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Abstract

Computer programs of rainfall–runoff models, such as HEC-1 and other related computer programs, utilize a set of algorithms that represent the effects of catchment subarea runoff, network link hydrograph flow routing, and detention/retention basin hydrograph routing. By subdividing a watershed into numerous subareas, and connecting the subareas by a network of links, a link-node model representation of the watershed is constructed. Each of the algorithms computationally solves a prescribed relation by subdividing storm duration into unit periods-of-time of constant duration (e.g. 5 min). Such link-node models are so flexible and comprehensive that one might conjecture that the algorithms and rules of linkage are sufficient to simulate any storm runoff problem that can be formally expressed in the model. In contrast, the single-area unit hydrograph (UH) modeling approach represents a watershed as a single subarea, and utilizes a single UH to represent all of the effects being modeled by a link-node model. The mathematical underpinnings of the single-area model UH are typically described in the literature as being a "black box". In this paper, a mathematical formalization of link-node models can be developed with the computer program HEC-1 and the related systems will be introduced. The formalization is then used to establish, and also dispel, some perceptions regarding application of such link-node models.

Keywords: Unit hydrographs; Runoff; Modeling; Estimation

Nomenclature

$[\alpha^i]$	Toeplitz matrix for storm i, used to equate E^i_M and Q^i_M
$[\beta^i_0]$	single-area UH model Toeplitz correction matrix, for storm i runoff
$[\beta^i]$	Toeplitz correction matrix for storm i runoff estimate, Q^i_M
$[\beta^i_1]$	$[\beta^i]$ correction matrix for subarea 1 and storm i
$[\gamma^i_{AB}]$	Toeplitz correction matrix, for storm i, for routing algorithm between nodes A and B
A_T	total watershed area
$\underline{e}^i$	unit effective rainfalls, for storm i
$\underline{e}^i_g$	reference (gauge) effective rainfall
E^i_M	model error, for storm i
e^i_j	subarea j effective rainfalls for storm i
F	loss vector
[I]	identity matrix
I_t	inflow, at time t
LN	link-node hydrograph mode
LR	linear reservoir
O_t	outflow, at time t
$P^i_g(t)$	gauged precipitation for storm i

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P_g^i	vector representation of $P_g^i(t)$
Q_M^i	modeled unit runoffs, for storm i
Q_1^i	storm i runoff from subarea 1
$\hat{Q}_1^i$	model estimate of storm i runoff from subarea 1
$[\mathbb{R}_{AB}]$	Toeplitz matrix representation of routing algorithm between nodes A and B
SA	single-area hydrograph model
$T(n)$	space of $n \times n$ Toeplitz matrices, of form Eq. (1)
$[T_j^i]$	Toeplitz matrix equating e_j^i and e_g^i
UH	unit hydrograph
$[U_M]$	UH model Toeplitz matrix representation of a UH
U_M	UH unit period values; first column vector of $[U_M]$
$[U_1]$	subarea 1 UH in Toeplitz matrix form
$[U_0]$	single-area UH model Toeplitz matrix structure

1. Introduction

A general modeling procedure applied by many agencies involves discretization of a watershed into numerous subareas (perhaps dozens to hundreds) connected by hydrologic-routing links (again, perhaps dozens to hundreds). Unit hydrographs (UH) are specified for each subarea as well as a loss function such as the widely used phi-index (or ϕ-index). The US Army Corps of Engineers provides documentation regarding application of the computer program HEC-1 (US Army Corps of Engineers, 1990), to such catchment modeling applications (e.g. DeVries, 1982). Hydrologic flow-routing algorithms that are applied to each modeling link typically use methods such as Muskingum, Convex, translation, or Modified-Puls (where outflow is proportional to storage, i.e. the linear reservoir method). Subarea UHs are typically developed using the Clark method, or regionalized statistical relations (e.g. McCuen, 1989, p. 49).

In Hromadka and Whitley (1989), the Stochastic Integral Equation Method (SIEM) is introduced and is applied to rainfall–runoff model structures such as the HEC-1 link-node UH model structure, and a mathematical description of the rainfall–runoff modeling approach is provided on a storm-class basis. A storm class is the set of all storms to which a particular rainfall–runoff-model parameter set is assumed to apply. When applied to the HEC-1 computer program setting, the SIEM reduces to a system of Toeplitz matrices that have several convenient mathematical properties including the key property of commutative matrix multiplication. This system of Toeplitz matrices precisely describes the subject link-node UH model structure as it is actually applied; namely, in discretized time step unit-period additions and multiplications.

