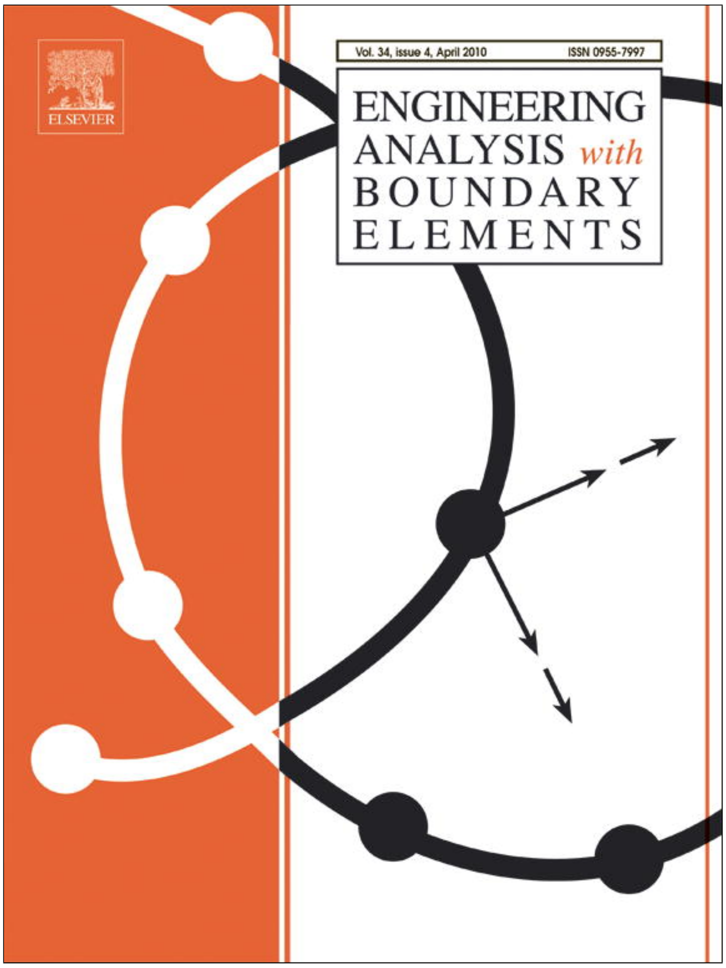
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Technical Note

A collocation CVBEM using program Mathematica

T.R. Dean [a], T.V. Hromadka II [b,*]

[a] Department of Electrical Engineering and Computer Science, United States Military Academy, West Point, NY 10996, USA
[b] Department of Mathematical Sciences, United States Military Academy, West Point, NY 10996, USA

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ABSTRACT

The well-known complex variable boundary element method (CVBEM) is extended for using collocation points not located at the usual boundary nodal point locations. In this work, several advancements to the implementation of the CVBEM are presented. The first advancement is enabling the CVBEM nodes to vary in location, impacting the modeling accuracy depending on chosen node locations. A second advancement is determining values of the CVBEM basis function complex coefficients by collocation at evaluation points defined on the problem boundary but separate and distinct from nodal point locations (if some or all nodes are located on the problem boundary). A third advancement is the implementation of these CVBEM modeling features on computer program Mathematica, in order to reduce programming requirements and to take advantage of Mathematica's library of mathematical capabilities and graphics features.

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1. Introduction

The complex variable boundary element method (CVBEM) has been reported in the literature extensively, with a recent Special Edition of the Journal of Engineering Analysis with Boundary Elements (see volume 30 (2006), issue 12) dedicated to the recent advances in CVBEM. The CVBEM is useful in modeling problems involving LaPlace or Poisson equations for an arbitrary target potential function (i.e., temperature, ideal fluid flow, electrostatics, groundwater flow, among many other topics). The underpinning of the CVBEM is the well-known Cauchy integral formula, which relates values of an analytic complex function in the interior of a simply connected domain to a contour integral of values of that analytic function (or continuous function) defined on a simple closed contour boundary that encloses the subject domain.

