Volume 13, Number 3/4, 2002

Boundary Element Communications

An International Journal

WITPRESS Southampton & Boston

A METHOD FOR SELECTING 3D CVBEM PROJECTION PLANE VECTORS

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Key Words: CVBEM, 3DCVBEM, three dimensions, Complex Variable Boundary Elements, transport processes

Abstract

The Complex Variable Boundary Element Method (CVBEM) has been extended into three dimensions (3D) for solving 3D problems of the LaPlace or Poisson Equation with boundary conditions. This new advance provides an entirely new field of opportunity and research for the CVBEM.

At the heart of the 3DCVBEM formulation is the selection of Projection Plane Vectors (PPV) that are used for the development of CVBEM basis functions. The more PPVs used, the more CVBEM basin functions are being utilized. Of course, the PPVs must all be mutually linearly independent vectors. The CVBEM basis functions depend on both the PPVs, and also the CVBEM nodes used in each 2D plane resulting from each PPV. The CVBEM nodal placement in each PPV plane is usually determined by consideration of the problem geometry and boundary conditions. The choice of the PPVs, however, are somewhat subjective. It is the goal of this paper to present a method for selecting PPV's.

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I. Background

The Complex Variable Boundary Element Method (or CVBEM) has been the subject of numerous publications. The recent book of Hromadka (2002) contains a review of a wide range of publications detailing CVBEM advances, including application of the CVBEM to problems in engineering, applied mathematics, and science. The CVBEM has found utility in studies involving structural engineering, transport processes (e.g., groundwater flow, sediment flow, contaminant transport, atmospheric transport, among other topics), atmospheric processes, among other topics. Like other Boundary Element Methods (BEM), the CVBEM requires nodal points for numerical solution only on the problem boundary, Γ. Unlike the real variable BEM, the CVBEM is based on the theory of complex variables and analytic functions. Until recently, the CVBEM, like the theory of analytic functions, was limited to two-dimensional (2D) problems domains. The extension of the CVBEM to three-dimensional (3D) problem domains was accomplished in Hromadka (2001). The term "3DCVBEM" is used herein to denote this new numerical technique, because problem formulation between the usual 2DCVBEM and the new 3DCVBEM differ substantially. Some example problem demonstrations of the 3DCVBEM can be found on the web site www.hromadka.net/3DCVBEM.html.

II. Mathematical Model

The 3DCVBEM is used to solve problems involving the 3D LaPlace or Poisson equations, with boundary conditions specified on boundary Γ, given by $\phi_b(x,y,z)$, where in problem domain Ω,

$$\nabla^2\phi = f(x,y,z);\ (x,y,z) \in \Omega \tag{1}$$

$$\phi\,(x,y,z) = \phi_b(x,y,z);\ (x,y,z) \in \Gamma \tag{2}$$

By determining a particular solution for (1), ϕ_p, then Eqs. (1) and (2) can be rewritten as

$$\nabla^2\hat{\phi} = O;\ (x,y,z) \in \Omega \tag{3}$$

$$\hat{\phi}\,(x,y,z) = \phi_b(x,y,z) - \phi_p(x,y,z);\ (x,y,z) \in \Gamma \tag{4}$$

where the final approximation to Eqs. (1) and (2) is ϕ^*, given by

$$\phi^*(x,y,z) = \hat{\phi}(x,y,z) + \phi_p(x,y,z) \tag{5}$$

where

$$\nabla^2\phi = f(x,y,z) \tag{6}$$

and where $\hat{\phi}(x,y,z)$ is approximated by the 3DCVBEM.

Equations (1) and (2) are widely used to mathematically describe a variety of problems such as transport processes and potential processes, among other topics.

III. 3DCVBEM Formulation

Let Ω be the 3D problem domain enclosed by simply connected boundary, Γ, of finite surface area. The 3DCVBEM procedure initiates by enclosing Ω within a sphere; S_Ω. The sphere is generally selected with a minimal radius, centered at approximately the geometric center of the geometry of Ω (this corresponds to about the center of mass if the density of Ω is uniformly constant). S_Ω is then translated to be located in the first octant such that the center of S_Ω is a minimum distance from the 3D coordinate axis origin. That is, S_Ω is tangent to the positive x-y, x-z, and x-y planes of the first octant.

