Approximating Solutions to the Dirichlet Problem in R^N Using One Analytic Function

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A simpler proof is given of the result of (Whitley and Hromadka II, Numer Methods Partial Differential Eq 21 (2005) 905–917) that, under very mild conditions, any solution to a Dirichlet problem with given continuous boundary data can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This can be applied to give a method for the numerical solution of potential problems in dimension three or higher. Numer Methods Partial Differential Eq 26: 1636–1641, 2010

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I. INTRODUCTION

A new method for the numerical solution of the Dirichlet problem was given in [1] in which it was shown that under mild conditions on a bounded open set Ω any solution to a Dirichlet problem with given continuous boundary data on $\partial\Omega$ can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This approximation has a form simple enough that it can be used in the numerical solution of Dirichlet problems in dimension three or higher. The proof given here, a substantial simplification of that given in [1], is obtained by proving the theorems of [1] in reverse order.

II. HARMONIC POLYNOMIALS

A complex-valued polynomial $P(x)$ on R^N can be written using the standard multi-index notation: $\alpha_1, \alpha_2, \ldots, \alpha_N$ non-negative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $x = (x_1, x_2, \ldots, x_N)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$, as the finite sum $P(x) = \sum_\alpha C_\alpha x^\alpha$. This sum can be written as $P(x) = \sum P_m(x)$, where $P_m(x) = \sum_{|\alpha|=m} C_\alpha x^\alpha$, with $|\alpha| = \alpha_1 + \cdots + \alpha_N$, is a homogeneous polynomial of

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degree m, i.e., $P_m(\lambda x) = \lambda^m P_m(x)$ for λ real. In this way of writing $P(x)$, the polynomials $P_m(x)$ are uniquely determined and $P(x)$ is harmonic if and only if each $P_m(x)$ is harmonic, all of which follows directly upon substituting λx for x and considering the resulting polynomial in λ [2, pp. 23–24]. Consequently, in studying harmonic polynomials on R^N it generally suffices to consider the harmonic polynomials on R^N which are homogeneous of degree m, denoted by $\mathcal{H}_m(R^N)$.

As noted in [3, 2.1], and [2, pp. 81–82], the space $\mathcal{H}_m(R^2)$ has a "very simple form" in that it has a basis consisting of the functions $Re(x_1 + i\ x_2)^m$ and $Im(x_1 + i\ x_2)^m$, which using complex-valued functions can be written

$$\mathcal{H}_m(R^2) = sp\{(x_1 + ix_2)^m, (x_1 - ix_2)^m)\}$$

The purpose of this section is to establish a similar representation for general N. The following elementary lemma [1] is useful.

Lemma 1. *Let h be an harmonic function of two variables defined on an open set U in R^2, and let a and b be two perpendicular vectors, $a \cdot b = 0$, of equal length $|a| = |b|$ in R^N. Then $h(a \cdot x, b \cdot x)$ is an harmonic function for x in R^N and $(a \cdot x, b \cdot x)$ in U.*

Proof. Set $H(x) = h(a \cdot x, b \cdot x)$. The result follows from the computation of the Laplacian $\Delta H(x)$:

$$h_{11}(a \cdot x, b \cdot x) \sum_1^N a_j^2 + h_{22}(a \cdot x, b \cdot x) \sum_1^N b_j^2 + 2h_{12}(a \cdot x, b \cdot x) \sum_1^N a_j b_j. \tag{1}$$

∎

Let the set A_N consist of the pairs a,b of orthogonal points in R^N, $a \cdot b = 0$, having unit length, $|a| = |b| = 1$, and define the complex vector space spanned by functions on R^N of the form $(a \cdot x + i\ b \cdot x)^m$:

$$\mathcal{A}_m^N = sp\{(a \cdot x + i\ b \cdot x)^m : \text{for pairs } a, b \text{ in } A_N\}$$

Each function in $\mathcal{A}_m^N$ is harmonic by Lemma 1.