Consequently, a mathematical formalization of link-node model structures, as computationally developed by the computer program HEC-1 and related systems, is provided by the Toeplitz matrix systems introduced in this paper. Given such a precise mathematical description, progress can be made towards answering questions regarding application and use of such models, including the evaluation of model performance and how to apply a link-node model, among other topics. The formalization developed in this paper provides another tool for use in future research on the efficiency and accuracy of rainfall–runoff models, (see, for example, Loague and Freeze (1985) and McCuen (1989, p. 747)).

2. Mathematical development: key Toeplitz matrix properties

A Toeplitz matrix $[U]$ is an $n \times n$ lower triangular matrix which has the form

$$[U] = \begin{vmatrix} u_1 & 0 & 0 & \cdots & 0 \\ u_2 & u_1 & 0 & \cdots & 0 \\ u_3 & u_2 & u_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u_n & u_{n-1} & u_{n-2} & \cdots & u_1 \end{vmatrix} \quad (1)$$

where the u_i are real constants, $i = 1, 2, \ldots, n$. The set of all $n \times n$ Toeplitz matrices is denoted by $T(n)$.

If $[A]$ and $[B]$ are elements of $T(n)$, then under the usual rules of matrix addition and multiplication the following results are obtained:

$$[A] + [B] = [C] \in T(n), \quad (2)$$

$$[A][B] = [D] \in T(n), \quad (3)$$

$$[A][B] = [B][A]. \tag{4}$$

Eqs. (2) and (3) show that the sum and product of Toeplitz matrices are Toeplitz matrices. Eq. (4) provides the property, unusual for matrix multiplication, that the product of two Toeplitz matrices is commutative, and the order of multiplication can be interchanged. To prove Eq. (4), the properties of Eqs. (1) and (3) are used and considering only column 1 of the resulting product Toeplitz matrix $[D]$, as the components of column 1 fully specify the matrix $[D]$. For component $(n, 1)$ of the product,

$$[A][B](n,1) = \sum_{k=1}^{n} a_{n+1-k}b_k = \sum_{k=1}^{n} b_{n+1-k}a_k$$

$$= [B][A](n,1) \tag{5}$$

similarly for component $(i, 1)$ of the product, for $1 \leq i < n$,

$$[A][B](i,1) = \sum_{k=1}^{i+1} a_{i+1-k}b_k = \sum_{k=1}^{i+1} b_{i+1-k}a_k$$

$$= [B][A](i,1) \tag{6}$$

where $a_{i+1-k} = 0$ and $b_{i+1-k} = 0$ for $k \geq i+1$.

Toeplitz matrices also satisfy key invertability relations. For example, let $[A] \in T(n)$ and $[B] \in T(n)$ such that $b_1 \neq 0$, and therefore that the main diagonal of $[B]$ is nonzero. Then solving by means of back substitution shows that there exists a $[C] \in T(n)$ such that $[A] = [C][B]$. Similarly, if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are $n \times 1$ column vectors such that the first component of $\underset{\sim}{Y}$ (i.e. $\underset{\sim}{Y}(1)$) is nonzero, then there exists a $[D] \in T(n)$ such that $\underset{\sim}{X} = [D]\underset{\sim}{Y}$.

The above properties of Toeplitz matrices will be useful in the matrix manipulations applied in the following mathematical development. Details regarding the theory of Toeplitz matrices can be found in Iohvidov (1982).

3. Toeplitz matrices applied to UH and hydrograph flood routing techniques

3.1. The UH method as a Toeplitz matrix system

The UH method for developing a runoff hydrograph can be expressed by use of a Toeplitz matrix system where, for storm i,

$$\underset{\sim}{Q}_M^i = [U_M]\underset{\sim}{e}_M^i \tag{7}$$

where the subscript M refers to the UH model; $\underset{\sim}{Q}_M^i$ is the $n \times 1$ column vector of unit-period model runoff values; $\underset{\sim}{e}_M^i$ is the $n \times 1$ column vector of modeled unit-period effective rainfalls (unit-period rainfalls less unit-period loss) for storm i; and $[U_M]$ is the model UH in $n \times n$ Toeplitz matrix form (see Eq. (1)). In Eq. (7), the common dimension, n, is achieved by proper extension of each vector or matrix with zeroes. From Eq. (1), the matrix $[U_M]$ is fully described knowing only the first column vector, $\underset{\sim}{U}_M$.

