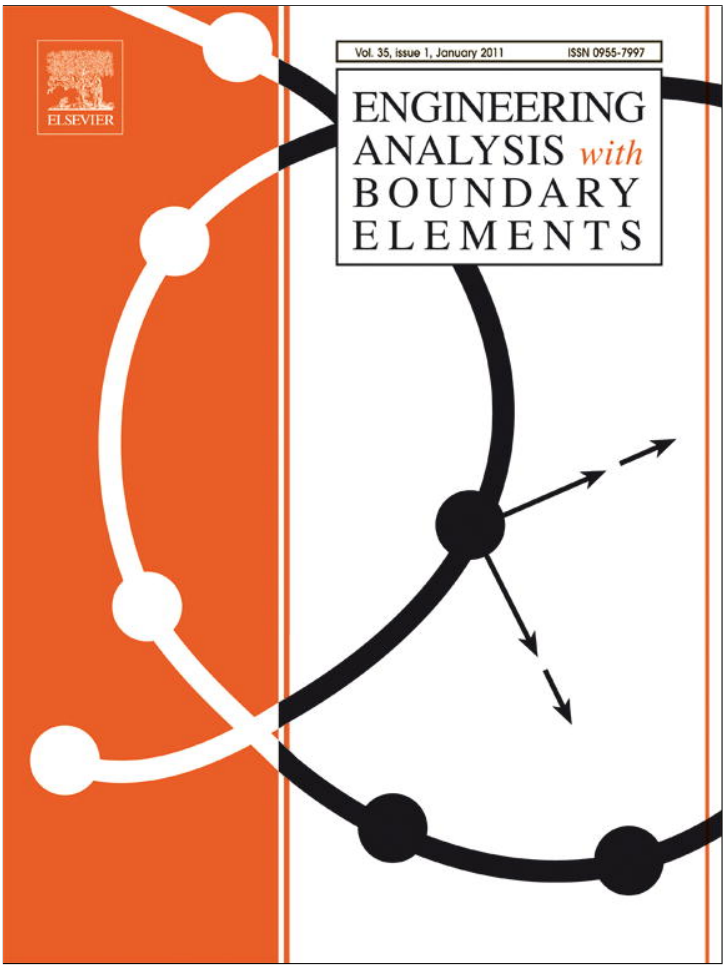

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An algorithm for optimizing CVBEM and BEM nodal point locations

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ABSTRACT

The Complex Variable Boundary Element Method or CVBEM is a numerical technique for approximating particular partial differential equations such as the Laplace or Poisson equations (which frequently occur in physics and engineering problems, among many other fields of study). The advantage in using the CVBEM over traditional domain methods such as finite difference or finite element based methods includes the properties that the resulting CVBEM approximation is a function: (i) defined throughout the entire plane, (ii) that is analytic throughout the problem domain and almost everywhere on the problem boundary and exterior of the problem domain union boundary; (iii) is composed of conjugate two-dimensional real variable functions that are both solutions to the Laplace equation and are orthogonal such as to provide the "flow net" of potential and stream functions, among many other features. In this paper, a procedure is advanced that locates CVBEM nodal point locations on and exterior of the problem boundary such that error in matching problem boundary conditions is reduced. That is, locating the nodal points is part of modeling optimization process, where nodes are not restricted to be located on the problem boundary (as is the typical case) but instead locations are optimized throughout the exterior of the problem domain as part of the modeling procedure. The presented procedure results in nodal locations that achieve considerable error reduction over the usual methods of placing nodes on the problem boundary such as at equally spaced locations or other such procedures. Because of the significant error reduction observed, the number of nodes needed in the model is significantly reduced. It is noted that similar results occur with the real variable boundary element method (or BEM).

The CVBEM and relevant nodal location optimization algorithm is programmed to run on program Mathematica, which provides extensive internal modeling and output graphing capabilities, and considerable levels of computational accuracy. The Mathematica source code is provided.

