approximation function. Likewise, it is possible to approximate the corresponding stream function by applying the coefficients to the imaginary part of the CVBEM approximation function.

The approximate transient solution is a linear combination of basis functions that are the products of a two-dimensional Fourier Sine Series and an exponential function.

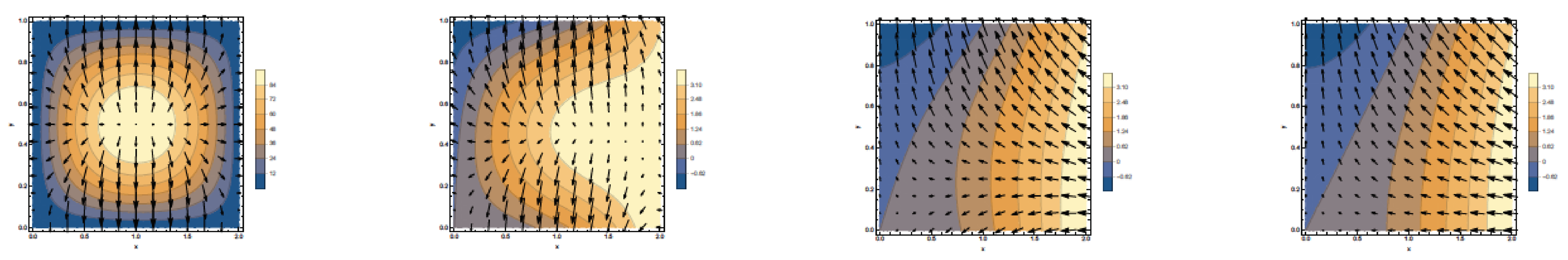
$$\hat{u}_2(x, y, t) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \sin\left(\frac{\pi x i}{L_1}\right) \sin\left(\frac{\pi y i}{L_2}\right) e^{-\pi^2\left(\frac{i^2}{L_1^2} + \frac{j^2}{L_2^2}\right)t}$$

where $\hat{u}_2$ is the approximate value of the potential quantity that is associated with the unsteady component of the problem at a particular location and time, x and y are spatial variables, t is the model time, *i* and *j* are indices, $a_{i,j}$ is the coefficient corresponding to the $(i, j)^{th}$ term of the series (to be determined by collocation with the given initial condition), and L_1 and L_2 are the length and width of the rectangular domain, respectively. The global approximation function is the sum of the CVBEM and the Fourier Series approximation.

Numerical Solutions to Test Problems

Groundwater flow vector gradients are determined as standard vector gradients of the resulting global potential function outcome. Since both the CVBEM outcome as well as the Fourier series approximation of the transient solution are functions, it is possible to calculate the gradient of their sum, which represents the global approximation function. This results in a vector field representing streamlines, which are orthogonal to the iso-potential lines.
The global approximation functions that were used in assessing the maximum error of the global approximation function for various time steps was created using eight terms in the CVBEM approximation function and eight terms in the transient solution approximation function. The maximum errors that are presented in this section were approximated by comparison of the global approximation with the analytic solution at 2,500 uniformly spaced points within the problem domain.

Figure 3 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem A for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing groundwater flow in a 90-degree bend (the steady-state solution).

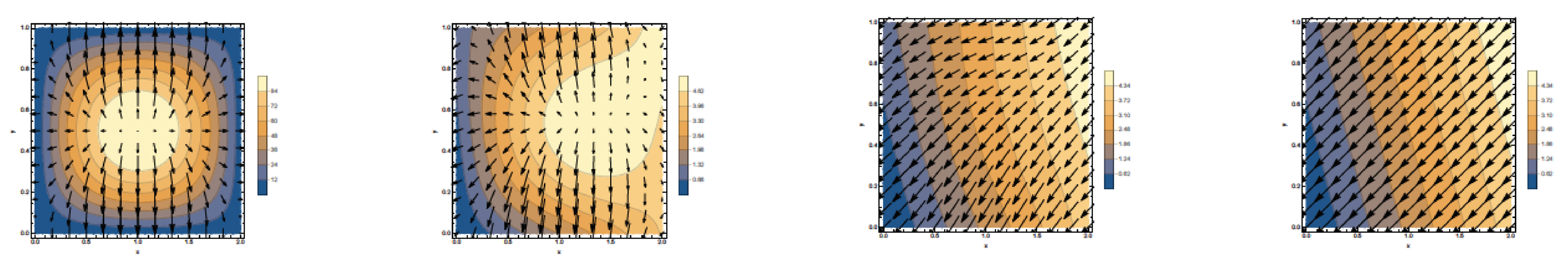


(a) Mound evolution at model time $t = 0.0$. (b) Mound evolution at model time $t = 0.3$. (c) Mound evolution at model time $t = 0.5$. (d) Mound evolution at model time $t = 1.0$.

Figure 3: Time evolution of groundwater mound with underlying flow around a 90-degree bend

Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	1.7763e−15
0.1	1.0658e−14	0.7	1.7763e−15
0.2	5.3290e−15	0.8	1.7763e−15
0.3	1.7763e−15	0.9	1.7763e−15
0.4	1.7763e−15	1.0	1.7763e−15
0.5	1.7763e−15	Steady-state	1.7763e−15

Figure 4 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem B for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing planar flow (the steady-state solution).



(a) Mound evolution at model time $t = 0.0$. (b) Mound evolution at model time $t = 0.3$. (c) Mound evolution at model time $t = 0.5$. (d) Mound evolution at model time $t = 1.0$.

Figure 4: Time evolution of groundwater mound with underlying planar flow

Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	2.6645e−15
0.1	7.1054e−15	0.7	2.6645e−15
0.2	3.5527e−15	0.8	2.6645e−15
0.3	2.6645e−15	0.9	2.6645e−15
0.4	2.6645e−15	1.0	2.6645e−15
0.5	2.6645e−15	Steady-state	2.6645e−15

Conclusion

In this work, we developed test problems in groundwater mounding for the purpose of assessing computational software. Further, we develop a numerical method for modelling groundwater mound evolution. Our method decomposes the problem into a steady-state component governed by the Laplace equation and a transient component governed by the diffusion equation. These are modeled by the CVBEM and a Fourier Sine series respectively. We were able to validate our method and determine error by applying the model to test problems with known analytic solutions.