Example 1. *Demonstration of the UH method as a Toeplitz matrix system:*

Let $\underset{\sim}{e}_M^i = (e_1, e_2)^T$; $\underset{\sim}{U}_M = (u_1, u_2, u_3)^T$. Then $\underset{\sim}{Q}_M^i$ is given by

$$\underset{\sim}{Q}_M^i = \begin{vmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{vmatrix} = \begin{vmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_1 & 0 & 0 \\ u_3 & u_2 & u_1 & 0 \\ 0 & u_3 & u_2 & u_1 \end{vmatrix} \begin{vmatrix} e_1 \\ e_2 \\ 0 \\ 0 \end{vmatrix} = [U_M]\underset{\sim}{e}_M^i$$

where q_i are unit-period runoffs, u_i are unit-hydrograph values, and e_i the unit-period effective rainfalls, $i = 1, 2, 3$, and the size of the matrix, $n = 4$, is determined by the desire to include all nonzero discharges.

3.2. UH model error analysis

In Eq. (7), the UH model runoff vector $\underset{\sim}{Q}_M^i$, differs from the true runoff vector, $\underset{\sim}{Q}^i$, where the storm i UH model error, $\underset{\sim}{E}_M^i$ is

$$\underset{\sim}{E}_M^i = \underset{\sim}{Q}^i - \underset{\sim}{Q}_M^i. \tag{8}$$

But, the "correct" UH for storm i, $\underset{\sim}{U}^i$, and the "correct" effective rainfall, $\underset{\sim}{e}^i$, are related to the true $\underset{\sim}{Q}^i$ by

$$\underset{\sim}{Q}^i = [U^i]\underset{\sim}{e}^i. \tag{9}$$

Relating Eqs. (7)–(9),

$$\underset{\sim}{Q}^i = [U_M + \Delta U^i](\underset{\sim}{e}^i_M + \Delta \underset{\sim}{e}^i)$$

$$= \underset{\sim}{Q}^i_M + ([\Delta U^i])\underset{\sim}{e}^i_M + [U_M]\Delta \underset{\sim}{e}^i + [\Delta U^i]\Delta \underset{\sim}{e}^i) \quad (10)$$

where $[\Delta U^i]$ and $\Delta \underset{\sim}{e}^i$ are the corrections to the UH model $[U_M]$ matrix and $\underset{\sim}{e}^i_M$ vector, respectively. The sum of the three components of the terms in parenthesis in Eq. (10) are denoted as the UH model error vector, $\underset{\sim}{E}^i_M$. The above formulation is analogous to the Stochastic Integral Equation Method representation of hydrograph modeling error as presented in Hromadka and Whitley (1997).

For storm i, $\underset{\sim}{E}^i_M$ can be related to the UH model estimate, $\underset{\sim}{Q}^i_M$, by setting (recalling that $Q_M(1) \neq 0$)

$$\underset{\sim}{E}^i_M = [\alpha^i]\underset{\sim}{Q}^i_M \quad (11)$$

where $[\alpha^i] \in T(n)$, and is associated to storm i. Thus, for m storms, there would be m Toeplitz matrices $[\alpha^i]$; $i = 1, 2, \ldots m$.(Example 2 demonstrates Eq. (11) by a constructive development.)

But from Eq. (8), $\underset{\sim}{Q}^i = \underset{\sim}{Q}^i_M + \underset{\sim}{E}^i_M$, so that

$$\underset{\sim}{Q}^i = \underset{\sim}{Q}^i_M + [\alpha^i]\underset{\sim}{Q}^i_M = ([\mathbf{I}] + [\alpha^i])\underset{\sim}{Q}^i_M = [\beta^i]\underset{\sim}{Q}^i_M \quad (12)$$

where $[\mathbf{I}]$ is the $n \times n$ identity matrix; and $[\beta^i] = [\mathbf{I}] + [\alpha^i]$ is another Toeplitz matrix in $T(n)$ and, like $[\alpha^i]$, varies for every storm i. The correction of various modeling estimates by use of the Toeplitz matrices, such as that used in Eq. (12), is a recurring result in the formalization presented in this paper.