III.1. Gram-Schmidt Inner Product

The Gram-Schmidt orthonormalization process involves the use of an inner-product, given in our case by

$$(f,g) = \int_\Gamma fg \, d\Gamma \tag{7}$$

where f and g are real-valued and integrable on the problem boundary, Γ (see Figure 1). In general, (7) is evaluated numerically, and the approach used herein is as follows:

> Step 1. Define a set of "integration points", $\{q_i\}$, of uniform density, on the problem boundary Γ, (see Figure 2). Number the integration points from 1 to NI. (The uniform density of the $\{q_i\}$ on Γ can be relaxed and the integral of (7) can be extended to include a weighting function.)

Step 2. Develop a geometry vector, called "GEOM", of dimension NI, composed of the coordinates of q_i, i=1,2,...,NI.

Step 3. For any function used in (7), say f, develop a vector F, of dimension NI, composed of the values $F=\{f(q_i);\ i=1,2,...NI\}$ (that is, develop a column or row vector).

Step 4. Approximate (7), by the vector dot product,

$$(f,g) \cong (F,G) = \sum_{i=1}^{NI} f(q_i)\, g(q_i)\, \Delta\Gamma \tag{8}$$

where $\Delta\Gamma$ is the measure of the incremental boundary containing q_i. Note that as $NI \to \infty$ and $\Delta\Gamma \to 0$, then $(F,G) \to (f,g)$. Also, the requirement that the set of integration points be uniformly distributed on Γ can be readily relaxed by defining $\Delta\Gamma$, in (8), as an appropriate function of the q_i.

III.2. Gram-Schmidt Orthonormalization Process

Given a set of m linearly independent basis functions $\{f_j;\ j=1,2,...,m\}$, orthonormalization is achieved by using the Gram-Schmidt process, except now we use the discrete (approximate) vector representation of the set $\{f_j\}$, noted as $\{F_j\}$, and we use the vector dot product (8). The resulting orthonormalized vectors are $\{H_j;\ j=1,2,...,m\}$.

III.3. Determining an Approximation of ϕ_b on Γ

To approximate a function ϕ_b on Γ, another vector, ϕ_B, of dimension NI, is developed as $\phi_B = \{\phi_b(q_i);\ i=1,2,...,NI\}$. The best approximation of $\hat{\phi}_b$ on Γ, noted as $\hat{\phi}^*$, is given by

$$\hat{\phi}^* = \sum_{j=1}^{m} (H_j, \phi_B)\, H_j\,. \tag{9}$$

By back-substitution, $\hat{\phi}^*$ can be rewritten in terms of the original vectors, F_j, giving

$$\hat{\phi}^* = \sum_{j=1}^{m} \alpha_j F_j \tag{10}$$

where the α_j are the coefficients determined from the usual Gram-Schmidt back-substitution process from the (H_j, ϕ_B) values (see Hromadka and Whitley, 1993). Note that as $NI \rightarrow \infty$ and $\Delta\Gamma \rightarrow 0$, the $\alpha_j \rightarrow \beta_j$ where β_j is the Gram-Schmidt coefficient corresponding to the original basis function, f_j, and in the original space spanned by the $\{f_j\}$,

$$\phi^* = \sum_{j=1}^{m} \beta_j f_j . \tag{11}$$

III.4. 2D Geometry CVBEM Basis Functions

The Complex Variable Boundary Element Method, or CVBEM, will be utilized to generate 2D geometry basis functions to be used in the 3D problem. Details regarding the CVBEM can be found in numerous papers and books (including Hromadka and Whitley, 1998) and consequently, these details will not be repeated here. For our purposes, it is sufficient to state the form of the CVBEM basis functions (for the case of a linear global trial function used in the CVBEM).

