Complex Variable Boundary Element Method in High Aspect Ratio Domains

Cadet Noah J. DeMoes and Cadet Gabriel T. Bann
Advisors: Dr. Theodore V. Hromadka II & LTC Randy Boucher

Abstract

The Complex Variable Boundary Element Method is one method that solves Laplace's equation. Laplace's equation is used to model many types of boundary value problems such as ideal fluid flow, ground water flow, and heat transfer. Also, the CVBEM has been proven to solve Laplace's equation in two or more dimensions. Typically, the CVBEM is applied to boundary value problems with low aspect ratio domains. This research focuses on the computational efficiency and accuracy of solving boundary value problems with high aspect ratios. Additionally, different basis function families are used to determine if one family of basis functions performs better in problems with high aspect ratio domains than other families.

CVBEM Basics

The CVBEM builds on the foundation of Cauchy's integral theorem. Cauchy's integral theorem states that if we let Γ be a simple closed contour, and let f be analytic everywhere within and on Γ, then for any point z_0 in the domain enclosed by Γ,

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)dz}{z - z_0}.$$

The CVBEM approximation $\hat{\omega}(z)$ is a linear summation of complex basis functions of the form,

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j g_j(z),$$

where $\hat{\omega}(z)$ is the approximate solution, c_j is the jth coefficient, $g_j(z)$ is the jth basis function, and n is the number of basis functions used to approximate the solution.

CVBEM Modeling Process

1. Form a simple closed contour within the problem domain using linear interpolation between known boundary conditions.
2. Determine the number and type of basis functions used in the CVBEM approximation function.
3. Create a linear combination of the chosen basis functions to form the CVBEM approximation function.
4. Generate a system of equations given the boundary conditions to solve for the coefficients of the approximation function.
5. Substitute in the known coefficients into the CVBEM approximation solution creating a function to determine both the potential and stream values at any location within the domain where the basis functions are analytic.

CVBEM Error Analysis Methodology

1. Chose n basis function families to form n approximation functions.
2. Set the number of collocation points each approximation function uses to form its solution.
3. Establish the comparison domain; this research focuses on the centerline.
4. Define error and compare each approximation function's error.
5. The basis function family that creates the CVBEM approximation function with least error. Follow steps 1-5, and apply to high aspect ratio domains.
6. Compare errors to determine the basis function that is strongest across high aspect ratio domains.

Basis Function Families Utilized

1. CVBEM: $g(z) = (z - z_j)\ln(z - z_j)$;
2. CPM: $g(z) = (z - z_j)^n$;
3. Complex poles: $g(z) = \frac{1}{(z - z_j)}$;
4. Laurent series: $g(z) = \frac{1}{(z - z_0)^n}$; $z_0 = 20 + 20i$.

Example Problem

Domain: $\Omega = \{(x, y) \in R^2 \mid 1 \le x \le L, 1 \le y \le 2\}$
PDE: $u_{xx} + u_{yy} = 0$
BC1: $u(1, y) = x^2 - y^2$
BC2: $u(L, y) = x^2 - y^2$
BC3: $u(x, 1) = x^2 - y^2$
BC4: $u(x, 2) = x^2 - y^2$
$L = 10; 25; 50$
$n = 32$ for basis families 1-3 and $n = 17$ for basis family 4

In this test problem, the boundary conditions are the same on every boundary facilitating the ability to check the approximation solution to the exact solution as the exact solution is the boundary conditions.

Results

Figure A plots the exact solution vs the CPM and CVBEM approximations. Figure B plots the exact solution against the Laurent series and complex poles approximations.

