Complex Variable Boundary Element Methods for the Diffusion and Wave Equations

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Introduction

The diffusion equation and the wave equation are similar in that they can be decomposed into steady-state and time-dependent components.

Diffusion Equation: $\Delta u = u_t \rightarrow \Delta(u_1 + u_2) = u_t$
Wave Equation: $\Delta u = u_{tt} \rightarrow \Delta(u_3 + u_4) = u_{tt}$

Where, $u_1, \ldots, u_4$ satisfy the following equations:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = \frac{\partial u_2}{\partial t}, \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0, \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = \frac{\partial^2 u_4}{\partial t^2}$$

Notice that u_1 and u_3 satisfy Laplace's equation while u_2 and u_4 are particular solutions of the diffusion and wave equations, respectively.

Initial-Boundary Value Problems

- Solving an initial-boundary value problem implies finding a function u that satisfies the following three criteria
  - The governing PDE
  - The boundary conditions
  - The initial conditions
- The initial-boundary value problems that are considered here have a special property. If we let Ω represent the problem domain, $f(x, y)$ denote the boundary conditions, and $g(x, y)$ denote the initial condition, then

$$f(x, y) = g(x, y) \quad \forall (x, y) \in \partial\Omega.$$

Why Decompose the Global Problem?

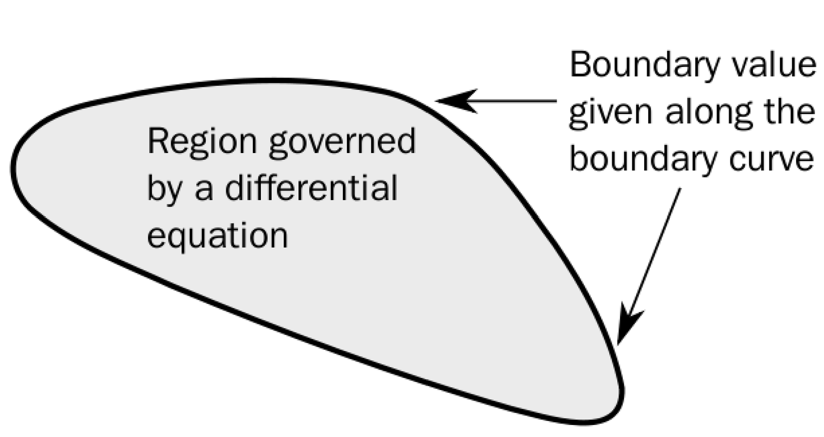
Decomposing the global problem into a steady-state component and a transient component has the desirable effect of transferring the non-homogeneous boundary conditions to a problem that is easier to solve. The result is that:

1. The problem governed by Laplace's equation has non-homogeneous boundary conditions.
2. The problem governed by the time-dependent PDE has homogeneous boundary conditions.

Fundamentals of the CVBEM

Theorem (The Cauchy Integral Theorem). *Let Γ be a simple closed contour, and let f be analytic everywhere within and on Γ. Then, for any point z_0 in the domain enclosed by Γ,*

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)dz}{z - z_0}.$$



Why is this Technique an Approximation?

- This technique is an approximation because the boundary conditions are approximated by interpolation between collocation points (the finite number of known boundary conditions).

The General CVBEM Approximation Function

- A CVBEM approximation function is a linear combination of analytic complex variable basis functions,

$$\hat{\omega}(z) = \sum_{j=1}^{n} c_j g_j(z),$$

where the c_j are complex coefficients, $g_j(z)$ is the j^{th} member of the family of analytic complex variable functions being used in the approximation, and n is the number of basis functions being used in the approximation.

The CVBEM Modeling Procedure

1. The CVBEM begins by interpolating between the given boundary conditions to generate a simple, closed contour along the boundary of the problem domain.
2. A CVBEM approximation function is created as a linear combination of the basis functions being used in the modeling process.
3. A system of equations using the given boundary conditions is used to determine the coefficients of the linear combination.
4. Once the coefficients are known, the CVBEM function can be used to approximate both the potential and stream functions anywhere in the plane where the basis functions are analytic.

Conditions for the Transient Component

- Due to the special property that $f(x, y) = g(x, y) \quad \forall (x, y) \in \partial\Omega$, the boundary conditions of the transient problem are homogeneous.
  - Recall that the CVBEM already satisfied the boundary conditions of the global problem!
- The initial condition of the transient component corresponds to the difference between the initial condition of the global problem and the steady-state solution.