The CVBEM utilizes a set of analytic functions as basis functions and then formulates a vector space of linear combinations of these basis functions using complex coefficients. Such linear combinations are therefore also analytic functions, where the real and imaginary parts of these linear combinations are two-dimensional real-valued functions, both satisfying the LaPlace equation over the entire problem domain and also on the problem boundary. Depending on the type of analytic basis functions used, branch-cuts or other singularities may be involved, which can be handled by locating relevant singularities outside the problem domain and perhaps also outside and displaced from the problem boundary. As a result, the real and imaginary components of the approximation function satisfy the LaPlace equation inside, on, and outside the problem boundary except at singularity points and related branch-cuts, if any. For example, Bohannon and Hromadka [1] use complex monomials as the basis function set (complex polynomial method) and develop approximation functions that are analytic in the entire plane, which do not involve singularities or branch-cuts anywhere in the plane. The CVBEM formulation results in a convex combination of products of complex polynomials with complex logarithms. This product of complex analytic functions can be evaluated at the nodes (i.e., the branch points of the complex logarithms) because the limit exists as the arbitrary coordinate point z approaches the node (if located on the problem boundary) from the interior of the problem domain. The limiting value can also be examined by the usual L'Hopital Rule derivation.

The idea of using series expansions of functions, such as the underpinning of the CVBEM, or the well-known Fourier Series expansions, or other such series expansions, is an important concept in the approximation of functions and partial differential equations such as the LaPlace equation, among other important relationships in engineering and science. Viewed in terms of a series expansion, the CVBEM approach presented in this work considers nodal points located in the exterior of the problem domain union boundary, which is a novel technique in such boundary element modeling. But the idea of expanding series about points located in the exterior of a problem domain union boundary is not new and is well examined in various works, including the work of Wang et al. [2], where the target problem domain is "embedded" within a larger circular domain

* Corresponding author. Tel.: +1 845 938 2036.
E-mail address: tedhromadka@yahoo.com (T.V. Hromadka II).

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(or other appropriate geometric shapes), on which a series expansion can be more readily developed, such as the Fourier type, among other possible expansions. When viewed as a series expansion, the CVBEM parallels other series expansions and the CVBEM can take advantage of other such important concepts such as those found in the embedding approach for modeling, among other concepts.

In the current work, basis functions of the form $(z-z_j)Ln(z-z_j)$ are used, where z_j are nodal points defined perhaps on the problem boundary or exterior of the problem boundary union problem domain. (Motivation for using such basis functions follows by the numerical integration of linear polynomial basis functions in the Cauchy integral formula. See Hromadka and Whitley [3].) The natural logarithm function is defined at each nodal point such that the branch-cut is positioned to lie entirely exterior of the problem domain union problem boundary. Therefore, the linear combination of such basis functions form the CVBEM approximation function, which is analytic everywhere except at nodal points z_j and the relevant branch-cuts. The real and imaginary parts of the CVBEM approximation function both exactly satisfy the LaPlace equation, where the approximation function is analytic as mentioned previously.

Complex coefficients of the form a_j+ib_j are used with each basis function where i is the usual square root of -1, and "a" and "b" are real constants to be determined under some selected rule to satisfy boundary condition values. In the current work, each basis function has two real-valued coefficients to be calibrated for each basis function. Additionally, the term $a_0+ib_0+z(a_1+ib_1)$ is added to the basis functions to increase the accuracy of the model.

The use of the complex monomial basis functions 1 and z improves modeling capability because these monomials are part of the CVBEM expansion resulting from using a linear trial function interpolation in the numerical integration of the Cauchy integral equation. It is noted that elimination of these complex monomials will still result in a complex analytic approximation function. It is also noted that had a higher order interpolation polynomial been used in the numerical integration of the Cauchy integral equation, a higher order set of complex monomials would result in the CVBEM formulation (see [3]).