The CVBEM in 2D involves the sum of products of complex coefficients C_j multiplied by certain analytic functions:

$$\hat{\omega}(z) = \sum_{j=1}^{N} C_j(z-z_j)\, Ln_j(z-z_j) \tag{12}$$

where $\hat{\omega}(z)$ is the CVBEM approximation in the (x,y) plane; N is the number of CVBEM basis functions; C_j are complex constants; z_j is $x_j + iy_j$, the (x,y) plane coordinate of node j; and Ln_j is the complex natural logarithm with branch cut oriented to lie exterior of Ω and not intersecting with other such branch cuts emanating from other CVBEM nodes. From Euler's formula,

$$z - z_j = r_j e^{i\theta_j} \tag{13}$$

where r_j is the 2D radial distance from z to z_j, and θ_j is the radial angle measured counterclockwise from the branch cut defined at CVBEM node j. From (12) and (13), the associated 2D CVBEM basis functions are of the form

$$a_j r_j (\cos\theta_j \ln r_j - \theta_j \sin\theta_j)$$

or (14)

$$b_j r_j (\sin\theta_j \ln r_j - \theta_j \cos\theta_j)$$

where the a_j and b_j are real constants. The choice of the a_j and b_j are determined by minimizing the usual least squares residual difference in matching boundary condition values on the problem boundary.

III.5. 3DCVBEM

The generalized 3DCVBEM approach can be summarized by the following procedural steps:

Step 1. Determine, approximately, the geometric center of gravity (i.e. supposing uniform mass density), P_c, of the problem domain, Ω.

Step 2. Define an enclosing sphere, with P_c, and with radius, R_0, such that the sphere's surface does not intersect the problem boundary, Γ (see Figure 3).

Step 3. Translate the enclosing sphere, and Ω, to the first octant, such that the sphere touches the three orthogonal planes of the 3-coordinate geometry.

Step 4. Define a set of V vectors, v_i, i=1,2,...V, each of length R_0, emanating from the center of the sphere. These vectors will terminate on the sphere's surface. Do not use vectors that are colinear. Figure 4 depicts five such orientation vectors in 3D space.

Step 5. Determine the plane, P_i, orthogonal to v_i, and touching the sphere at the endpoint of v_i for each vector, v_i (see Figure 5).

Step 6. For each plane, P_i, project the vector "GEOM" of ordered integration points of Ω onto the plane (see Figure 6). Note that the projection of the sphere's boundary, on P_i, is a circle.

Step 7. For each plane, locate CVBEM nodes (the points z_j of equation (12)) on the projected circle of step 6, where each CVBEM basis function branch-cut (see Equations 12 to 14) lies on an outward ray from the center of the circle (see Figure 6). For notational simplicity assume that N nodes are defined for each plane, P_i.

Step 8. For each basis function defined on P_i, develop a vector $\mathbb{F}_{ij}$ where i = plane number, j = basis function number (or node number), and $\mathbb{F}_{ij}$ is composed

of the basis function evaluated at each of the projected points, from GEOM, on P_i.

Step 9. At this point of the method, there are VN nodes, VN basis functions, and VN vectors, $\mathbb{F}_{ij}$, i=1,2,...,V; j=1,2,...,N.

Step 10. Using the Gram-Schmidt method, determine the best approximation to the problem boundary condition values defined on Γ. The resulting best approximation is the 3DCVBEM approximation function. It is noted that the minimization of residual error, in matching known boundary condition values over Γ, is accomplished with respect to all VN vectors, $\mathbb{F}_{ij}$, simultaneously.

IV. 3DCVBEM Projection Plane Vector Selection

At the heart of the 3DCVBEM formulation is the selection of Projection Plane Vectors (PPV) that are used for the development of 2DCVBEM basis functions (Hromadka, 2001, 2002). The more PPVs used, the more CVBEM basin functions are being utilized and the better the approximation in a Hilbert space setting. Of course, the PPVs must all be mutually linearly independent vectors. The CVBEM basis functions depend on both the PPVs, and also the CVBEM nodes used in each 2D plane resulting from each PPV (see previous CVBEM Formulation discussion). The CVBEM nodal placement in each PPV plane is usually determined by consideration of the problem geometry and boundary conditions, the choice of the PPVs, however, are somewhat subjective. It is the goal of this paper to present a method for selecting PPV's.

IV.1. PPV Selection Approach

The enclosing sphere, S_Ω, is used to develop the PPV projection planes (Hromadka, 2001, 2002). Let S_Ω be populated with a nearly uniform density of points, $\{P_k, k = 1,2,\ldots,M\}$ located on the surface of only one hemisphere of S_Ω. These points on S_Ω will be used for PPV development and selection.

