Advances in Computational Methods for Modeling Problems in Geoscience

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Abstract

In this work, we propose a numerical scheme for modeling the evolution in time of a groundwater mound on a rectangular domain. The global initial-boundary value problem is assumed to have specified Dirichlet boundary conditions. To model this phenomenon, the global problem is decomposed into two components. Specifically, a steady-state component and a transient component, which are governed by the Laplace and diffusion partial differential equations, respectively. The Complex Variable Boundary Element Method (CVBEM) is used to develop an approximation of the steady-state solution. A linear combination of basis functions that are the product of a two-dimensional Fourier sine series and an exponential function is used to develop an approximation of the transient solution. The global approximation function is the sum of the CVBEM approximation function for the steady-state component and the Fourier series approximation function for the transient part of the global problem. This work focuses on two important problems in computational geoscience that can be used to assess the validity of computational groundwater models.

Description of Groundwater Test Problems

The use of computational methods for modeling fluid flow is becoming increasingly useful in problems of various size and complexity. Numerous computational modeling software packages have been developed that can provide approximate solutions to initial-boundary value problems that are governed by the fluid flow partial differential equations. In this investigation, groundwater flow is formulated into two problems suitable to be solved using computational methods. These problems will act as tests for evaluating the computational method developed in this study. Additionally, these problems could serve as benchmarks in the field of computational engineering mathematics in general to compare different modelling approaches.

Test Problem A models a background regime consisting of flow with a 90 bend. Problem A is formally stated as;

Governing PDE: $u_{xx} + u_{yy} = u_t$ on $\Omega = [0, 2] \times [0, 1]$

Boundary conditions: $u(x, y, t) = [z^2] = x^2 - y^2$ on Γ

Initial condition: $u(x, y, 0) = 100 \sin \frac{\pi x}{2} \sin(\pi y) + x^2 - y^2$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain. The analytic solution of the initial-boundary value problem is

$$u(x, y, t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2(5/4)t} + x^2 - y^2.$$

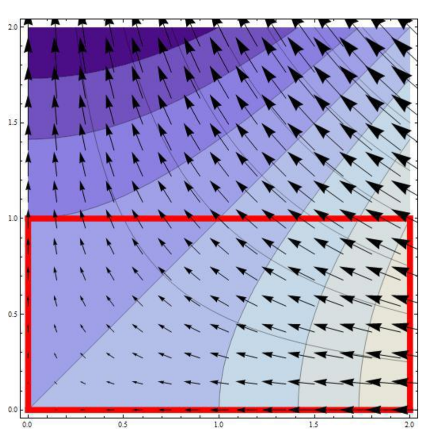


Figure 2: A Contour plot of the steady state surface corresponding to a 90° turn flow background regime.

Test Problem B models a background flow regime consisting of planar flow. Problem B is formally stated as;

Governing PDE: $u_{xx} + u_{yy} = u_t$ on $\Omega = [0, 2] \times [0, 1]$

Boundary conditions: $u(x, y, t) = 2x + y$ on Γ

Initial condition: $u(x, y, 0) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y) + 2x + y$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain.

As the mound dissipates with time, the global flow regime (i.e., background flow plus mound) is restored to just the background flow regime (the steady-state solution). The analytic solution of this initial-boundary value problem is

$$u(x, y, t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2(5/4)t} + 2x + y.$$

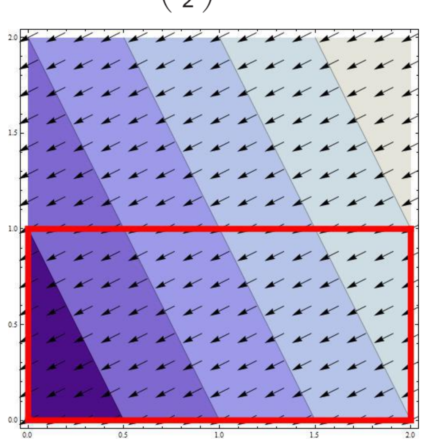


Figure 2: A Contour plot of the steady state surface corresponding to the planar flow background regime.

Numerical Method Development

The purpose of this investigation is to develop a numerical method to model the evolution of groundwater mounds. To this end, the global problem is decomposed into two components; a steady-state component and a transient component. The steady state component is governed by the Laplace partial differential equation (PDE), $\Delta u1 = 0$, and the transient component is governed by the diffusion PDE, $\Delta u2 = \partial u/\partial t$. The global solution is the sum, $u = u1 + u2$ of steady-state and transient solutions. The Complex Variable Boundary Element Method (CVBEM) is used to develop the approximate potential function description of the steady-state condition. The CVBEM is a linear combination of analytic complex variable basis functions of the form

$$\hat{w}(z) = \sum_{k=1}^{p} c_k g_k(z)$$

where c_k is the k^{th} complex coefficient, $g_k(z)$ is the k^{th} member of the family of basis functions being used in the approximation, and p is the number of basis functions being used in the approximation. To approximate a solution to the steady-state problem, i.e. determine suitable c_k coefficients, the CVBEM is applied to the boundary conditions of the global BVP. Once the coefficients are known, it is possible to approximate the potential function of the steady-state situation by applying the coefficients to the real part of the CVBEM approximation function. Likewise, it is possible to approximate the corresponding stream function by applying the coefficients to the imaginary part of the CVBEM approximation function.

The approximate transient solution is a linear combination of basis functions that are the products of a two-dimensional Fourier Sine Series and an exponential function.