1. Introduction

The Complex Variable Boundary Element Method or CVBEM is a well documented numerical procedure that numerically solves boundary value problems involving the Laplace or Poisson equations in two-dimensional space (for example, see Hromadka and Lai [1], and Hromadka [2]). The CVBEM was subsequently extended to three dimensional and higher spatial dimensions by use of spatial projections and Hilbert space considerations [3]. More recently, a special issue of the journal Engineering Analysis with Boundary Elements highlighted new advances made with the CVBEM [4]. The CVBEM has been applied to a spectrum of problems spanning torsion in structural mechanics, to soil water phase change effects in porous media, to the analysis of heat and groundwater movement in two and three dimensions on arbitrarily shaped domains. Research into the underpinnings of the CVBEM continues along several lines of development.

In the current paper, the focus is on methods of adding nodal points in the CVBEM modeling effort such that the number of nodal points is kept small while retaining or improving modeling accuracy. Hromadka [2] describes an algorithmic procedure for adding nodes on the problem boundary such as to rapidly reduce modeling error in matching boundary conditions. However, in that referenced work, nodes are only added to the problem boundary to improve CVBEM accuracy (it is noted that the class of problems under study are potential problems and therefore the modeling magnitude of error will be a maximum on the problem boundary and, therefore, focusing on matching boundary conditions provides error containment within the problem domain enclosed by the problem boundary). The current paper develops an algorithm to optimize the location of CVBEM nodes not only on the problem boundary but also on the exterior of the problem domain union boundary. The algorithm initiates with the first node used in the CVBEM model, and then reduces approximation

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error by selecting the nodal location that results in a CVBEM approximation function that best fits problem boundary conditions. That is, the algorithm locates the nodal position, which results in the least modeling error in fitting problem boundary conditions. Holding the selected first node location fixed, the second node location is then determined that minimizes error in the 2-node CVBEM model. Then, holding the first two nodal locations fixed, the third node location is determined, and so forth.

The algorithm is implemented on the computer program Mathematica, which provides considerable advantages in coding effort reduction by use of the program's internal mathematical functions. The Mathematica output features and graphing capabilities also provide considerable advantages over compiled software languages such as FORTRAN, among others. Mathematica also provides a significant level of computational accuracy enhancement. The Mathematica code for this paper is provided in Supplementary Material. Example problem results are also provided to demonstrate the nodal placement algorithm.

It is noted that the developed algorithm may also be applied to real variable boundary element analogs such as those developed by Brebbia [5] among others.

2. Nodal placement optimization algorithm

Consider a two-dimensional problem domain that is simply connected and enclosed by a simple closed boundary, with the domain centroid located at the origin in R^2 (extension to three or higher dimensions follows directly as those extensions are a linear combination of two-dimensional CVBEM approximation functions [3]). The nodal placement optimization algorithm used in this paper is a modified version of the Mathematica code developed by Dean and Hromadka [4], and is described as follows.

Step 1 consists of finding an optimal nodal point on[1] the domain boundary. This is accomplished by subdividing the domain in angular fashion. Let $\theta_i, i=1,2,\ldots,n$, be the angular measurement of each subdivision i, where $\theta_n=2\pi$ and $\theta_1=\Delta\theta=\theta_{i+1}-\theta_i, \forall i$, thus creating equally spaced subdivisions. Associated with each θ_i is a point (r_i,θ_i) lying on footnote 1 the domain boundary. For $i=1,2,\ldots,n$, a CVBEM model approximation is determined for each candidate nodal point (r_i,θ_i) and the corresponding total error (or "integrated error") between the CVBEM model and the boundary conditions is computed. In determining the CVBEM model approximation, the logarithmic branch cut is defined to emanate from the candidate node along the ith radial at an angle of θ_i from the horizontal. The candidate nodal point with the lowest total error is identified and labeled $(\hat{r},\hat{\theta})$. This completes the first iteration of Step 1.

