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# Real time boundary element node location optimization

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ABSTRACT

Boundary Element Method (BEM) computer models typically involve use of nodal points that are the locations of singular potential functions such as the logarithm or reciprocal of the Euclidean distance function. These singular functions are typically associated with the nodes themselves as far as identification. The Complex Variable Boundary Element Method (CVBEM) is another application of similar types of singular potential functions and includes other functions that are not singular but are fundamental solutions of the governing partial differential equation (PDE). These various singular potential functions form a basis whose span of linear combinations (either real or complex space, as appropriate) is a vector space. As part of the approximation approach, one determines that element in the vector space that is closest (usually in a least squares residual measure) to the exact solution of the PDE and related boundary conditions. Recent research on the types of basis functions used in a BEM or CVBEM approximation has shown that considerable improvement in computational accuracy and efficiency can be achieved by optimizing the location of the singular basis functions with respect to possible locations on the problem boundary and also locations exterior of the problem boundary (in general, exterior of the problem domain). To develop such optimum locations for the modeling nodes (and associated singular basis functions), the approach presented in this paper is to develop a Real Time Boundary Element Node Location module that enables the program user to click and drag nodes (one at a time) throughout the exterior of the problem domain (that is, nodes are allowed to be positioned on or arbitrarily close to the problem boundary, and also to be positioned exterior of the problem domain union boundary). The provided module interfaces with the CVBEM program, built within computer program Mathematica, so that various types of information flows to the display module as the node is moved, in real time. The information displayed includes a graphic of the problem boundary and domain, the exterior of the domain union boundary, evaluation points used to represent problem boundary conditions, nodal locations, modeling error in $L_2$ and also $L_\infty$ norms, and a plot of problem boundary conditions versus modeling estimates on the problem boundary to enable a visualization of closeness of fit of the model to the problem boundary conditions. As the target node is moved on the screen, these various information forms change and are displayed to the program user, enabling the user to quickly navigate the target node towards a preferred location. Once a node is established at some optimized location, another node can then be clicked upon and dragged to new locations, while reducing modeling error in the process.



## 1. Introduction

### 1.1. The CVBEM

The Complex Variable Boundary Element Method (CVBEM) is a boundary element technique that was originally developed using a numerical solution of the well-known Cauchy Integral Equation such as developed by Hromadka and Guymon [3]. Subsequent to the original CVBEM development, the CVBEM was extended into a Hilbert space setting by writing the CVBEM approximation function as a linear combination of analytic function basis functions and then selecting the complex coefficients used in the linear combination such as to minimize the usual least-squares error between the problem boundary conditions and the approximation function evaluated on the problem boundary [1]. The usual basis functions are products of complex polynomials with complex logarithm functions expanded about "nodal points" placed upon the problem boundary. Other basis function families, such as complex polynomials, or other analytic functions may be used

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or mixed together, including Laurent series expansion involving monomials with negative powers. The CVBEM was extended to three and higher dimensions by Hromadka and Whitley [1].

### 1.2. Similar techniques

Other similar but different numerical modeling techniques include the method of fundamental solutions (e.g. [2]), and the Complex polynomial method (Hromadka [3] and more recently Bohannon [4]). Because of the similarities between these numerical approximation techniques, there are common features among those approximations that involve basis functions (or approximation functions) that are singular. Such singular functions of interest typically include functions involving the logarithm or the reciprocal function, among other possibilities (for example, the negative powered complex monomials used in the Laurent series provide a wide family of relevant potential singular functions for use in approximations of the Laplace equation). These singular basis functions are typically defined with respect to a particular location, which is a pole or singularity of that particular basis function. In general, these singularities or poles are defined with respect to modeling nodal points (or nodes). Boundary element numerical methods typically locate such nodes on the problem boundary. For example, the CVBEM nodes are typically found on the problem boundary as a result of the numerical integration of the Cauchy Integral equation. The current paper also provides a link between boundary element methods and other modeling techniques such as the method of fundamental solutions [5], where consideration is made of (i.e., "fundamental solutions") singularities or poles to be located in the exterior of the problem domain.