The choice of whether to consider real-valued harmonic functions [3] or complex-valued harmonic functions [2] is merely a question of which notation is more convenient, like the choice of considering Fourier series as series in $\{\sin(n\theta), \cos(n\theta)\}$ or as a series in $\{e^{in\theta}\}$. For $\mathcal{A}_m^N$, the choice of complex-valued harmonic functions gives the simpler notation, keeping in mind that if the pair a, b belongs to A_N, then so does the pair $a, -b$, i.e., both $(a \cdot x + ib \cdot x)^m$ and $(a \cdot x - ib \cdot x)^m$, and therefore $Re(a \cdot x + ib \cdot x)^m$ and $Im(a \cdot x + ib \cdot x)^m$ are in $\mathcal{A}_m^N$

As $\mathcal{H}_m(R^N)$ is finite-dimensional, all norms on it are equivalent; below the supremum norm will be used:

$$\|p(x)\| = \sup\{|p(x)| : |x| \le 1\}.$$

For an harmonic function, the maximum principle implies that the supremum can be restricted to x on the surface of the unit sphere in R^N.

Lemma 2. $\mathcal{A}_m^N = sp\{(a \cdot x + i\ b \cdot x)^m$*: for pairs a,b in A_N with $a_1 + ib_1 \neq 0$*$\}$

Consider $(a \cdot x + ib \cdot x)^m$ in $\mathcal{A}_m^N$ in the case where $a_1 + ib_1 = 0$. Define, for each $0 < \delta < 1$, $a(\delta)$ in R^N by

$$a(\delta) = (\delta, \sqrt{1-\delta^2}a_2, \ldots, \sqrt{1-\delta^2}a_N). \tag{2}$$

As $|a(\delta)| = 1 = |b|$ and $a(\delta)\cdot b = 0$, the pair $a(\delta), b$ belongs to A_N and therefore $(a(\delta)\cdot x + ib\cdot x)^m$ belongs to

$$L = sp\{(a \cdot x + i\ b \cdot x)^m : \text{for pairs } a, b \text{ in } A_N \text{ with } a_1 + ib_1 \neq 0\},$$

a subspace of $\mathcal{A}_m^N$. As $|a(\delta) - a|$ converges to zero as $\delta \to 0$,

$$\|(a(\delta) \cdot x + ib \cdot x)^m - (a \cdot x + ib \cdot x)^m\| \to 0.$$

Thus for a finite sum $p(x) = \sum c_j (a^{(j)} \cdot x + ib^{(j)} \cdot x)^m$ in $\mathcal{A}_m^N$, if in each term where $a_1^{(j)} + b_1^{(j)} = 0$ the element a is modified as in (2), the resulting modified sum $\hat{p}(x)$ is in L and $\|p - \hat{p}\|$ can be made less than any preassigned $\epsilon > 0$ for δ chosen small enough. This shows that L is dense in $\mathcal{A}_m^N$, but L being finite dimensional is closed so it must equal $\mathcal{A}_m^N$. ∎

Theorem 1. *For all m and N,*

$$\mathcal{H}_m(R^N) = \mathcal{A}_m^N. \tag{3}$$

Proof. The proof will be by induction on the dimension $N \geq 2$, and then a further induction on those m for which the statement (3) holds for the N under consideration.

It has been noted that (3) holds for $N = 2$ and all $m = 0, 1, \ldots$. For any N, (3) is obviously true for $m = 0$; for $m = 1$, the corresponding homogeneous polynomials of degree one are given by $x_j = [(x_j + ix_k) + (x_j + i(-1)x_k)]/2$ for $k \neq j$.

To start the induction, suppose that (3) holds for some N and for that N, for all m. Consider N+1, and as (3) holds for $m = 0$ and $m = 1$, it will be supposed that it holds for some m and it will be shown then that (3) holds for m+1.