The second iteration of Step 1 refines the result obtained in the first. To do so, we consider only three of the original candidate nodes: $(\hat{r},\hat{\theta})$, and its two adjacent candidate nodes located at $\hat{\theta}+\Delta\theta$ and $\hat{\theta}-\Delta\theta$, thus narrowing our search for an optimal node to two intervals of consideration: $\lfloor \hat{\theta}-\Delta\theta,\ \hat{\theta} \rfloor$ and $\lfloor \hat{\theta},\ \hat{\theta}+\Delta\theta \rfloor$. Two additional candidate nodes are constructed by halving these two intervals, creating two new candidate nodes located at $\hat{\theta}-(\Delta\theta/2)$ and $\hat{\theta}+(\Delta\theta/2)$. Total error is calculated for these two nodes (again, with appropriate logarithmic branch cut), and compared to the error for the three remaining original node candidates. The point with the lowest total error is selected and renamed $(\hat{r},\hat{\theta})$.

Subsequent iterations of Step 1 are completed in like fashion, narrowing the search region down to two intervals: $[\hat{\theta}-(\Delta\theta/2^{j-2}),\ \hat{\theta}]$ and $[\hat{\theta},\ \hat{\theta}+(\Delta\theta/2^{j-2})]$ for iteration j, halving these intervals to create two new candidate nodal points, computing the total error, comparing these with the total error at the endpoints of the two intervals, and selecting the point with the lowest total error. This process is ended when the error improvement between subsequent iterations falls below some threshold, and the resultant optimal boundary nodal point is labeled $(\hat{r},\theta^*)$. This completes Step 1.

Step 2 consists of improving upon the result obtained in Step 1 by searching along a radial emanating from the domain centroid at an angle θ^* outside the domain boundary. This is accomplished by choosing an appropriate distance Δr, and m, the number of radial subdivisions. Total error is then calculated for each of the $m+1$ candidate nodal points $(\hat{r}+k\Delta r, \theta^*),\ k=0,1,2,\ldots,m^2$ and the point with the lowest total error is recorded and labeled $(\hat{r}_1,\theta^*)$.[3] This completes the first iteration of Step 2. Subsequent iterations proceed in much the same manner as in Step 1. For each iteration j, eliminate all but three candidate nodal points: $(\hat{r}_{j-1},\theta^*)$ and its two adjacent candidate nodes, $(\hat{r}_{j-1}-(\Delta r/2^{j-2}),\theta^*)$ and $(\hat{r}_{j-1}+(\Delta r/2^{j-2}),\theta^*)$. These three points form the endpoints of the two remaining subintervals, $[\hat{r}_{j-1}-(\Delta r/2^{j-2}),\ \hat{r}]$ and $[\hat{r}_{j-1}-(\Delta r/2^{j-2}),\hat{r}]$. These subintervals are halved, forming two new candidate nodal points, $(\hat{r}_{j-1}-(\Delta r/2^{j-2}),\theta^*)$ and $(\hat{r}_{j-1}+(\Delta r/2^{j-2}),\theta^*)$. The total error is calculated at these two new points, and compared to the other three remaining candidate points. The point with the lowest total is selected and labeled $(\hat{r}_j,\theta^*)$. Iterations are continued in this fashion until the error improvement between subsequent iterations falls below some threshold. The resultant optimal nodal point is labeled (r^*,θ^*), which is the optimal location for a single nodal point. This completes Step 2.

Step 3 results in pairing a second nodal point with (r^*,θ^*) from Step 2 to form the optimal *pair* of nodal points. This is accomplished by fixing (r^*,θ^*) and repeating Steps 1 and 2 to find a second point. In each case, the total error is computed using both nodal points. Step 2 is complete when the optimal companion node to (r^*,θ^*) is found.

The process then continues for three and more nodal points, until we have the number of nodes desired. Although the presented algorithm is based upon a simply connected domain, the algorithm can be extended to a multiply-connected domain under many situations. Currently, care is needed to ensure that CVBEM nodal point branch-cuts lie exterior of the problem domain. Consequently, multiply connected domains (e.g., including holes and so forth) become problematic due to the positioning of nodal point branch-cuts. One approach to handling the branch-cut issue is to split the problem domain into subdomains, where branch-cuts may lie exterior of the subdomains. This approach necessitates development of additional boundary conditions along the new "boundaries" created in the splitting process. Further research is needed in exploring application of the methods presented in the subject paper to multiply connected domains.