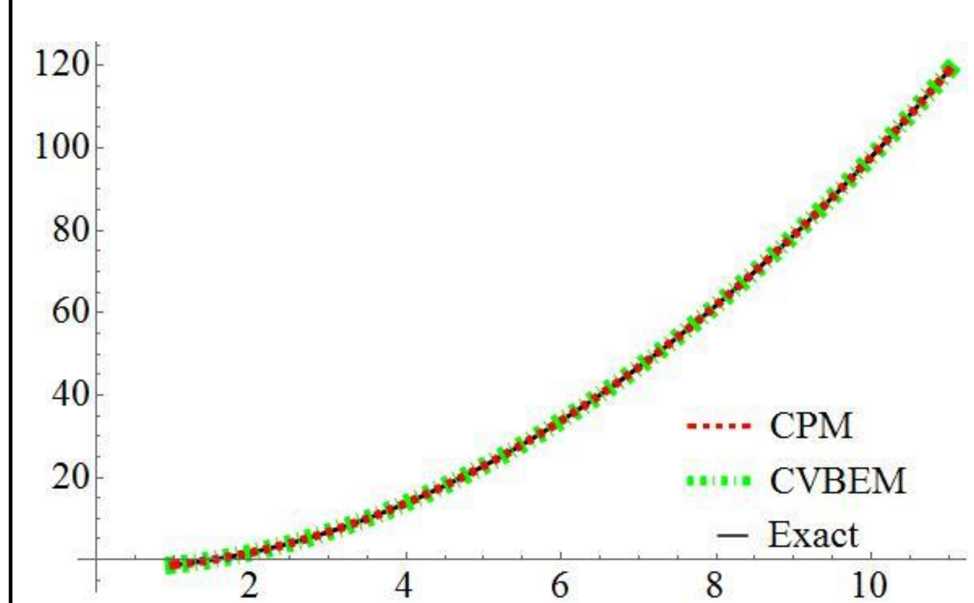

Figure A

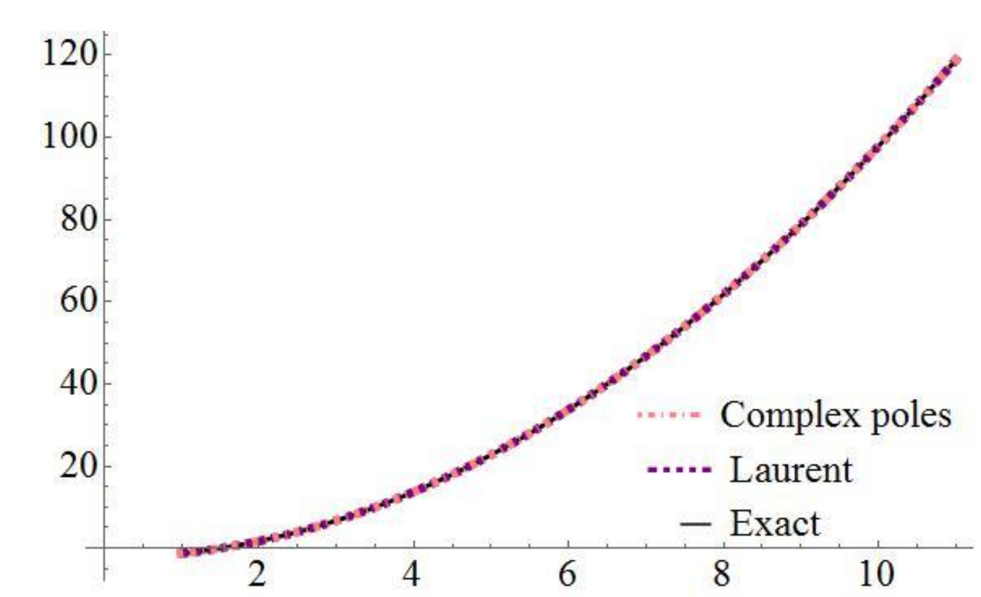

Figure B

Computational error is defined as the absolute value of the exact solution minus the approximate solution divided by the exact solution.