The procedure begins by developing 3DCVBEM models based upon only 1 PPV. Here, a PPV is defined as a vector originating from the center of S_Ω, and intersecting S_Ω at one of the points in $\{P_k\}$. Because there are M points on S_Ω, there

are M 3DCVBEM possible models to consider (assuming all such PPV are linearly independent). The best 3DCVBEM model, based upon 1 PPV, is that model with the smallest least squares residual in matching the problem boundary conditions, ϕ_b, on Γ. This best model is associated with PPV1.

Now, the best 2-PPV 3DCVBEM model can be determined for the given set of points $\{P_k\}$ defined on S_Ω, by considering the (M-1) 3DCVBEM 2-PPV models, all including PPV1 (again, all PPV are assumed to be linearly independent). The second best PPV is that PPV resulting in the smallest residual in matching ϕ_b on Γ. The best 2-PPV 3DCVBEM model is based upon use of the vectors {PPV1, PPV2).

Continuing, there are M-2 remaining points of $\{P_k\}$ to consider in determining the third best PPV, given that PPV1 and PV2 are also used, for the best 3-PPV 3DCVBEM model. Similar to the above, PPV3 is selected, and the best 3-PPV 3DCVBEM model is determined.

The above process is continued until either all points on S_Ω are used (a M-PPV 3DCVBEM model), or an acceptable residual error is achieved. (It is this author's experience that a 5-PPV 3DCVBEM model oftentimes produced acceptable results).

V. Applications

Several application problems, or demonstrations, of the above approach can be found at the web address www.hromadka.net/3DCVBEM.html. The reader is referenced to this web address.

VI. Conclusions

In the 3DCVBEM vectors are used to develop 2D plane surfaces upon which the problem boundary and domain are projected onto. In general, under mild conditions, the more 2D planes used, the better the 3DCVBEM approximation. A method to select such projection planes is presented in this paper. Using this method, a 3DCVBEM model can be developed with a goal of reducing computational effort. The method has been applied to several 3D problems, with good results. This approach is readily programmable.

References

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3. Hromadka II, T.V. and Whitley, R.J., 2001, A Note on Approximating Three-Dimensional Potential Problems Using the CVBEM and Domain Rotations, Boundary Element Communications, Vol. 12, No. 4, pgs. 21-30.