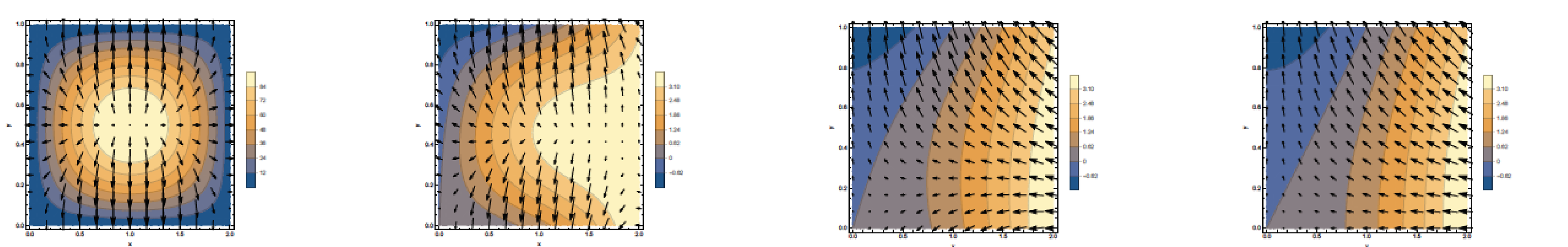
$$\hat{u}_2(x, y, t) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \sin\left(\frac{\pi x i}{L_1}\right) \sin\left(\frac{\pi y i}{L_2}\right) e^{-\pi^2\left(\frac{i^2}{L_1^2} + \frac{j^2}{L_2^2}\right)t}$$

where $\hat{u}_2$ is the approximate value of the potential quantity that is associated with the unsteady component of the problem at a particular location and time, x and y are spatial variables, t is the model time, *i* and *j* are indices, $a_{i,j}$ is the coefficient corresponding to the $(i, j)^{th}$ term of the series (to be determined by collocation with the given initial condition), and L_1 and L_2 are the length and width of the rectangular domain, respectively. The global approximation function is the sum of the CVBEM and the Fourier Series approximation.

Numerical Solutions to Test Problems

Groundwater flow vector gradients are determined as standard vector gradients of the resulting global potential function outcome. Since both the CVBEM outcome as well as the Fourier series approximation of the transient solution are functions, it is possible to calculate the gradient of their sum, which represents the global approximation function. This results in a vector field representing streamlines, which are orthogonal to the iso-potential lines.
The global approximation functions that were used in assessing the maximum error of the global approximation function for various time steps was created using eight terms in the CVBEM approximation function and eight terms in the transient solution approximation function. The maximum errors that are presented in this section were approximated by comparison of the global approximation with the analytic solution at 2,500 uniformly spaced points within the problem domain.

Figure 3 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem A for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing groundwater flow in a 90-degree bend (the steady-state solution).

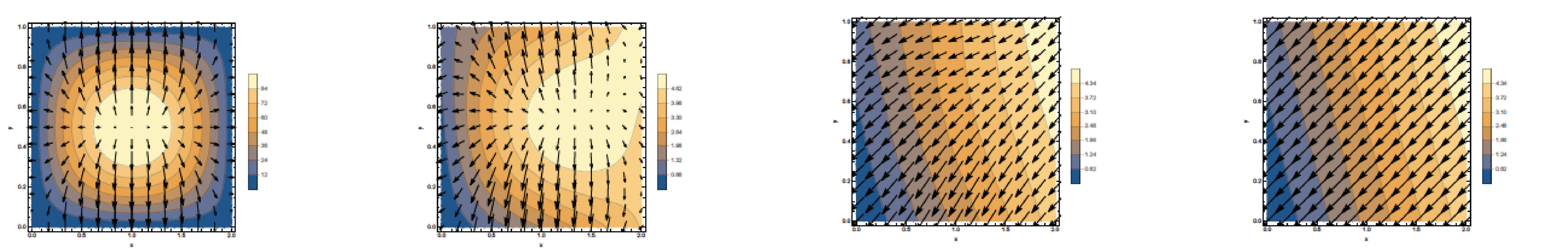


(a) Mound evolution at model time $t = 0.0$. (b) Mound evolution at model time $t = 0.3$. (c) Mound evolution at model time $t = 0.5$. (d) Mound evolution at model time $t = 1.0$.

Figure 2: Time evolution of groundwater mound with underlying flow around a 90-degree bend

Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	1.7763e−15
0.1	1.0658e−14	0.7	1.7763e−15
0.2	5.3290e−15	0.8	1.7763e−15
0.3	1.7763e−15	0.9	1.7763e−15
0.4	1.7763e−15	1.0	1.7763e−15
0.5	1.7763e−15	Steady-state	1.7763e−15

Figure 4 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem B for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing planar flow (the steady-state solution).



(a) Mound evolution at model time $t = 0.0$. (b) Mound evolution at model time $t = 0.3$. (c) Mound evolution at model time $t = 0.5$. (d) Mound evolution at model time $t = 1.0$.

Figure 2: Time evolution of groundwater mound with underlying planar flow

Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	2.6645e−15
0.1	7.1054e−15	0.7	2.6645e−15
0.2	3.5527e−15	0.8	2.6645e−15
0.3	2.6645e−15	0.9	2.6645e−15
0.4	2.6645e−15	1.0	2.6645e−15
0.5	2.6645e−15	Steady-state	2.6645e−15

Conclusion

In this work, we developed test problems in groundwater mounding for the purpose of assessing computational software. Further, we develop a numerical method for modelling groundwater mound evolution. Our method decomposes the problem into a steady-state component governed by the Laplace equation and a transient component governed by the Diffusion equation. These are modeled by the CVBEM and a Fourier Sine series respectively. We were able to validate our method by applying it to our test problems.

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