Let u be a function in $\mathcal{H}_{m+1}(R^{N+1})$ The partial derivative of u with respect to the first variable x_1, $D_1 u$, is a harmonic polynomial homogeneous of degree m, and by the induction hypothesis can be written as the finite sum

$$D_1 u = \sum c_j (a^{(j)} \cdot x + i\ b^{(j)} \cdot x)^m, \tag{4}$$

c_j complex constants and each pair $a^{(j)}, b^{(j)}$ belonging to A_{N+1}. Applying Lemma 2, it can be further be assumed that $a_1^{(j)} + ib_1^{(j)} \neq 0$ for each j. Define

$$v = \sum c'_j (a^{(j)} \cdot x + i\ b^{(j)} \cdot x)^{m+1}, \tag{5}$$

with

$$c'_j = \frac{c_j}{(m+1)(a_1^{(j)} + i\ b_1^{(j)})}.$$

Then v belongs to $\mathcal{A}_{m+1}^{N+1}$ and $u - v$ belongs to $\mathcal{H}_{m+1}(R^{N+1})$ with $D_1(u - v)$ zero. Write $u - v = \sum_{|\alpha|=m+1} C_\alpha x^\alpha$, then $D_1(u - v) = \sum_{|\alpha|=m+1} \alpha_1 C_\alpha x^{\alpha - e_1}$, $e_1 = (1, 0, \ldots, 0)$, showing that $u - v$ is an harmonic polynomial in the variables $x_2, x_3, \ldots, x_{N+1}$, having the form

$$u - v = \sum C_\alpha x^\alpha = \sum C_{(0,\alpha_2,\ldots,\alpha_{N+1})} x_2^{\alpha_2} \ldots x_{N+1}^{\alpha_{N+1}}, \tag{6}$$

the sum taken over all $\alpha_2 + \cdots + \alpha_{N+1} = m + 1$. This makes it clear that $u - v$ is an harmonic function, homogeneous of degree $m + 1$, in the N- variables $x_2, \ldots, x_{N+1}$, and as such by the induction hypothesis can be written as a linear combination of the functions of the form

$$[(a_2, \ldots, a_{N+1}) \cdot (x_2, \ldots x_{N+1}) + i\ (b_2, \ldots, b_{N+1}) \cdot (x_2, \ldots x_{N+1})]^{m+1},$$

the pair $(a_2, \ldots, a_{N+1})$, $(b_2, \ldots, b_{N+1})$ belonging to A_N. Each of these terms can be written

$$[(0, a_2, \ldots, a_{N+1}) \cdot (x_1, \ldots, x_{N+1}) + i\ (0, b_2, \ldots, b_{N+1}) \cdot (x_1, x_2, \ldots, x_{N+1})]^{m+1},$$

the pairs $(0, a_2, \ldots, a_{N+1}), (0, b_2, \ldots, b_{N+1})$ belonging to A_{N+1}. Thus the sum representing $u - v$ belongs to $\mathcal{A}_{m+1}^{N+1}$, as does v, and therefore so does u. ■

III. THE DIRICHLET PROBLEM

The basic Dirichlet problem for a domain Ω in R^N is: Given a function g defined and continuous on the boundary $\partial\Omega$ of Ω, find a function u harmonic in Ω and continuous on the closure $\overline{\Omega}$ with $u = g$ on the boundary.

Lemma 3. *Let f be analytic on the disc $D(z_1, r) = \{z : |z - z_1| < r\}$. If f is not a polynomial, there is a point z_0 in this disk where f and every derivative of f is not zero:*

$$f^{(n)}(z_0) \neq 0 \, for\ n = 0, 1, \ldots. \tag{7}$$

Proof. See [4, ex. 2, p. 227] or [5]. Let $D_n = \{z : f^{(n)}(z) = 0\}$. If the lemma is false, $D(z_1, r) \subset \cup(D_n)$ and any closed uncountable subset F of $D(z_1, r)$ intersects at least one D_n in an infinite set with a limit point in F; by the identity theorem $f^{(n)}$ is identically zero in $D(z_1, r)$ and f is a polynomial. ■

With reference to the above lemma, a linear change of variable applied to any function analytic and not a polynomial on some disk will give a function satisfying the conditions on the function f in Theorem 2 below.