## 2. Nodal location optimization

### 2.1. Optimization method

The general procedure used in this paper for searching for optimal nodal locations (that reduce all three considered approximation error measures) is to introduce a nodal point upon the problem boundary and then to move the target node into the exterior of the problem domain further and further away until the three error measures appear to not be significantly improving the approximation. Similarly, the target node is also moved into the interior of the problem domain. Once that particular line of search is assessed, the target node is then re-introduced into the CVBEM approximation function, but this time at a different location on the problem boundary, and the previously mentioned procedure repeated. After examining all the considered pathways, the optimal node location is concluded. Then, a new target node may be introduced and the entire process repeated. In the problems considered, movement of the target node did not appear to follow generalized trends such as linearly decreased approximation error versus departure distance from the problem boundary.

With the recent advance of optimization of nodal point locations on the problem boundary and also exterior of the problem domain, additional degrees of freedom are available to better approximate the problem boundary conditions, resulting in a significantly improved CVBEM approximation function. Because nodal point location optimization is achieved by searching the region located outside of the problem domain, having measures of approximation error displayed while moving the target nodal point provides an environment that speeds up the optimization process. Further research is needed, however, to develop more direct methods in finding such optimized nodal point locations. In the current paper, three measures of error are displayed during the iterative process of searching for an optimized location for a target node. Namely, a least-squares error, a maximum error measure, and a plot of the problem, boundary conditions versus the current status of the CVBEM approximation function evaluated at the problem boundary.

The finalized so-called "optimal" locations are concluded based upon the three types of error measures used in the nodal point location optimization. Although the search for optimal nodal point locations is not at all exhaustive in the current paper, sufficient extent exterior of the problem domain is examined such as to demonstrate the utility of nodal point location optimization using the three approximation error measures considered. Of course, other measures of approximation error may result in different concluded optimal locations. Further research is needed to assess better error measures to base the process used for optimizing nodal point locations.

### 2.2. Optimization procedure using mathematica

In the current paper, the above described Node Location Optimization method is implemented using the computer program Mathematica (relevant Mathematica code is available from the correspondence author). The ability to locate nodes throughout the exterior of the problem domain is explored and a computational module is presented that enables the program user to click upon any nodal point, drag that target node to any new location in the problem exterior (except to be coincident with another node location). The computational engine of the model, (the CVBEM) determines new modeling results for the newly established nodal location, including modeling error in matching problem boundary conditions, so that the program user can assess a desirable new location for that target node. Once the target node is relocated, the program user then proceeds to any of the other modeling nodes and then navigates that new target node throughout the problem exterior (while the computational engine re-determines the corresponding modeling results using the usual CVBEM least-squares error minimization approach in matching problem boundary conditions) until an improved CVBEM model is concluded. Each node can be likewise repositioned. Similarly, re-positioned nodes can be re-navigated at a later occasion in the optimization process.

The application develops the latest error information in matching the problem boundary conditions with respect to both the least squares error residual and also the maximum absolute value of error along the problem boundary. This error information is returned to the program user, along with a plot of problem boundary conditions as well as the latest developed CVBEM approximation function evaluated continuously along the problem boundary to provide a visual measure of modeling "goodness of fit" to the problem boundary conditions. Because the CVBEM exactly solves the governing partial differential equations (for example, the Laplace equation in the demonstration problems), the modeling error is assessed by the "goodness of fit" with the problem boundary conditions. As the program user continues to move the selected node to other positions, the corresponding error measures and graphs are redeveloped so that the program user can use the application as feedback in real time as guidance in locating an optimum nodal position in reducing modeling error. Once the selected node's position is finalized, the program user selects the next node for assessing an improved location, and the above process is repeated. In this fashion, some or all of the initially defined modeling nodes are positioned to reduce modeling error.

