A New Advancement in the Complex Variable Boundary Element Method with a Potential Flow Application

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Overview

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2. Review of CVBEM Methodology
3. An Optimization Algorithm
4. Results

Section 1

Overview of the Application

Potential Flow Around a Cylinder

- We consider the flow of an ideal, irrotational, incompressible fluid around a cylinder.
- Characteristics of this particular flow:
  - Far away from the cylinder, the flow is unidirectional and uniform.
  - The flow has no vorticity, and so the velocity field $\boldsymbol{V}$ is irrotational, hence $\nabla \times \boldsymbol{V} = 0$.
  - Being irrotational, there exists a velocity potential φ satisfying $\boldsymbol{V} = \nabla\varphi$.
  - Being incompressible, $\nabla \cdot V = 0$.
- We conclude that φ must satisfy Laplace's equation, $\nabla^2\varphi = 0$.

Depiction of Analytic Solution

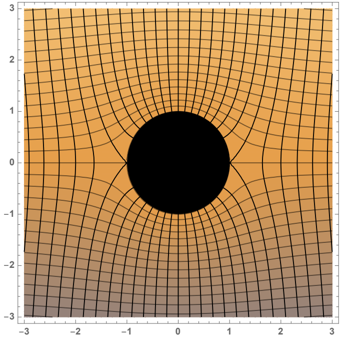

Figure: Zoomed-in

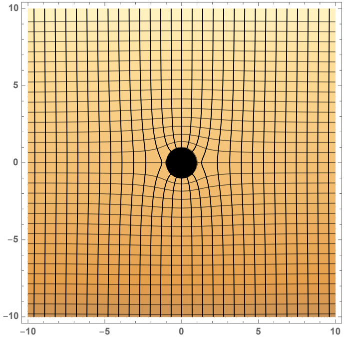

Figure: Zoomed-out

Two perspectives of the analytic solution. Notice how the flow approaches uniform flow at points farther away from the location of the cylinder.

Modeling Considerations

- Due to symmetry, we can choose to model only the upper half plane. We chose the upper half plane as opposed to modeling just a single quadrant because we wanted the approximation to include the two stagnation points at $(1, 0)$ and $(-1, 0)$.
- Boundary conditions came from the imaginary part of the complex variable function $z + \frac{1}{z}$

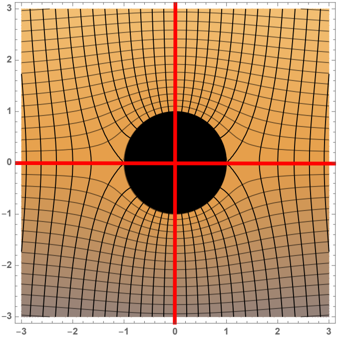

Figure: Vertical and horizontal axes of symmetry. The problem symmetry allows for modeling simplification.

Section 2

Review of CVBEM Methodology

CVBEM Fundamentals

Theorem (The Cauchy Integral Theorem)

Let Γ be a simple closed contour, and let f be analytic everywhere within and on Γ. Then, for any point, z_0 in the domain enclosed by Γ,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)dz}{z - z_0}. \tag{1}$$

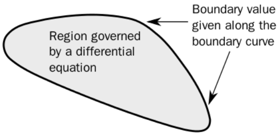

Figure: Boundary Value Problem Depicting Simple Closed Contour

The General CVBEM Approximation Function

- A CVBEM approximation function is a linear combination of analytic complex variable functions.

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j g_j(z), \tag{2}$$

where

- $c_j \in \mathbb{C}$ are complex coefficients,
- $g_j(z)$ are the complex variable basis functions being used in the approximation,
- n is the number of basis functions being used in the approximation

We note there are $2n$ degrees of freedom since each complex coefficient has an unknown real part as well as an unknown imaginary part.

The CVBEM Modeling Procedure

1. The CVBEM begins by interpolating between the given boundary values to generate a simple closed contour along the boundary of the problem domain. When linear interpolation is used, numerical integration of the Cauchy integral equation yields basis functions of the form:

$$(z - z_j)\ln(z - z_j)$$

2. The modeler selects the z_j that are to be used in the approximation. These points are the branch points of the logarithm and are often referred to as "nodes."
3. Collocation with known boundary conditions is used to determine the coefficients of the approximation function.
4. Once the coefficients are known, the approximate equipotential and stream lines can be evaluated continuously in the plane as the real and imaginary parts of the CVBEM function, respectively, at all points at which the basis functions are analytic.