For N nodal points defined in the approximation function, there are $2N+4$ real-valued coefficients to be determined. The rule used in the current work is to equate the approximation function to the known boundary condition values at $2N+4$ "evaluation points" located on the problem boundary. The resulting CVBEM approximation function will therefore match boundary conditions at each of the $2N+4$ evaluation points located on the problem boundary, and the real and imaginary components of the CVBEM approximation function will both satisfy the LaPlace equation within the entire problem domain and also on the problem boundary (for the situation where all nodal points and associated branch-cuts are located exterior of the problem domain union boundary). This approach for evaluating basis function coefficients differs from the usual CVBEM collocation approach, where coefficients are determined at nodal points in the limit, and nodes are defined on the problem boundary. Again, in the current work, all nodes are displaced away from the problem domain union boundary.

In locating nodal points, it is advantageous to locate nodes outside the problem domain. This is because the CVBEM basis functions are products of complex polynomials and complex logarithm functions (developed from the Cauchy integral numerical solution, see [3]) and therefore the complex logarithm functions involve branch-cuts that cause discontinuities along the branch-cuts. Locating nodes outside the problem domain (i.e., along the problem boundary or outside the problem domain union problem boundary) removes the complex logarithm branch point (located at the node) from the problem domain, and by rotating the branch-cut to lie exterior of the problem domain union boundary results in the CVBEM approximation being analytic throughout the problem domain. In the examination of node locations exterior of the problem domain union boundary, estimation of resulting approximation error in matching problem boundary conditions is used to locate node locations where the total absolute value of error is minimized in matching boundary conditions integrated along the problem boundary. Because error is a maximum on the problem boundary, such an error assessment bounds the modeling error in the problem domain itself. In the current work, node locations are examined by trial and error until a minimum value of total error on the boundary is achieved. Then, holding the target node fixed, the next node is introduced and its location determined by trial and error, for positions on and exterior of the problem boundary.

Error analysis proceeds by assessment of goodness of fit between approximation function values evaluated on the problem boundary versus boundary condition values defined over the entire problem boundary. Because of the collocation technique being used, there is no error in matching boundary condition values at the $2N+4$ evaluation points. Furthermore, there is no modeling error in satisfying the governing partial differential equation (i.e., the LaPlace equation or many forms of the Poisson equation) throughout the problem domain because the approximation function is analytic over the entire problem domain (this feature of the CVBEM is not achieved by domain methods such as finite difference or finite element). Boundary condition values can be evaluated continuously and exactly for the approximation function over the entire problem boundary for direct comparison to the given boundary condition values. Additional interpolation is not needed. Because of the Maximum Modulus Theorem, the maximum error of approximation throughout the problem domain and boundary occurs on the problem boundary when using the CVBEM, and therefore only the problem boundary needs to be assessed in order to evaluate modeling error magnitude. (Other methods to reduce modeling error include least-squares type minimization, which is a topic for future research).

In the following, further details of the implementation of the above procedures are presented along with Mathematica code and example problems. Graphics are presented using program Mathematica, demonstrating the convenience afforded by programs of that type. Other such programs include, but are not limited to, MatLAB and Maple, among others.

2. Problem formulation

Given a two-dimensional simply connected domain enclosed by a simple closed contour boundary, "N" CVBEM basis function nodes are defined in the exterior of the problem domain union boundary. Branch-cuts are located for each node to lie in the exterior of the problem domain union boundary. Holes in the problem domain or other such non-homogeneities can be handled as is accomplished in the usual CVBEM [3]. For N nodes used in the CVBEM approximation function, $2N+4$ evaluation points are located on the problem boundary (which contains no nodes). For presentation purposes, the LaPlace equation is assumed to be applicable with only boundary conditions of the potential known. (Flux type boundary conditions or stream function boundary conditions can also be assumed in this modeling approach, involving both the real and imaginary components of the CVBEM approximation function.) Since we know only the potential boundary conditions, the solution of the flux will be left with some arbitrary constant, b_0, which here will be assumed to be zero. This will reduce the number of evaluation points needed to $2N+3$. The relevant boundary value problem satisfies the LaPlace equation over the problem domain with the potential function values given on the problem boundary.