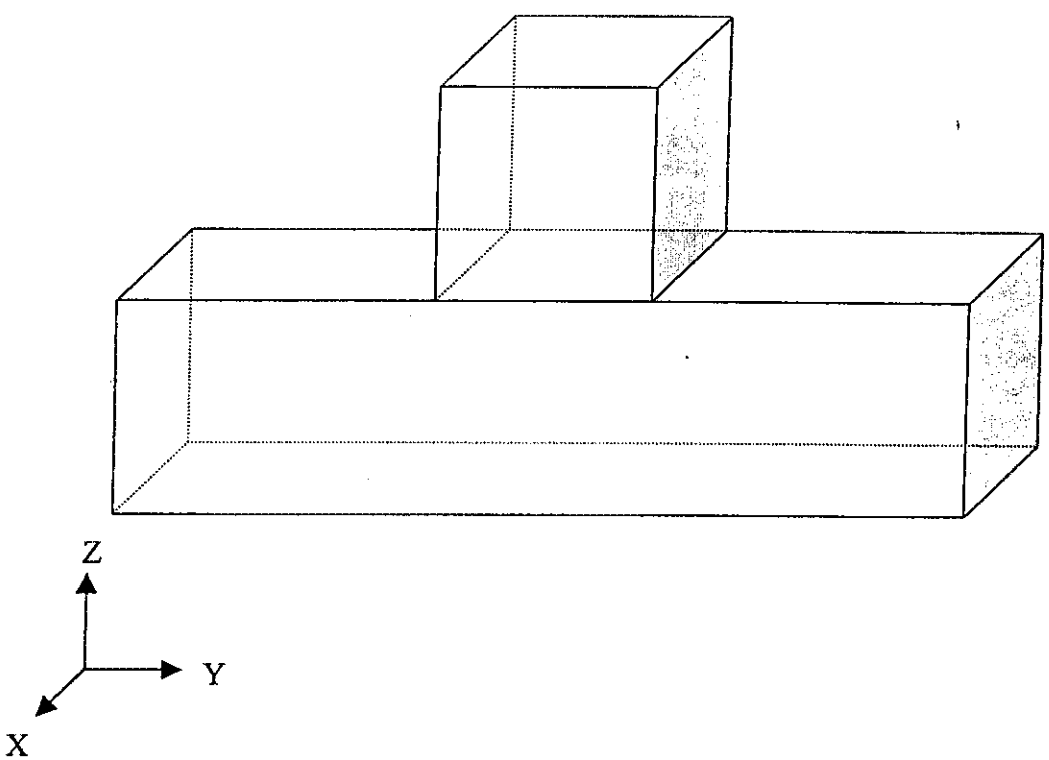

Figure 1. **Example 3-D Domain**

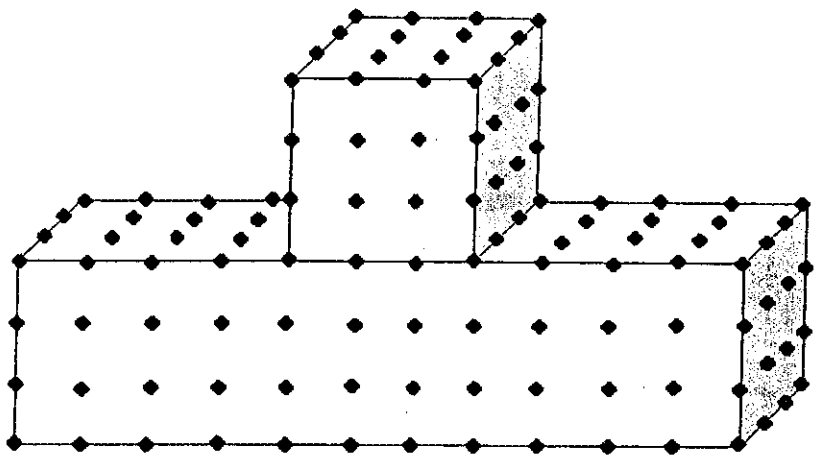