Being able to interactively design a modeling nodal scheme, for nodes to be positioned on the problem boundary or in the

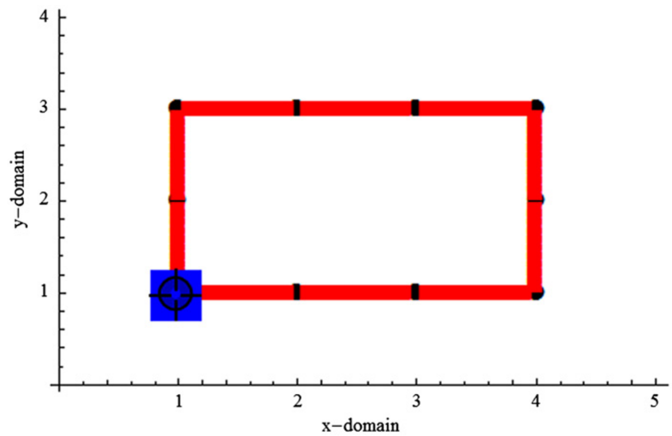


**Fig. 1.** Single node at the bottom-left vertex.

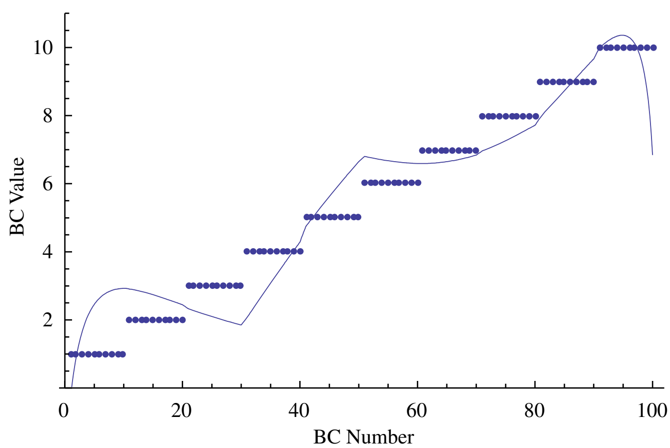


**Fig. 2.** Fit of function to boundary conditions. $L_2=86.2094$, $L_\infty=4.75691$.

exterior of the problem domain, creates a new modeling environment. Furthermore, visualization of the modeling error in matching problem boundary conditions provides a good feedback between the modeling approach and the model user, so that modeling efficiency is increased and modeling accuracy improved.

### *2.3. Demonstration problem in mathematica*

As a demonstration problem, a ten-step Dirichlet type boundary condition is used in that introduces singularities in the boundary conditions at the interfaces between the boundary condition stepped values. Because the CVBEM results in an approximation function that is analytic everywhere except along branch cuts and at nodal points, and furthermore is continuous everywhere except at the branch cuts but is continuous at the nodal points, then the imposed singularities of the boundary condition cannot be matched but will be approximated as a typical Fourier series representation along a stepped target function. This behavior (similar to a Fourier series) is the result of the approximation process involved with the Hilbert space setting. Fig. 2 displays a typical comparison between the ten-stepped problem boundary conditions along the problem boundary and the corresponding CVBEM approximation function evaluated along the problem boundary, for a single node CVBEM approximation function with nodes located as shown in Fig. 1. As more nodes are introduced into the CVBEM approximation function, the resulting comparison between CVBEM and given boundary condition values will converge similar to what is seen as more terms are added to a generalized Fourier series approximation (see Figs. 3–8). This is evidenced by the decreasing $L_2$ and $L_\infty$ norms as the number of nodes increases.

In Fig. 8, we see substantial improvement in the approximation fit to the problem boundary conditions can be achieved by adding more nodal points to the approximation function effort. Analogous to generalized Fourier series approximations, an