At each evaluation point location, the value of the potential is given as a boundary condition. The modeling procedure now moves

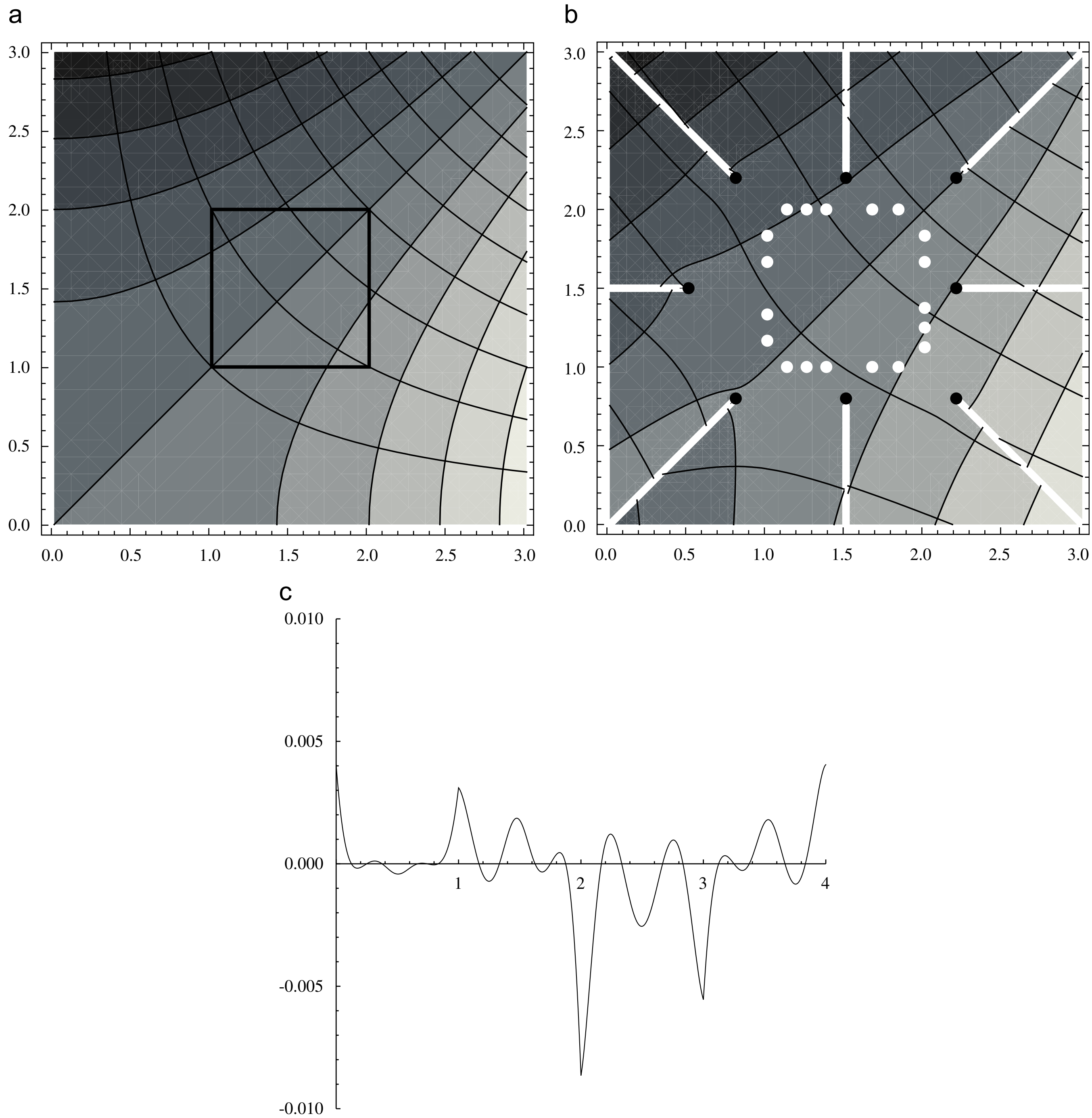


Fig. 1. (a) Exact solution of z^2 representing ideal fluid flow around a 90° bend. The real portion of the function (equipotential lines) is shown by the shaded contours. The imaginary portion (streamlines) is shown by the solid lines. The black box represents the problem domain for the CVBEM approximation. (b) CVBEM approximation of the function z^2. White dots represent the evaluation points where the approximation is forced to match the function. Black dots represent the nodal points and the white lines represent branch-cuts of the logarithms (note: the evaluation points all lie on the problem boundary shown in Fig. 1(a). (c) Error in the potential function along the problem boundary in a counter-clockwise direction. The maximum and minimum magnitudes of error occur along the boundary as guaranteed by the Maximum Principle.