Figure 2. **3-D Domain and Integration Points**

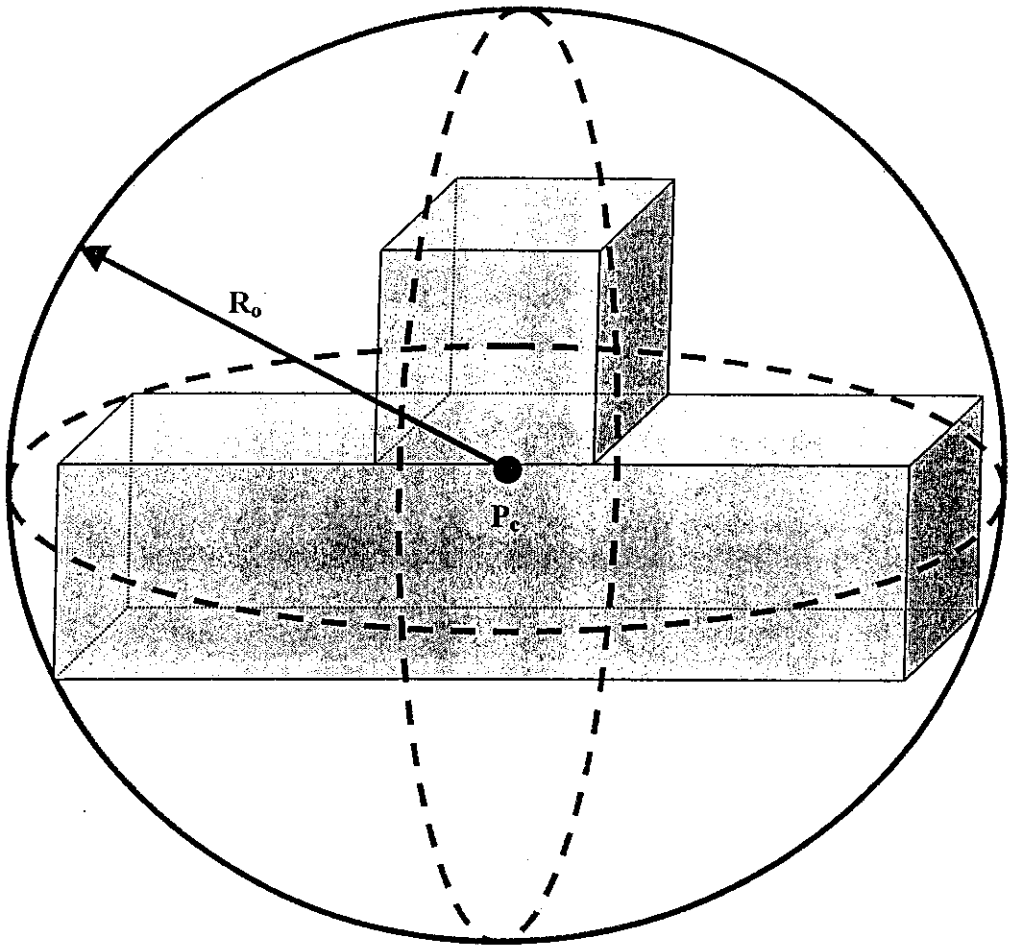

Figure 3. **3-D Domain and Enclosing Sphere**