3. Implementation on program mathematica

The described algorithm is implemented on computer program Mathematica using a modified version of the Mathematica code found in Dean and Hromadka [4] as a basis.

The algorithmic coding additions were introduced into the referenced code with the final product shown in Supplementary

[1] Actually, a point ε distance beyond the boundary, for some small ε.

[2] Note $k=0$ corresponds to the optimal nodal location from Step 1.

[3] If min total error is observed at $\hat{r}=m\Delta r$ (i.e., at the point farthest from the domain centroid), we proceed to extend the search beyond this point until a min error can be found that is strictly between the domain centroid and max. radial distance considered.

Material. The coding requirements are not substantial due to the internal mathematical function capabilities of program Mathematica. Other such programs include MATLAB, among others.

4. Example problems

Example problem 1. Ideal fluid flow over a cylinder in a 90° bend

In order to demonstrate the advantage in considering nodal point location as a variable to be optimized in the CVBEM (and also the BEM), a problem is modeled involving ideal fluid flow of an incompressible irrotational fluid in a 90° bend over a unit radius cylinder. The exact solution to this potential problem is given by the complex variable analytic function $\omega(z)=z^2+z^{-2}$, where the resulting streamline along the problem's unit circle boundary forms a boundary of the problem domain. Because the exact solution is known, it can be used to assess the CVBEM model success afforded by optimizing node locations. The subject problem is modeled by the first quadrant of the exact solution display as shown in Fig. 1, which contains the solution equipotential contours (lines of constant potential function values) and streamline contours. For this example problem, the study domain is a simple circular region of radius one located in the first quadrant and slightly away from the cylinder (see Fig. 1). It is noted that the CVBEM develops not only an approximation of the potential function but also the conjugate streamline function. Furthermore, these two approximation functions are orthogonal and therefore can be directly used in building traditional flownets for further detail. Since both approximation functions exactly solve the Laplace equation, assessment of modeling error on the problem boundary will adequately measure modeling accuracy in the interior of the problem domain.

For this example problem, two cases are considered: (1) an 8 node CVBEM model (with all nodes located on the problem boundary), (2) a single node CVBEM model with the location of the single node optimized using the supplied algorithm. For both models, collocation is used to determine the relevant complex coefficient values of either approximation function.

The 8-node CVBEM model has modeling nodes evenly spaced along the circular problem boundary, shown as the points emitting branch cut lines in Fig. 2. Collocation points are also shown as the other points on the boundary in this figure. Modeling error in matching boundary conditions is shown in Fig. 3, where collocation occurs at the relevant evaluation point with shown identification (ID). For the single node model, numerous test locations were examined using the node location algorithm discussed above. Upon completing the algorithm, the optimum node location was determined to be positioned far to the upper right of the problem domain, located about 20-times the problem domain diameter away from the problem domain (see Fig. 4). In comparing modeling accuracy between the above two CVBEM models,