towards solving a matrix system to determine the $2N+3$ CVBEM real-valued coefficient components by collocating the approximation function to equal the boundary condition values at each of the $2N+3$ evaluation points. After solving the square $(2N+3)\times(2N+3)$ matrix system, the real-valued coefficients are substituted back into the underlying CVBEM approximation function, resulting in an analytic function defined over the problem domain union problem boundary. Furthermore, these same $2N+3$ coefficient real values can be directly used to develop the conjugate stream function, for use in developing flow nets and other graphical plots. Further information on the CVBEM formulation is provided in the following discussions of the included program Mathematica code, demonstrated by example problems.

3. Mathematica code development

Previous methods of implementing the CVBEM have translated the basis functions into a polar coordinate system. Since Mathematica has many built in features to handle functions of

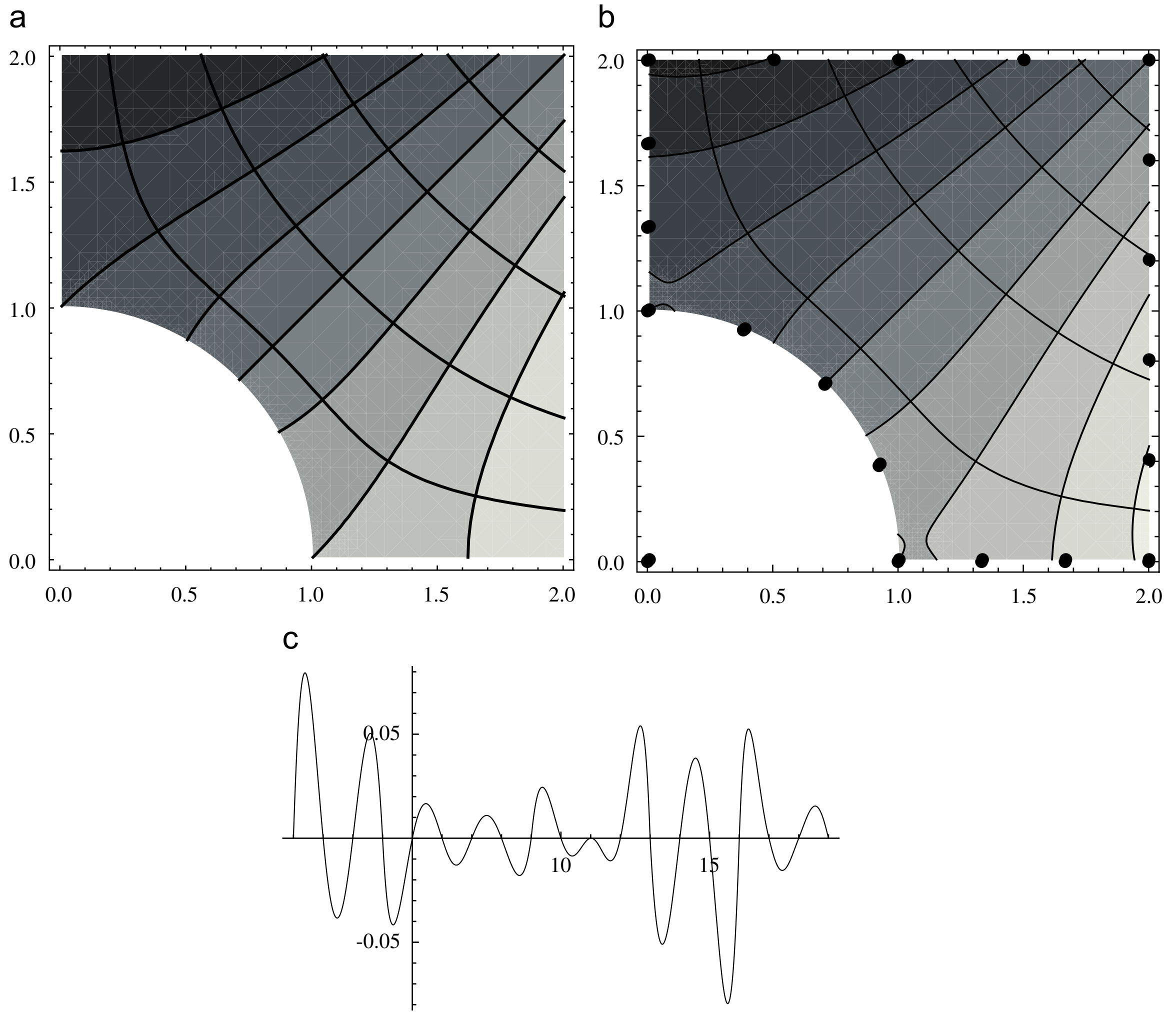


Fig. 2. (a) Exact solution of z^2+z^{-2}. This problem represents ideal fluid flow around a cylinder in a 90° bend. The real portion of the function (equipotential lines) is shown by the shaded contours. The imaginary portion (streamlines) is shown by the solid lines. The cylinder is located at the origin and has a radius of 1. (b) CVBEM approximation of z^2+z^{-2}. Dots around the problem boundary are collocation points. (c) Error in the potential function along the problem boundary. The maximum error across the entire domain is 0.0798 units.