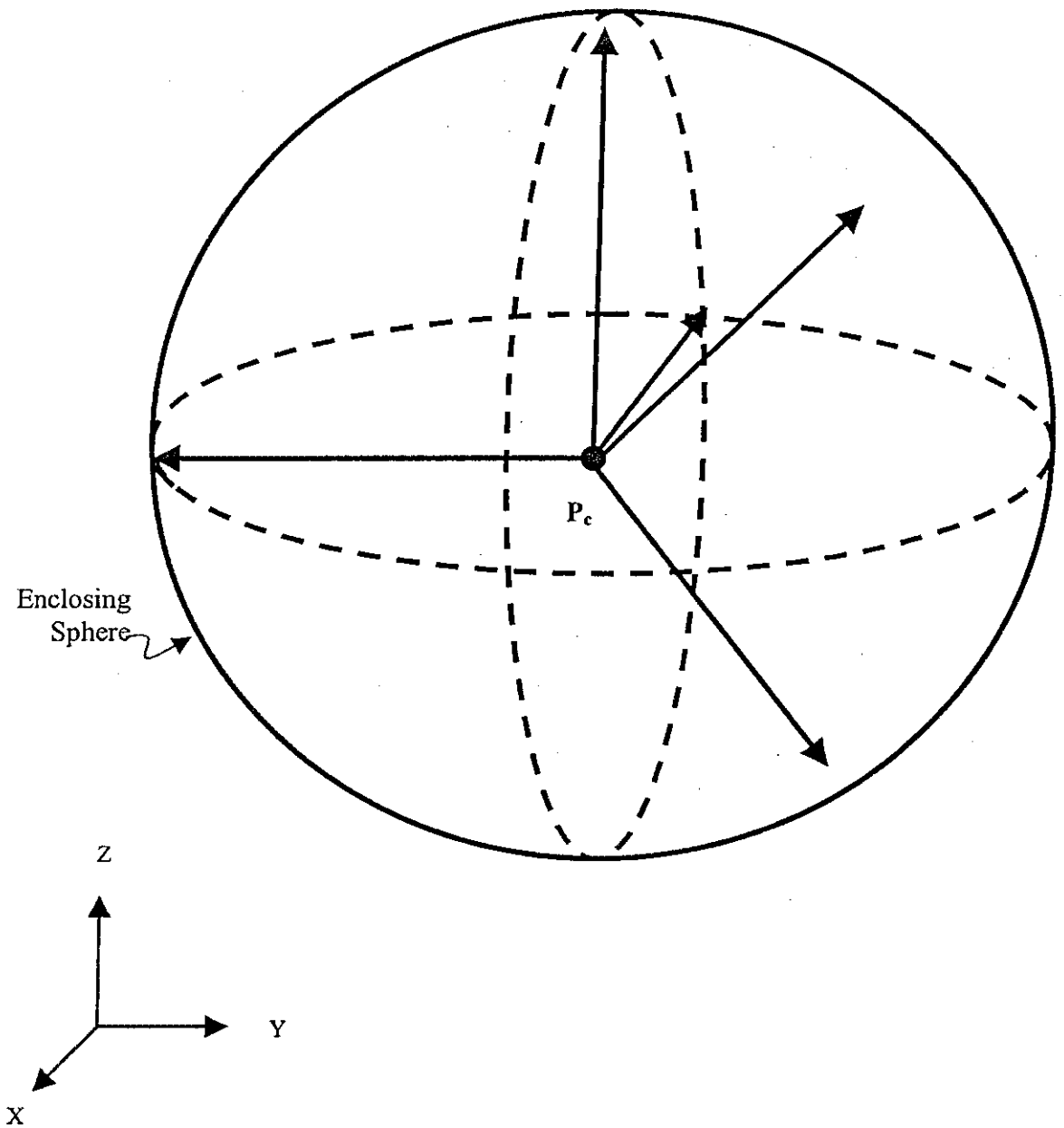

Figure 4. **5-Projection Vectors**

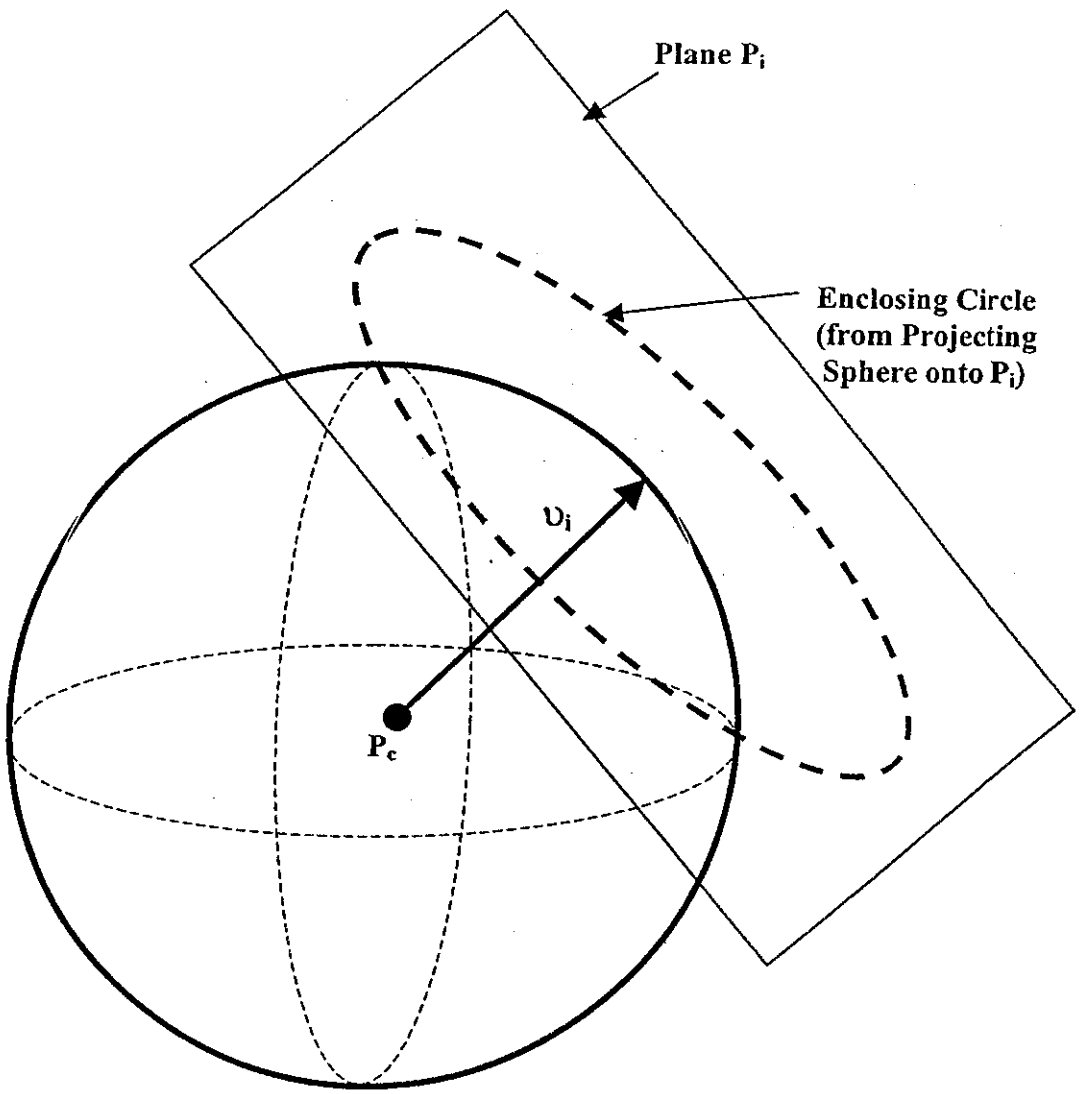

Figure 5. Example Projection Plane, P_i