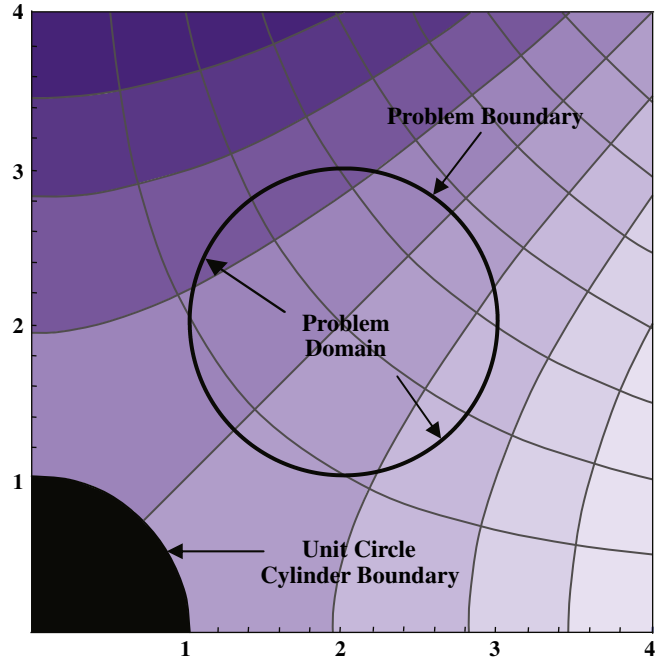

Fig. 1. Flownet of the Example Problem 1 Solution: The problem domain is the area enclosed in the problem boundary as shown. The problem domain is centered at (2,2). The unit circle cylinder boundary (shown as the boundary of the shaded quarter disk) is part of the boundary to the underlying ideal fluid flow around a 90 degree bend over a unit radius cylinder.

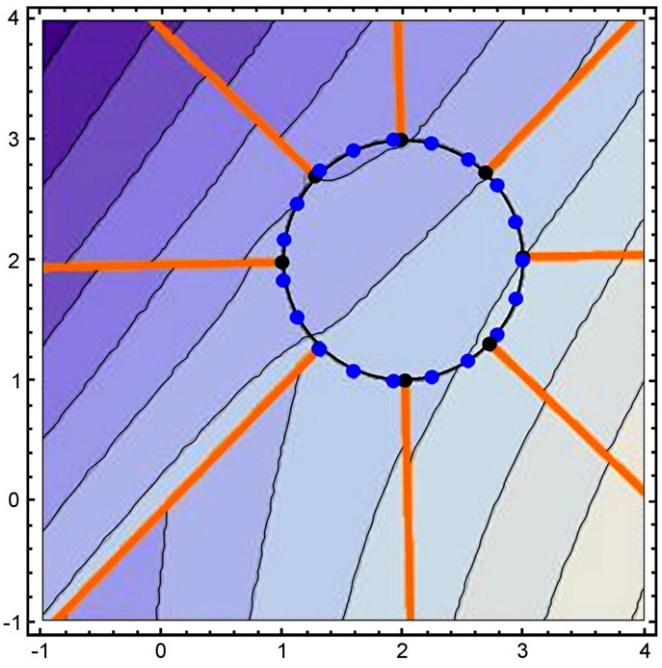

Fig. 2. 8-Node CVBEM Approximation. Nodes are shown with Branch cuts extending outwards.

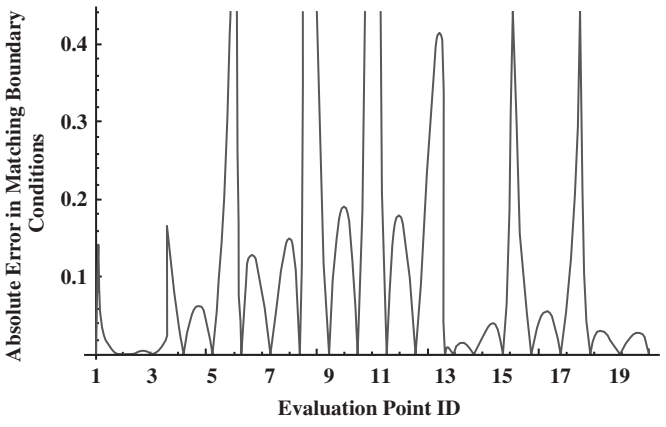

Fig. 3. Error in matching the boundary conditions using 8-Node CVBEM Approximation. (Integrated Error=.135).