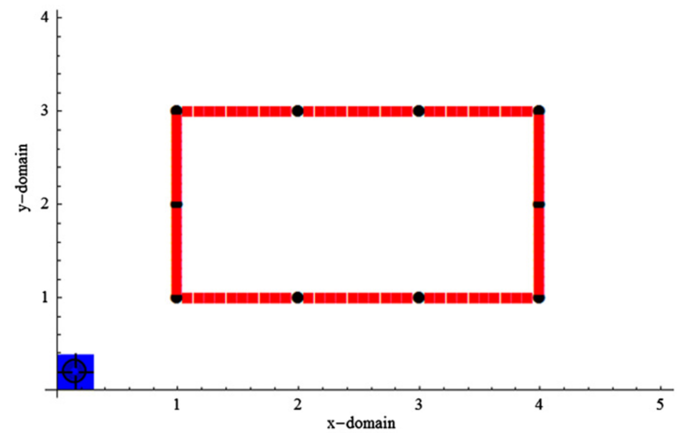


**Fig. 3.** Adjusted nodal location.

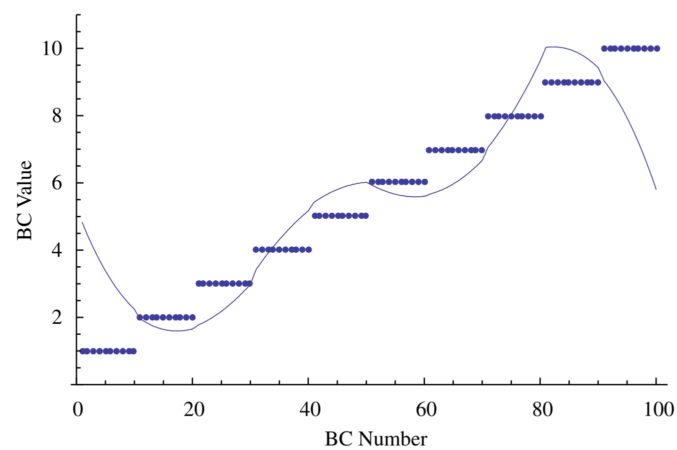


**Fig. 4.** Fit of function to boundary conditions. $L_2=99.0174$, $L_\infty=4.18971$.

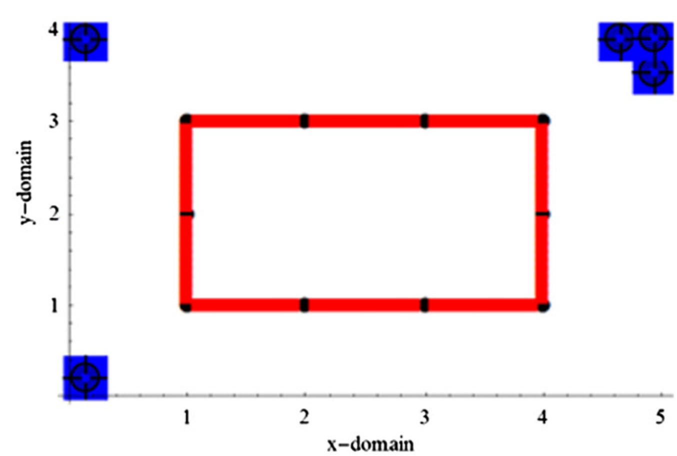


**Fig. 5.** Five node model.

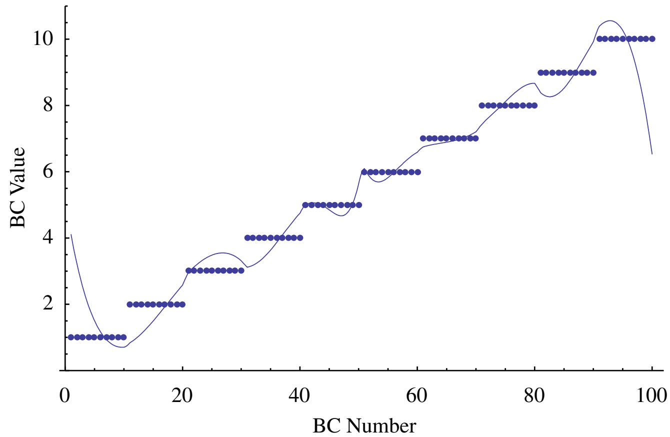


**Fig. 6.** Fit of function to boundary conditions. $L_2 = 48.2625$, $L_\infty = 3.38377$.

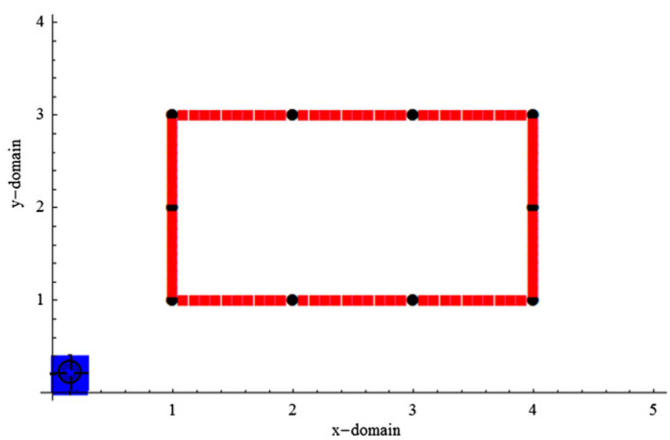


**Fig. 7.** Optimized clustered 10 node model.

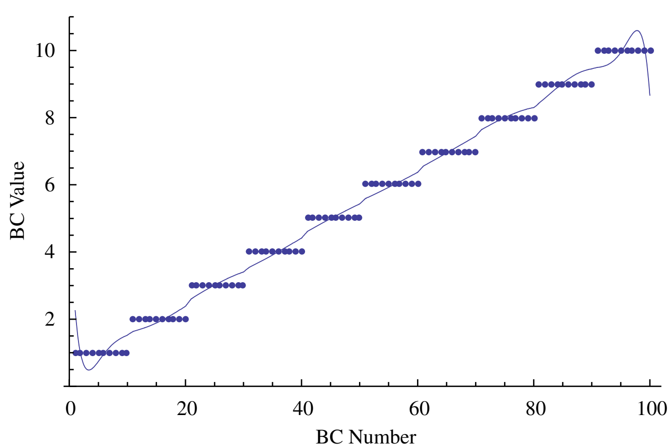


**Fig. 8.** Fit of function to boundary conditions. $L_2 = 26.5259$, $L_\infty = 1.26019$

increase in nodal point number used in the CVBEM approximation function is like increasing the dimension of the basis function set used in the generalized Fourier series expansion, decreasing approximation error accordingly. In Fig. 8, the modeling effort at that junction is displayed so as to demonstrate the CVBEM approximation function fit to the problem boundary conditions in a ten stepped discontinuous boundary condition problem with the use of only a few nodal points that have locations optimized by the presented procedure.

## 3. Conclusions

In this paper, a new approach to CVBEM (and BEM) modeling is presented using real time CVBEM program modeling error feedback. The modeling error feedback is provided by a new application developed in program Mathematica that enables a program user ‘‘click and drag’’ capability for repositioning CVBEM modeling nodes throughout the exterior of the problem domain. Using CVBEM modeling error measure feedback as a guide, produced in real time, the program user can ‘‘click and drag’’ any of the CVBEM modeling nodes while observing the