Theorem 2. *Let Ω be a bounded open subset of R^N, with the property that given any continuous function g on its boundary, for each $\epsilon > 0$, there is a harmonic polynomial p with $|p(x) - g(x)| < \epsilon$ for all x in $\partial\Omega$.*

Let f be analytic in the disk $D(0, r)$ which contains Ω, and further suppose that $f^{(j)}(0) \neq 0$, for $j = 0, 1, \ldots$. Let a continuous function g be given on $\partial\Omega$. For any $\epsilon > 0$, there are a finite number of pairs of elements a^k, b^k in A_N, λ_k real, $|\lambda_k| \leq 1/4$ and complex coefficients c_k, with

$$|g(x) - \sum c_k f(\lambda_k(a^k \cdot x + ib^k \cdot x))| \leq \epsilon \, for\ all\ x\ in\ \partial\Omega. \tag{8}$$

Proof. Consider the Banach space $C(\partial\Omega)$ of all continuous (complex- valued) functions defined on $\partial\Omega$, taken with the supremum norm. The theorem states that the subspace M spanned by all sums of the form given in (8) is dense in $C(\partial\Omega)$. If this is not so, there is a function g in $C(\partial\Omega)$ not in the closure of M. By the Hahn-Banach theorem, there is a continuous linear functional x^* which is zero on M and has $x^*(g) \neq 0$.

As x^* annihilates the subspace M, it must annihilate $f(\lambda(a \cdot x + ib \cdot x))$ for all a, b in A_N and all λ, $|\lambda| \leq 1/4$. On the closure of the the ball $B(0, r/2)$ the power series $f(z) = \sum c'_j z^j$ for f converges uniformly. By the continuity of x^*,

$$0 = x^*[f(\lambda(a \cdot x + ib \cdot x))] = \sum c'_j \lambda^j x^*((a \cdot x + ib \cdot x)^j) \tag{9}$$

for all λ and a, b of the prescribed type.

As none of the coefficients c'_j are zero, by regarding the series (9) as a power series in λ (clearly a smaller set of λ than all those satisfying $|\lambda| \leq r/4$ will suffice) it is seen that $x^*((a \cdot x + ib \cdot x)^j) = 0$ for all j and all pairs a, b in A_N. From Theorem 1, $x^*(p_j) = 0$ for all harmonic polynomials p_j, homogeneous of degree j, and hence $x^*(p) = 0$ for all harmonic polynomials. By hypothesis, such polynomials are dense in $C(\partial\Omega)$ and x^* is zero, contrary to assumption.

The Dirichlet problem is solvable for any continuous boundary function g for a domain with the property hypothesized in the theorem. For if $p^{(k)}$ are harmonic polynomials with $|g(x) - p^{(k)}(x)|$ converging to zero uniformly on $\partial\Omega$, the sequence of polynomials $\{p^{(k)}\}$ is Cauchy in $C(\partial\Omega)$ and so converges uniformly on $\partial\Omega$ to g, and by the maximum principle uniformly on the closure of Ω to a function u which is harmonic in Ω [2, p. 16], continuous on the closure, and equal to g on the boundary. The maximum principle implies that two harmonic functions which are close on the boundary of Ω are also close throughout Ω, and so u is approximated by the sum in (8) throughout the closure of Ω. ∎

If Ω is a bounded open set of R^N, with $R^N - \overline{\Omega}$ connected, the condition above on Ω, that any Dirichlet problem with continuous boundary data has a solution which can be approximated by an harmonic polynomial, is equivalent to the condition that $R^N - \overline{\Omega}$ is not thin at each point of $\partial\Omega$, see [6, Theorem 1.15], [3, Theorem 7.9.7], [1, Theorem 1], and the proof of this is more technically difficult than the results proved in this note. A condition that suffices for applications is that the domain has the hypothesized property of Theorem 2 if it satisfies the Poincare exterior cone condition: at each point ζ in the boundary of Ω, there is an open truncated cone C with vertex ζ and $C - \{\zeta\}$ lying in $R^N - \Omega$ [2, Chapter 11], [3, Chapter 6].

Theorem 2 gives a method for the numerical solution of the Dirichlet problem in R^3 (or R^N) which is particularly simple when using a programming language with a complex data type and built-in subroutines for some analytic functions. One chooses an analytic function f, some points in A_N, and fits a sum of the type described in the Theorem, say by least squares, to the given function g on $\partial\Omega$. See [7] for references to numerical results and a discussion of prior work.

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