complex numbers, the basis functions are easier handled in a rectangular coordinate system. In order to be able to leave this function in rectangular coordinates and still rotate branch-cuts to ensure that our entire domain remains analytic, we must apply a rotation at each node. This is accomplished by making a substitution for each x and y variable within the summation:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

This function is then expanded into its real and imaginary components using the ComplexExpand[] function. In order to get the solution specific to our problem, we must now evaluate the function at the locations where our boundary is known and then solve for our coefficients. This is done through generating tables and then using Mathematica's ability to solve matrices.

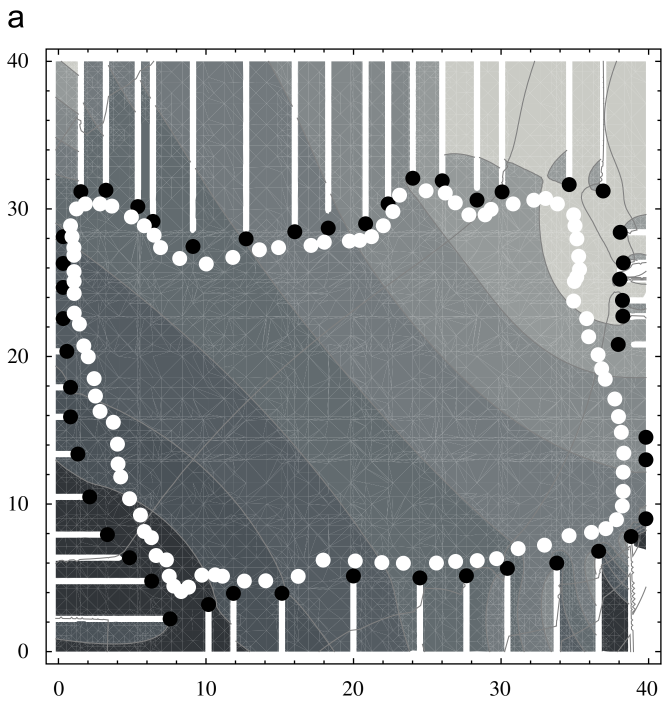
In order to use this code, the user must simply manipulate the arrays that appear in the first several lines of the code. The first array of points represents the coordinate of each node and the second represents the angle at which the branch-cut is rotated in a counter-clockwise direction from the negative x-axis. The third array of points is the location at which the boundary is to be evaluated. The final array represents the boundary values at these locations. The full Mathematica code is found in the appendix.