The Time-Dependent Approximation Function

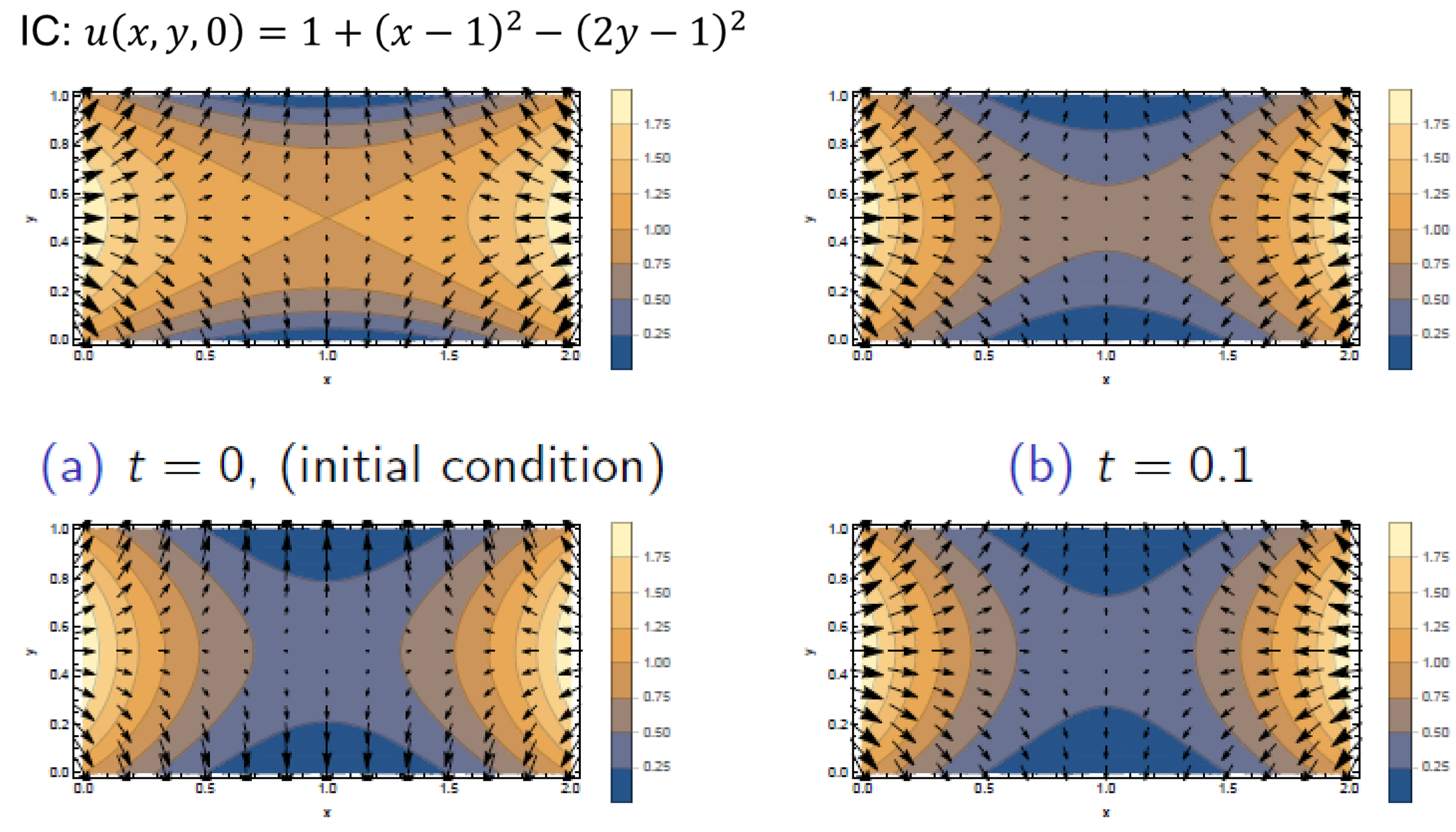
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \sin\left(\frac{i\pi x}{L_1}\right) \sin\left(\frac{j\pi y}{L_2}\right) e^{-\pi^2\left(\frac{i^2}{L_1^2} + \frac{j^2}{L_2^2}\right)t}$$

Example Problem:

PDE: $u_{xx} + u_{yy} = u\_t$

BC: $u(0, y) = 2 - (2y - 1)^2$,
$u(2, y) = 2 - (2y - 1)^2$,
$u(x, 0) = (x - 1)^2$,
$u(x, 1) = (x - 1)^2$.

IC: $u(x, y, 0) = 1 + (x - 1)^2 - (2y - 1)^2$



(a) $t = 0$, (initial condition)
(b) $t = 0.1$
(c) $t = 0.2$
(d) $t \to \infty$, (steady-state)

Figure: This figure depicts the evolution of the global approximation function at various model times (specified by t). The streamline vector trajectories are developed using the usual gradient procedures.

Table: Maximum absolute and relative error in the approximation of the initial condition for various numbers of basis functions used in the transient approximation function.

Number of basis functions (n) used in the transient approximation function	Maximum absolute error in approximating the initial condition	Maximum relative error in approximating the initial condition
2	2.4894	3.0213
4	0.3619	0.7950
8	0.0840	0.2509
16	0.0218	0.0627
32	0.0070	0.0152
64	0.0051	0.0092

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