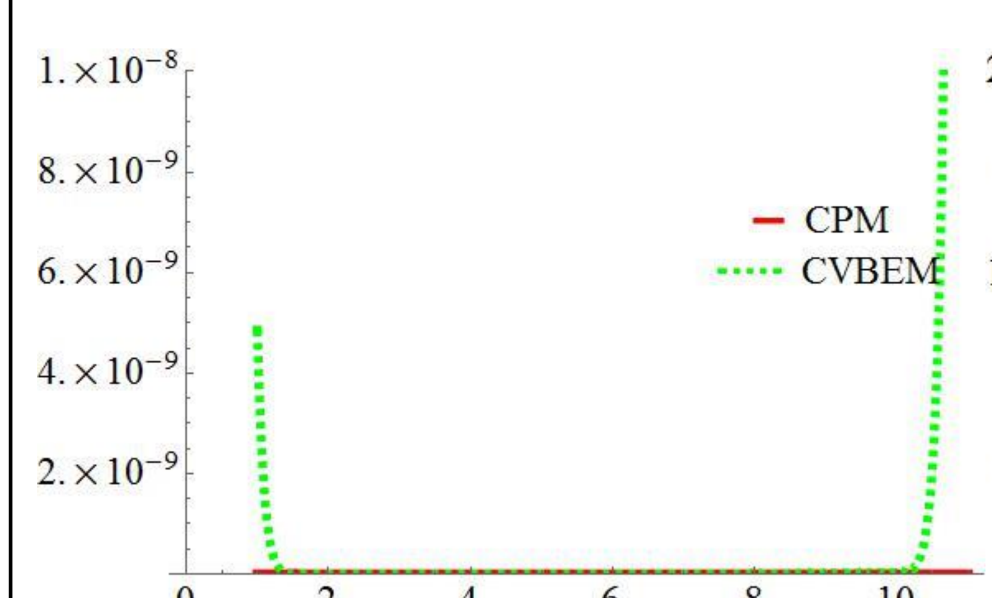

Figure C

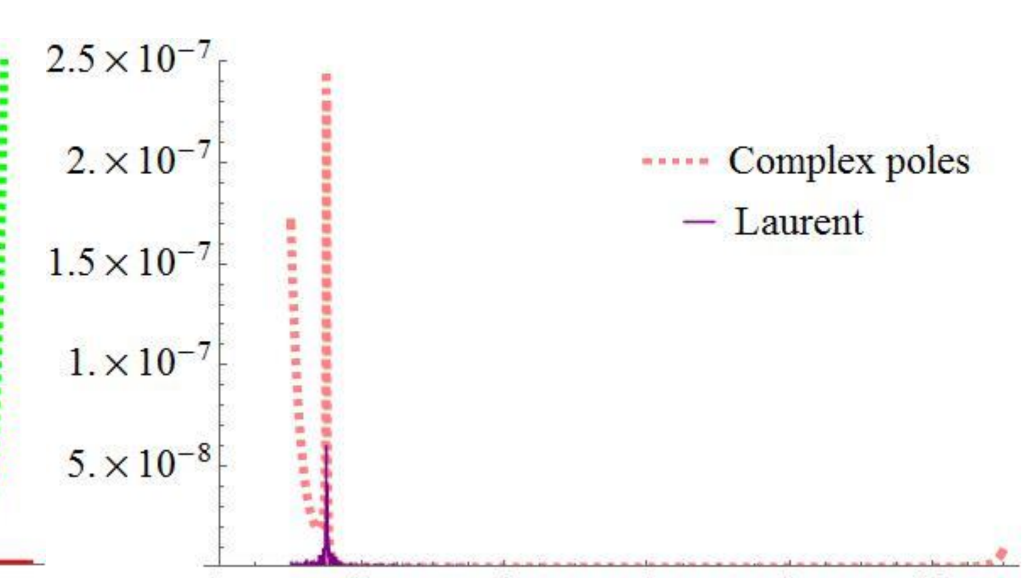

Figure D

Figures C and D depict that the greatest error occurs on the boundaries for all approximations, but the CVBEM is most accurate.

Maximum Relative Error on the Boundary $n = 128$					
Domain Ratio	CPM	CVBEM	complex poles	Laurent	FEM
1:10	1.023182×10^{-12}	6.821210×10^{-13}	1.463718×10^{-12}	3.412850×10^{-09}	3.97904×10^{-13}
1:25	1.743956×10^{-10}	2.046363×10^{-12}	3.798739×10^{-12}	1.942368×10^{-07}	1.93268×10^{-12}
1:50	3.919467×10^{-09}	2.971907×10^{-10}	9.780505×10^{-10}	2.512224×10^{-05}	6.36646×10^{-12}

Table 1: Display of error for different basis families

The tradition CVBEM approximation function has the least error and is the strongest in its accuracy across the high aspect ratio domains compared to the other three basis functions and is comparable to the traditional FEM used in industry.

Acknowledgements

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