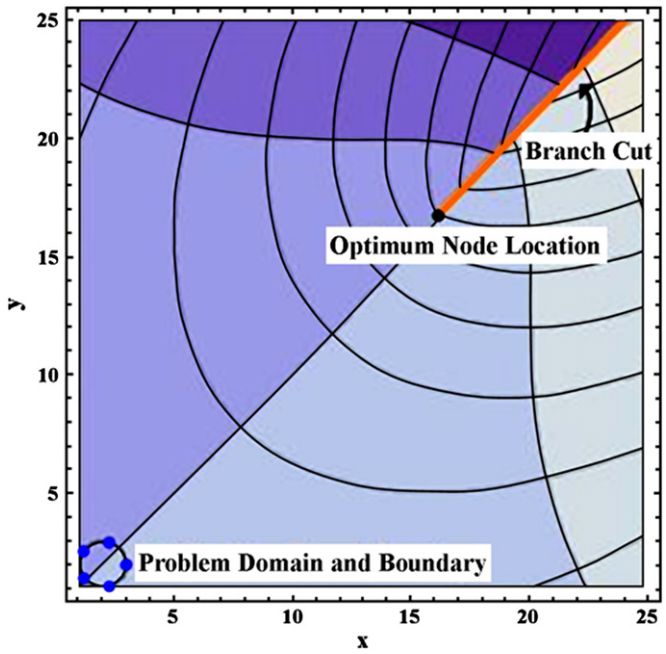

Fig. 4. Optimally placed single node using optimization algorithm. The node is shown far from the boundary with its corresponding branch cut.

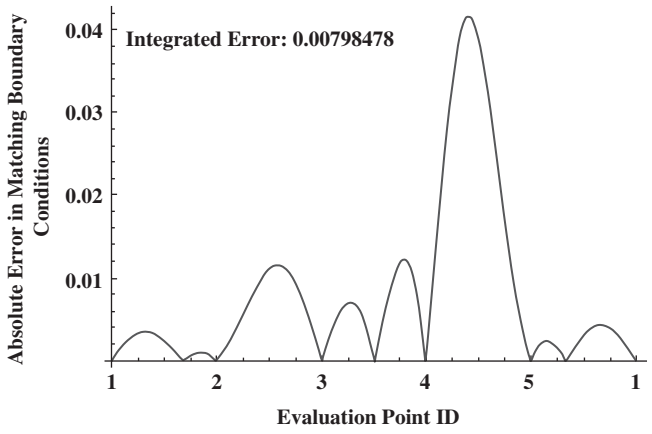

Fig. 5. Error in matching the boundary conditions using single node CVBEM approximation with optimized location. (Integrated Error=.008).

the single node CVBEM model (with optimized node location) provided significantly (17 times) better accuracy than did the 8-node CVBEM model. Consequently, consideration of node locations not only on the problem boundary but also on the exterior of the problem domain union boundary provides significant improvement of computational modeling accuracy. For the single node CVBEM model, 5 collocation points are necessary in order to determine complex coefficients for the CVBEM single node model (see [1,2]). A plot of modeling error in matching boundary condition values of the potential function is provided in Fig. 5.

The single node CVBEM model can be further improved by introducing a second nodal point and again employing the subject location optimization algorithm. The resulting two-node CVBEM model provided further significant improvement in modeling accuracy (see Appendix A: Figs. 7 and 8). Therefore, including the nodal point location as part of the modeling solution effort provides significant improvement in modeling accuracy, resulting in the reduction in the number of nodes required to achieve the similar level of modeling accuracy as obtained using many more boundary nodal points.

Example problem 2. Ideal fluid flow around a 90° bend

For comparison purposes, the described algorithm is used in the same example problem presented in Dean and Hromadka [4]. For this example, it is known that the function $\omega(z)=z^2$ is the solution for ideal fluid flow around a 90° corner. As in Dean and Hromadka, the problem boundary will be defined as the unit square with corners at (1,1) and (2,2). This example differs from that shown in Dean and Hromadka [4] in that where Dean and Hromadka [4] used 8 nodal locations on the problem boundary, this paper will use only one node.

Before running the algorithm to determine the optimized nodal location, the Dean and Hromadka approximation was analyzed to get a comparable modeling error. Fig. 6 shows the absolute error in matching the boundary conditions from the Dean and Hromadka [4] solution, which had a total error of .004 on the boundary. Note that this model achieves significantly low error with 8 nodes.