Example 2. *Demonstration of equating UH modeling error to the UH model output:*

For storm i, assume that the true effective rainfall vector (for the UH model) is $\underset{\sim}{e}^i$. The UH model assumes the UH to be the matrix $[U_M]$ but, in reality, the UH should be, for storm i, $[U^i]$. Then the UH model error in the runoff estimate is given by the vector, $\underset{\sim}{E}^i_M$, where

$$\underset{\sim}{E}^i_M = ([U^i] - [U_M])\underset{\sim}{e}^i. \quad (13)$$

But from Eq. (12), the true runoff, $\underset{\sim}{Q}^i$, is given by

$$\underset{\sim}{Q}^i = [\beta^i]\underset{\sim}{Q}^i_M = ([\beta^i][U_M])\underset{\sim}{e}^i = [U^i]\underset{\sim}{e}^i \quad (14)$$

so that the true UH, $[U^i]$, is related to the model UH, $[U_M]$, by

$$[U^i] = [\beta^i][U_M]. \quad (15)$$

This situation is

$$[\beta^i][U_M]\underset{\sim}{e}^i = [U^i]\underset{\sim}{e}^i$$

so that

$$([\beta^i][U_M] - [U^i])\underset{\sim}{e}^i = \underset{\sim}{0}$$

where $\underset{\sim}{0}$ is the zero column vector, and because $\underset{\sim}{e}^i(1) \neq 0$, $([\beta^i][U_M] - [U^i])$ is the zero Toeplitz matrix.

Combining Eqs. (12), (13), and (15),

$$\underset{\sim}{E}^i_M = ([\beta^i][U_M] - [U_M])\underset{\sim}{e}^i = ([\beta^i] - [\mathbf{I}])[U_M]\underset{\sim}{e}^i$$

$$= [\alpha^i][U_M]\underset{\sim}{e}^i. \quad (16)$$

Thus, from Eq. (7),

$$\underset{\sim}{E}^i_M = [\alpha^i]\underset{\sim}{Q}^i_M \quad (17)$$

which is the result derived from Eq. (11).

3.3. *Hydrologic hydrograph-routing methods and Toeplitz matrix representations*

Several flood-hydrograph-routing techniques that have been widely used are, in fact, convolutions (Hromadka and Whitley, 1989). Three popular routing methods are the Convex method, the Muskingum method, and hydrograph translation. Another popular hydrograph routing technique (especially when storage attenuation effects are significant) is the modified-Puls method. A special case of the modified-Puls method (and also the Muskingum method), known as the linear-reservoir method, is analyzed in Topic 3 (Section 4.3). Toeplitz matrix representations will be developed for these methods. The development of Toeplitz matrix representations for other convolution techniques is similar.

For the Convex method, which is used for many studies involving the computer program TR-20 (SCS, 1984), outflow (O) and inflow (I) unit values at typical model unit-period times t and $t + \Delta t$ are

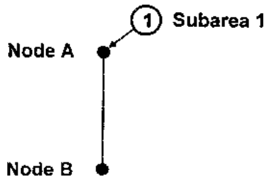

Fig. 1. A single subarea (#1) UH model runoff hydrograph with a Muskingum routing link between nodes *A* and *B*.

equated by

$$O_{t+\Delta t} = (1 - C)O_t + CI_{t+\Delta t} \tag{18}$$

where C is a constant parameter that depends on the channel, flow, and Δt characteristics, with $0 < C < 1$. By expanding the term O_t in terms of the prior values $O_{t-\Delta t}$ and $I_{t-\Delta t}$ gives

$$O_{t+\Delta t} = (1 - C)[(1 - C)O_{t-\Delta t} + CI_t] + CI_{t+\Delta t}. \tag{19}$$

Writing $O_{t-\Delta t}$ in terms of prior values:

$$O_{t+\Delta t} = (1 - C)^3 O_{t-2\Delta t} + (1 - C)^2 CI_{t-\Delta t} + (1 - C)CI_t + CI_{t+\Delta t}. \tag{20}$$

Continuing the expansion as in Eqs. (18)–(20) gives, in matrix form,

$$\begin{vmatrix} O_1 \\ O_2 \\ O_3 \\ \vdots \\ O_n \\ O_{n+1} \end{vmatrix} = \begin{vmatrix} C & 0 & 0 & \cdots & 0 \\ (1-C)C & C & 0 & \cdots & 0 \\ (1-C)^2 C & (1-C)C & C & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (1-C)^{n-1}C & (1-C)^{n-2}C & (1-C)^{n-3}C & \cdots & 0 \\ (1-C)^n C & (1-C)^{n-1}C & (1-C)^{n-2}C & \cdots & C \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_n \\ 0 \end{vmatrix} \tag{21}$$

where, in Eq. (21), the dimensions of the matrix and column vectors correspond to a single unit routing interval addition, Δt. The dimensions in Eq. (21), as in all equations, are consistent. From Eq. (21), the Convex routing technique is another application of a Toeplitz Matrix.