3.1. Language advantages

Mathematica contains a very powerful symbolic language. This can be leveraged to make an almost entirely symbolic evaluation of the CVBEM and leave the problem in rectangular coordinates. Previous implementations of the CVBEM have instead transformed the domain into polar coordinates. It is also very easy to manipulate functions and matrices in Mathematica. The combination of these two features makes symbolically evaluating the approximation point at each evaluation point a very simple task. Once the appropriate matrices are formulated, Mathematica's built-in functions for solving linear equations can be utilized to obtain a solution for all the necessary coefficients.

3.2. Output features

Perhaps one of the most powerful features of Mathematica is the tools it contains to visualize data. Through very simple commands, Mathematica can create many different types of plots of functions with one or two independent variables. Mathematica also makes it very easy to manipulate plots and add features to them. Two very



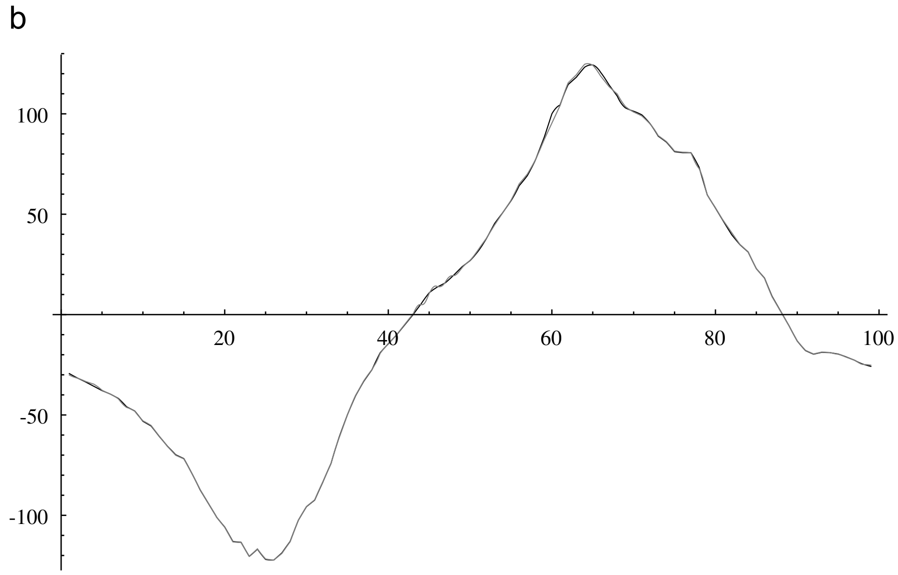


Fig. 3. (a) CVBEM solution modeling ground water levels based on bore readings around the boundary. White dots represent locations of bore readings, which is where the collocation points occur. Black dots represent the locations of the nodes (note: collocation points all lie on the problem boundary and nodes lie outside the problem domain). (b) Potential values along the boundary as modeled by the CVBEM (grey) and the CPM (black). The model differs by no more than two units across the entire domain.

powerful features that are useful in visualizing solutions found via the CVBEM are the contour and density plots, which it can create.

4. Example problems

4.1. Problem 1

The function z^2, which exactly represents ideal fluid flow through a 90° bend, will be analyzed in the first quadrant using the collocation CVBEM presented. The problem domain is a unit square with corners at (1,1) and (2,2). For this example problem, 19 evaluation points are specified along the problem boundary where boundary conditions are matched, and 8 CVBEM basis functions (as described above) are used in the CVBEM analog. This means that 8 CVBEM nodes and branch-cuts are used in the model. Figs. 1a–c show the various details of the CVBEM model.

The effect of the logarithm branch-cuts (as discontinuities) can be seen in the region located exterior of the problem domain. It can also be seen that within the problem domain, streamlines and equipotential lines and corresponding values are known continuously throughout the problem domain as well as on the problem boundary. Furthermore, the CVBEM approximation function exactly solves the governing partial differential equation (LaPlace equation) throughout the problem domain. Because the CVBEM approximation is a function defined throughout the entire plane, the approximation function values are known continuously along the problem boundary, which enables a direct comparison to be made with the problem boundary conditions. Flux values can also be directly determined anywhere in the problem domain and on the problem boundary using the Cauchy–Riemann equations to compute normal flux as the gradient of the approximation stream function.