CVBEM model being continuously rebuilt based upon the assembly of the latest nodal point locations, and the corresponding modeling error measures. In this fashion, an optimum positioning of the CVBEM modeling nodes can be obtained, in a new modeling interactive/feedback mode of operation. This modeling approach can be applied to other numerical techniques; particularly those that use functions that are position dependent, such as singular functions (i.e., negatively powered monomials, logarithms, etc). Because the CVBEM exactly solves the governing PDE over the problem domain, the modeling goal is simply the minimization of modeling error in matching problem boundary conditions, and, because the success in matching problem boundary conditions depends on the positioning of model nodal points, the provided approach for optimizing nodal point locations may be useful in developing better CVBEM (and BEM) models.

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## References

[1] Hromadka TV, Whitley RJ. Advances in the complex variable boundary element method. NY, USA: Springer-Verlag; 1998. 400 pages.
[2] Ramachandran PA. Method of fundamental solutions: singular value decomposition analysis. Commun Numer Methods Eng November 2002;18(11):789–801.
[3] Hromadka TV, Guymon GL. Complex polynomial approximation of the Laplace equation. ASCE J Hydraulic Eng 1984;110(3):329–39.
[4] Bohannon AW, Hromadka II TV. The complex polynomial method with a least-squares fit to boundary problems. Eng Anal Boundary Elem 2009, doi:10.1016/j.enganabound.2009.02.005.
[5] Fairweather G, Karageorghi A. The method of fundamental solutions for elliptic boundary value problems. Advances in Computational Mathematics 1998;9:69–95.