The Muskingum routing method is a widely used technique available in the computer program, HEC-1. In general, outflow (O) and inflow (I) are related by

$$O_{t+\Delta t} = C_0 I_{t+\Delta t} + C_1 I_t + C_2 O_t \tag{22}$$

or, upon expanding the term O_t,

$$O_{t+\Delta t} = C_0 I_{t+\Delta t} + C_1 I_t + C_2(C_0 I_t + C_1 I_{t-\Delta t} + C_2 O_{t-\Delta t}) \tag{23}$$

(see Hromadka and Whitley (1989) regarding topics concerning coefficient constraints).

In Toeplitz matrix form, the Muskingum method (again, for simplicity, neglecting a higher problem dimension for vectors and assuming only a single Δt time step movement in time)

$$\begin{vmatrix} O_1 \\ O_2 \\ O_3 \\ \vdots \\ O_{n+1} \end{vmatrix} = \begin{vmatrix} C_0 & 0 & 0 & \cdots & 0 \\ (C_1 + C_0C_2) & C_0 & 0 & \cdots & 0 \\ C_2(C_1 + C_0C_2) & (C_1 + C_0C_2) & C_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C_2^{n-1}(C_1 + C_0C_2) & C_2^{n-2}(C_1 + C_0C_2) & C_2^{n-3}(C_1 + C_0C_2) & \cdots & C_0 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ 0 \end{vmatrix}. \tag{24}$$

Given a particular routing method, the resulting Toeplitz matrix to be applied to the model link connecting upstream node A to downstream node B is noted as $[\mathbb{R}_{\mathbb{AB}}]$, where the matrix depends on the routing algorithm used.

Example 3. *A single subarea runoff hydrograph with Muskingum routing:*

Fig. 1 depicts the UH model schematic of a single subarea with a Muskingum routing link. The subarea runoff hydrograph, from subarea #1, tributary to model node A, is

$$\hat{\underset{\sim}{Q}}{}_1^i = [U_1]\underset{\sim}{e}{}_1^i \tag{25}$$

where $\hat{\underset{\sim}{Q}}{}_1^i$ is the column vector of subarea #1 unit runoff estimates; $[U_1]$ is the subarea #1 UH as per Eq. (7), and $\underset{\sim}{e}{}_1^i$ is the subarea #1 effective rainfall for storm i.

Let $[\mathbb{R}_{\mathbb{AB}}]$ be the Muskingum routing Toeplitz matrix analog as per Eq. (24), then the runoff

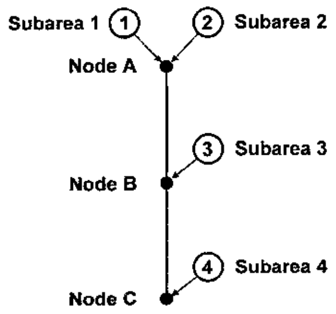

Fig. 2. A four-subarea link-node UH model schematic.

hydrograph at model node B is

$$\hat{\underset{\sim}{Q}}{}_B^i = [\mathbb{R}_{\mathbb{AB}}]\hat{\underset{\sim}{Q}}{}_A^i = [\mathbb{R}_{\mathbb{AB}}][U_1]\underset{\sim}{e}{}_1^i \quad (26)$$

where the hats in Eqs. (25) and (26) indicate model estimates (because Toeplitz correction matrices are not included).

Example 4. *Hydrograph routing modeling error representation:*

The Muskingum routing algorithm is only an approximation, needing the Toeplitz correction matrix $[\gamma_{\mathbb{AB}}]$ such that the "correct" convolution routing matrix would be the matrix product of $[\gamma_{\mathbb{AB}}]$ and $[\mathbb{R}_{\mathbb{AB}}]$. If the Muskingum technique was correct, then $[\gamma_{\mathbb{AB}}]$ is the identity matrix. In general, $[\gamma_{\mathbb{AB}}]$ will differ, stochastically, for every storm, and can be denoted as such by $[\gamma^i_{\mathbb{AB}}]$ for storm i. Of course, without runoff data, the various UH and routing correction Toeplitz matrices are not known, and hence modeling error is introduced with each algorithm.