Example Problem Situation

In the un-optimized implementation of the CVBEM algorithm, there is not much thought given to the location of nodes and collocation points. We just chose locations that "seemed reasonable." So, collocation points were equally spaced along the problem domain, and the nodes were arranged in a circle around the problem domain.

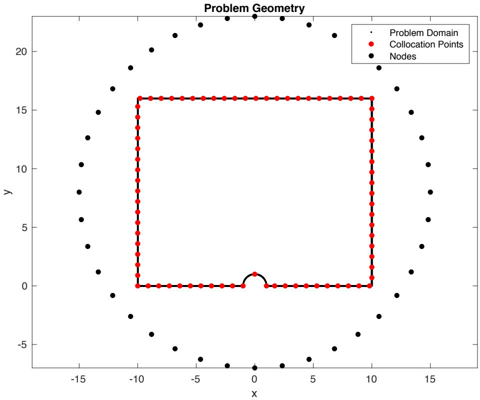

Figure: Red dots correspond to collocation points, black dots are nodes.

Section 3

An Optimization Algorithm

Description of Optimization Algorithm

Step 1

Many candidate nodes and collocation points are generated.

Step 2

The algorithm is initialized by the arbitrary selection of two collocation points.

Step 3

Then, each of the candidate nodes is checked to see which node results in the 1-node model of least error given the initial selection of two collocation points.

Description of Optimization Algorithm

Step 4

The node corresponding to the model of least maximum error is selected as the next node.

Step 5

With the newly-selected node now incorporated in the model, the error is now re-assessed along the entire boundary. The two locations of greatest error are selected as the locations of the next two collocation points.

Step 6

Then, each of the remaining candidate nodes is checked to see which node results in the model of least error given the previous selections of collocation points and nodes.

Step 7

Steps 4-6 repeat until all the required number of nodes and collocation points have been selected.

Refinement Procedure

At this point, we have selected all of the nodes and collocation points that we need. But we can still reduce the error in our approximation.

Step 8

Starting with the first selected node, each node is re-examined one-by-one to see if a different selection from the pool of remaining candidate nodes would result in a model with smaller maximum error.

Step 9

If selecting a different node would result in a smaller overall error, that node replaces the currently-used node and we proceed to check the next node. Otherwise, the currently-used node is kept and the next node is checked. At this point, note that the maximum error is decreasing monotonically since either no change is made to the model or a change resulting in a smaller maximum error is made.

Step 10

Refinement continues for a pre-specified number of iterations or until the maximum error is no longer decreasing.

Overview of the Application

Review of CVBEM Methodology

An Optimization Algorithm

Results

Section 4

Results

Example Problem Details

Parameters	Values
Degrees of freedom:	40
Length:	20
Height:	16
Number of candidate nodes for optimization algorithm:	2000
Number of candidate collocation points for optimization algorithm:	1500

Table: Parameter Details

Method	Maximum Absolute Error
Un-optimized:	7.9637×10^{-1}
Optimized:	7.8426×10^{-6}

Table: Error Details

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Optimized vs. Un-optimized Results