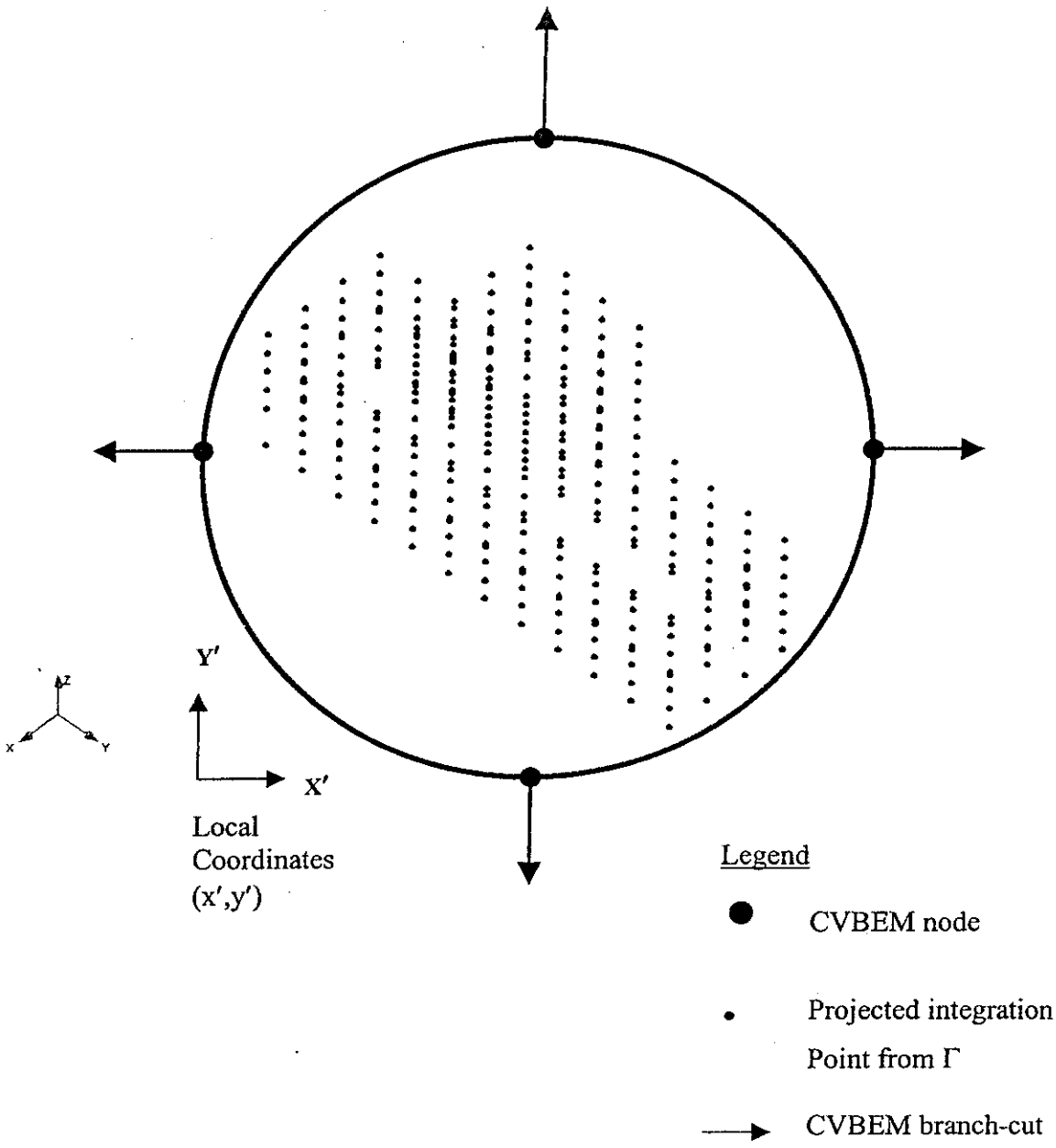

Figure 6. Projected Integration Points and Enclosing Circle. Note Local Coordinates (x′,y′).