The error achieved by the algorithm for example problem 2 is .00001. The algorithm places the single node at polar coordinates (3.837 rad, 20,000) from the problem domain center. The integrated error that the algorithm achieves with one node is 400 times smaller than that of the cited regular CVBEM model with 8 nodes evenly spaced on the boundary.

The example problems presented focus on the CVBEM computational results obtained by the optimization of nodal point locations for the simple case of using only one node in the modeling process. From the examples, considerable modeling accuracy is achieved by including the node location as another degree of freedom to be optimized. For more than one node, the procedure used by the authors is simply to include additional nodes to the problem solution, one node at a time, with node location optimization accomplished after each node is added to the set of nodes. Currently, the node locations already optimized are held fixed with the addition of another node. (Research is needed to ascertain whether all nodal points should be re-optimized with the addition of another node.) As may be expected, the second node added to the model solution provides significant CVBEM modeling accuracy improvement by optimizing the second node location (while holding the first node optimized location fixed), but the improvement in location optimization is typically not as significant as observed with the first node location optimization. For many problems, depending

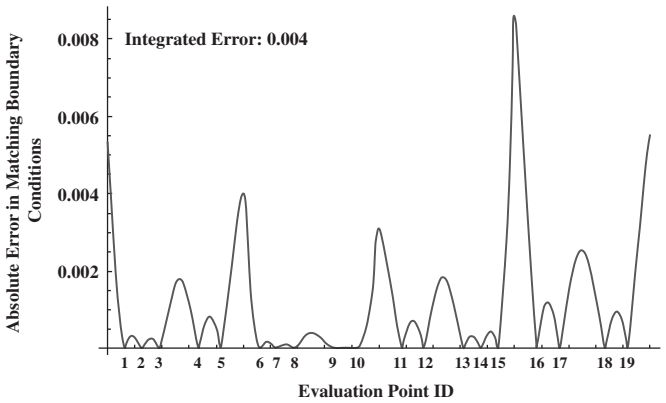

Fig. 6. The error on the boundary using the Dean and Hromadka [4] model.

on domain configuration and boundary condition complexity, the advantages afforded by including additional nodes and optimizing their locations, in reducing modeling error in matching problem boundary conditions, reduce with the inclusion of additional nodes.

In assessing the utility of optimizing nodal point locations, other tests considered include adding nodal points, one at a time, optimizing each new added nodal point location while holding the prior optimized nodal point locations fixed. As expected, the first nodal point included in the CVBEM model had the greatest reduction in modeling error due to nodal point location optimization. Each additional nodal point added reduced modeling error, but demonstrated, in general, less modeling accuracy improvement with respect to optimization of added nodal point location. In all cases considered, nodal point location optimization, particularly when location optimization occurs exterior of the problem domain, resulted in a significantly more accurate model than had the nodes been restricted to be located only on the problem boundary, as is the typical situation with boundary element techniques. The relative modeling improvement not only depended on the number of nodes already optimized (with respect to location) and in-place in the problem region, but also on the complexity of problem boundary conditions. A third factor to be considered is the problem domain geometry itself, namely, more complicated problem geometries, involving severe angle points and multiple segmented geometries, result in greater modeling improvement by optimization of added nodal point locations than when the problem domain is a simpler convex domain geometry. Further research is needed to qualify and quantify these considerations, among others, when predicting CVBEM modeling performance improvement due to the optimization procedures presented.

5. Conclusions

In this paper, an algorithm is presented that facilitates the search for placement of Complex Variable Boundary Element Method (CVBEM) or real variable Boundary Element Method (BEM) nodal points on the problem boundary or exterior of the problem domain union boundary. For CVBEM and BEM trial functions involving singularities, such as logarithmic functions or similar type of basis functions, the location of these trial function singularities (or "nodes") is shown to result in significant differences in resulting approximation functions and their modeling accuracy. In the past, the general procedure for use of CVBEM and BEM modeling of boundary value problems (such as the Laplace or Poisson equations, for example) is to locate nodes directly on the problem boundary. In this paper, however, it is shown that locating nodes exterior of the problem domain (i.e., exterior of the problem domain union problem boundary) can result in considerable improvement in modeling accuracy. Consequently, optimization of nodal point locations provides another dimension in improving CVBEM and BEM approximation accuracy. If the basis functions satisfy the governing partial differential equation (such as achieved by the real and imaginary parts of the CVBEM approximation analytic function for the Laplace and Poisson equations), then the approximation function development effort reduces to minimizing modeling error in matching problem boundary conditions. The presented algorithm optimizes node locations by assessing individual node location versus success in matching problem boundary conditions, optimizing one nodal point location at a time. The paper demonstrates the algorithm using two problems where exact solutions are known. The algorithm is implemented on computer program Mathematica.