Including both the UH (see Eq. (15)) and hydrograph-routing technique Toeplitz correction matrices, the storm i runoff vector, at node B, is

$$\underset{\sim}{Q}{}_B^i = [\gamma^i_{\mathbb{AB}}][\mathbb{R}_{\mathbb{AB}}]\underset{\sim}{Q}{}_A^i = [\gamma^i_{\mathbb{AB}}][\mathbb{R}_{\mathbb{AB}}][\beta^i_1][U_1]\underset{\sim}{e}{}_1^i \quad (27)$$

or

$$\underset{\sim}{Q}{}_B^i = [\gamma^i_{\mathbb{AB}}][\beta^i_1][\mathbb{R}_{\mathbb{AB}}]\hat{\underset{\sim}{Q}}{}_1^i = [\gamma^i_{\mathbb{AB}}][\beta^i_1]\hat{\underset{\sim}{Q}}{}_B^i \quad (28)$$

where in Eq. (28), analogous to Eq. (12), all Toeplitz correction matrices are assembled as an up-front matrix product correction of the model's approximation of the runoff hydrograph from Eq. (26).

Example 5. *Translation in time of a runoff hydrograph:*

As another example of modeling the hydrograph flow-routing process, consider "pure translation" of the hydrograph such that the upstream hydrograph, at node A, is $\underset{\sim}{Q}{}_A^{\mathrm{T}} = (q_1, q_2, q_3, 0, 0, 0)$ and the downstream hydrograph, at node B, is $\underset{\sim}{Q}{}_B^{\mathrm{T}} = (0, 0, q_1, q_2, q_3, 0)$; i.e. the hydrograph is translated forward in time by $2\Delta t$, where Δt is the unit time period (such as 5 min), and there is no attenuation of the hydrograph. Translation routing simulates a kinematric-wave routing response. The corresponding Toeplitz matrix is $[\mathbb{R}_{\mathbb{AB}}]$, where now

$$[\mathbb{R}_{\mathbb{AB}}] = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

and $\underset{\sim}{Q}_B = [\mathbb{R}_{\mathbb{AB}}]\underset{\sim}{Q}_A$. The "translation" Toeplitz matrix structure will be also used in Section 3.7 to relate the variation in rainfall across the catchment with respect to rainfall gauge data.

3.4. Link-node model structure representation as a Toeplitz matrix system

A common practice in the application of the computer program HEC-1 and related systems is to subdivide the catchment into m subareas, Ω_j, $j = 1, 2, \ldots, m$, generate runoff hydrographs $\underset{\sim}{Q}{}_j^i(t)$ in each subarea using subarea effective rainfalls $\underset{\sim}{e}{}_j^i(t)$, and the subarea UH, $U_j(t)$, and route and combine subarea runoff hydrographs by use of a hydrograph-routing algorithm such as the Muskingum method. Such models will be called "link-node" UH models.

Fig. 2 depicts a four-subarea link-node model. Given subarea effective rainfalls, $e_j^i(t)$, $j = 1, 2, 3, 4$, the approximation of the runoff hydrograph, $\hat{\underset{\sim}{Q}}{}_C^i(t)$, at node C is given by a direct extension of Example

4 results:

$$\hat{\underset{\sim}{Q}}{}^i_C = [\mathbb{R}_{\mathbb{BC}}]([\mathbb{R}_{\mathbb{AB}}]([U_1]\underset{\sim}{e}{}^i_1 + [U_2]\underset{\sim}{e}{}^i_2) + [U_3]\underset{\sim}{e}{}^i_3) + [U_4]\underset{\sim}{e}{}^i_4 \quad (29)$$

where the hat indicates an approximation because Toeplitz correction matrices are not yet included.