Since $\phi(z)$ and $\hat{\phi}(z)$ are both real components of respective analytic functions, they are both two-dimensional real-valued harmonic functions over the entire problem domain and therefore the Maximum Principle (analogous to the Maximum Modulus Theorem for analytic functions) applies. Because the difference $\phi(z)$-$\hat{\phi}(z)$ is also a harmonic function in the problem domain, the Maximum Principle applies and therefore the maximum error of approximation occurs on the problem boundary, Γ. A good way to visualize the amount of error in the function is therefore to look at the error along the boundary. Fig 1c shows the error in the potential along Γ moving in a counter-clockwise direction starting at (2,1).

4.2. Example problem 2

The function z^2+z^{-2} represents ideal fluid flow around a cylindrical corner. The CVBEM approximation model used in Example 2 is based on the same number of evaluation points, nodes, and branch-cuts as used in Example 1. Figs. 2a–c show various details of the model.

4.3. Example problem 3

The CVBEM can also be used to solve much more complex problems. In this problem, taken from the Journal of the Professional Geologist [4], the CVBEM is used to model a groundwater basin where several ground water supply wells are in use. The groundwater flow within the domain is assumed to be homogeneous and isotropic. Non-homogeneous or anisotropic properties can be accounted for by re-scaling or solving simultaneous sub-problems (see [3] for examples). Once groundwater levels are found from groundwater wells located along the problem boundary, the CVBEM can be used to approximate the ground water levels within the domain.Fig. 3a shows the result of this approximation. In this figure, the white dots represent the well sites and the black dots represent the nodes. Lighter colors represent higher water levels.

Fig. 3b shows the water levels along the boundary as found by both the CPM approximation presented in the journal and the CVBEM presented here. The differences between CVBEM approximation values and known boundary condition values of groundwater levels evaluated along the problem boundary are the largest when CVBEM nodes are located on the problem boundary (CVBEM nodes should be located on the problem boundary or exterior of the problem boundary union domain in order to avoid branch-cut discontinuities). This difference in modeling versus known boundary condition values can be readily reduced by simply moving the CVBEM nodes to be exterior of the problem domain union boundary when constructing the CVBEM approximation function. However, in order to show maximum modeling error in this paper, the CVBEM nodes are placed on the problem boundary, which provides a more informative assessment of CVBEM modeling error (if nodes are specified on the problem boundary). Furthermore, the CVBEM modeling error in matching boundary condition values is due to the internal approach used in Mathematica in handling the product of the functions $(z-z_j)$ and $Ln(z-z_j)$. Mathematically, L'Hopital's Rule can be used to derive the limiting value of the product $(z-z_j)Ln(z-z_j)$, which results in no similar modeling error at nodal points. However, the product itself is not evaluated in this manner in the limit by the computer software, and therefore the modeling error manifested is due to the software.

5. Conclusions

The complex variable boundary element method (CVBEM) is extended for using collocation points not located at the usual boundary nodal point locations. The CVBEM is improved to vary in location, impacting the modeling accuracy depending on chosen node locations. The CVBEM basis function complex coefficients are determined by collocation at evaluation points defined on the problem boundary in-between the CVBEM nodal points, which is an advancement over the usual collocation approach of using limits at nodal points defined on the problem boundary. A further advancement is the implementation of these CVBEM modeling features on computer program Mathematica. Example problems are provided that demonstrate this new CVBEM procedure. Mathematica code is also provided, showing the brevity of programming requirements.

Further research is needed in a variety of areas, including different methods for assessing and reducing modeling error in matching problem boundary conditions. One particular method that appears to be most promising is the graphical demonstration of modeling error in matching boundary conditions by the ''approximate boundary'' technique [3]. Program Mathematica may be particularly useful in assessing error by the approximate boundary approach due to the available graphical options inherent to that program.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.enganabound.2009.10.007.

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