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Several faculty and Cadets are acknowledged in this research effort including, but by no means limited to, COL Michael Phillips, Department Head, Department of Mathematical-Sciences, United States Military Academy at West Point, New York; COL Steven Horton, Deputy Department Head; LTC Michael Jaye; MAJ Anthony Williams; Cadet Thomas Dean and all of the United States Military Academy at West Point.

Appendix A

See Figs. 7 and 8.

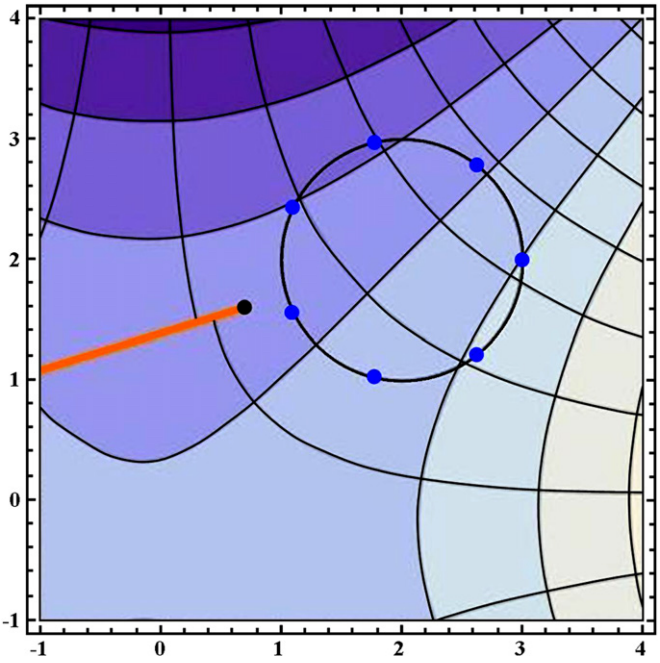

Fig. 7. Solution obtained using the algorithm to place 2 nodes in Example 1. Visible here is the problem domain, the new set of evaluation points and the 2nd node and branch cut location. The first node is located as shown in Fig. 4. The node here is shown with its corresponding branch cut.

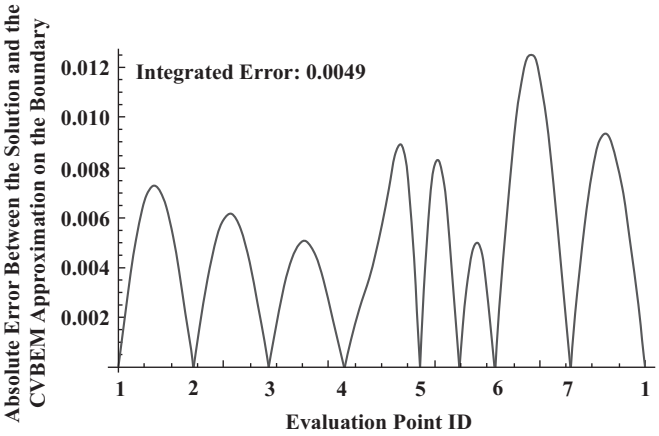

Fig. 8. Resultant modeling error on the problem boundary using the algorithm to locate 2 nodes in Example 1. This 2 node approximation produces a little over half of the error of the one node approximation (see Fig. 5).

Appendix B. Supplementary materials

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.enganabound.2011.11.008.

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