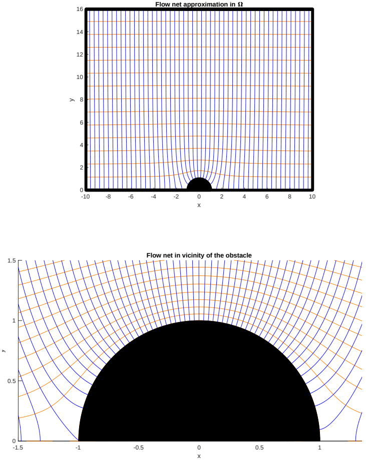


Figure: Optimized

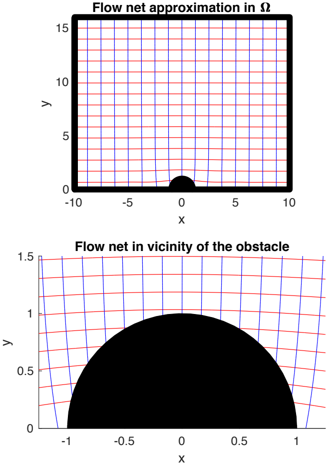
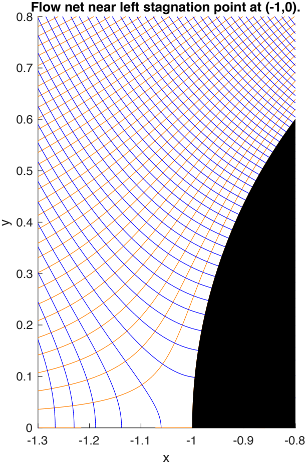
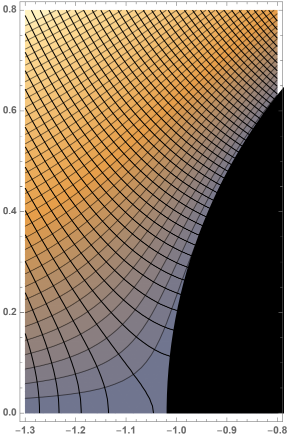


Figure: Un-optimized

Optimized Results - Left Stagnation Point

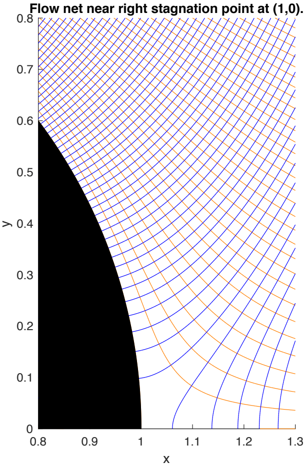


Approximation

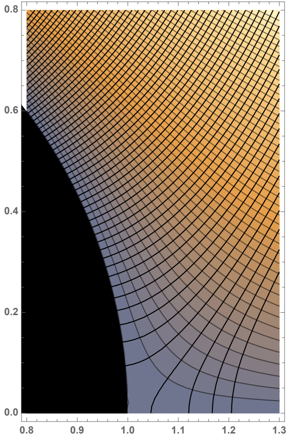


Analytic

Optimized Results - Right Stagnation Point

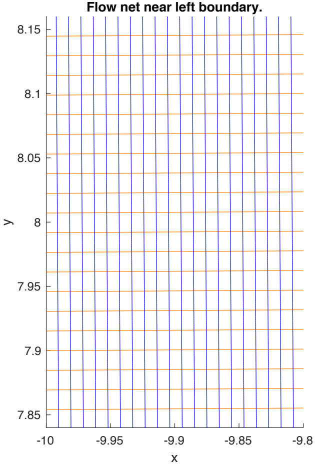


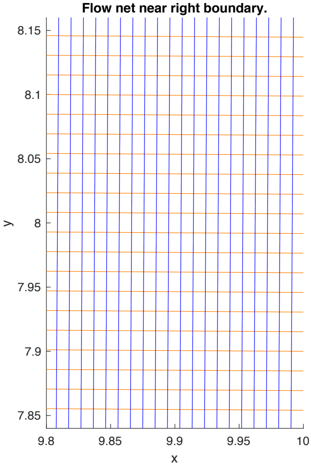
Approximation



Analytic

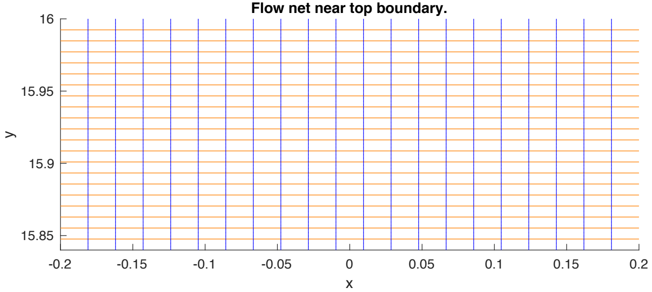
Optimized Results - Flow Near Boundaries





At the boundaries, the flow approaches uniform flow, as desired.

Optimized Results - Flow Near Boundaries



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