In comparison, analogous to the error analysis for Example 4, the true runoff hydrograph, for storm i, $Q^i_C(t)$, at model node C is given by the inclusion of Toeplitz correction matrices,

$$\underset{\sim}{Q}{}^i_C = [\gamma^i_{\mathbb{BC}}][\mathbb{R}_{\mathbb{BC}}]([\gamma^i_{\mathbb{AB}}][\mathbb{R}_{\mathbb{AB}}]([\beta^i_1][U_1]\underset{\sim}{e}{}^i_1 + [\beta^i_2] \times [U_2]\underset{\sim}{e}{}^i_2) + [\beta^i_3][U_3]\underset{\sim}{e}{}^i_3) + [\beta^i_4][U_4]\underset{\sim}{e}{}^i_4. \quad (30)$$

In Eqs. (29) and (30), $[\mathbb{R}_{\mathbb{BC}}]$ and $[\mathbb{R}_{\mathbb{AB}}]$ are hydrograph-routing Toeplitz matrices for model network links $\mathbb{BC}$ and $\mathbb{AB}$, respectively; $[U_1]$ to $[U_4]$ are Toeplitz UH matrices and $\underset{\sim}{e}{}^i_1$ to $\underset{\sim}{e}{}^i_4$ are effective rainfall vectors, for subareas 1 to 4, respectively, and for storm i; the Toeplitz correction matrices follow from Eqs. (27) and (28).

3.5. Comparison of a link-node UH model structure to a single-area UH model structure

The link-node model structure (shown in Eq. (30)) can now be compared to a single-area UH model structure by correlating subarea effective rainfalls and re-grouping terms. The subarea effective rainfalls, $e^i_j(t)$, can be equated to a reference effective rainfall, $e^i_g(t)$, (for $\underset{\sim}{e}{}^i_g$ (1) $\neq 0$), by another set of Toeplitz matrices,

$$\underset{\sim}{e}{}^i_j = [T^i_j]\underset{\sim}{e}{}^i_g; \qquad j = 1, 2, \ldots, m \quad (31)$$

where $[T^i_j]$ is a Toeplitz matrix which applies to storm i (and varies for each storm) and subarea j. If $e^i_j(t) = e^i_g(t)$, then $[T^i_j]$ is the identity matrix. Again, all matrices and vectors have consistent dimensions. Variations in storm magnitude and pattern are included in Eq. (31), as well as translation in time (refer to the translation hydrograph of Example 5).

Combining Eqs. (30) and (31) provides a unified UH model structure that still includes the identified modeling error Toeplitz correction matrices (for the example problem of Fig. 2),

$$\underset{\sim}{Q}{}^i_C = ([\gamma^i_{\mathbb{BC}}][\gamma^i_{\mathbb{AB}}][\beta^i_1][T^i_1][\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}][U_1] + [\gamma^i_{\mathbb{BC}}] \times [\gamma^i_{\mathbb{AB}}][\beta^i_2][T^i_2][\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}][U_2] + [\gamma^i_{\mathbb{BC}}][\beta^i_3] \times [T^i_3][\mathbb{R}_{\mathbb{BC}}][U_3] + [\beta^i_4][T^i_4][U_4])\underset{\sim}{e}{}^i_g. \quad (32)$$

Eq. (32) is a sum of products of Toeplitz matrices in $T(n)$ and, from the properties of Toeplitz matrices, simplifies to another Toeplitz matrix in $T(n)$. Thus, even though Eq. (32) represents numerous network link routing algorithms and subarea runoff hydrographs, with each algorithm's output modified by respective Toeplitz correction matrices, and all combined at model node C, the link-node model simplifies to having a single-area UH model Toeplitz matrix structure,

$$\underset{\sim}{Q}{}^i_C = ([\beta^i_0][U_0])\underset{\sim}{e}{}^i_g \quad (33)$$

where the matrix product $([\beta^i_0][U_0])$ is the sum of several matrix products within parenthesis in Eq. (32). In Eqs. (30) and (32), the matrix product commutability property associated with Toeplitz matrices is applied extensively, enabling the previous rewriting of terms. In Eqs. (32) and (33), the various sources of uncertainty and stochastic variations in the subarea UH, link routing, and subarea effective rainfall algorithms, are fully accounted for on a storm basis, i. Additionally, the link-node model structure of Eq. (32) includes several sources of uncertainty that cannot be isolated because they are a matrix product of several random processes.

The results of Eqs. (32) and (33) are extendable to any link-node model that can be formally expressed in the computer program HEC-1 or related systems. Consequently, it is now possible to mathematically describe and manipulate the subarea UH runoff hydrographs and the link hydrograph routing submodels used in a typical link-node model structure of a catchment, such as the UH and Muskingum routing options in HEC-1 (see US Army Corps of Engineers, 1990), or the UH and Convex routing options used in TR-20 (SCS, 1984), among many others.

3.6. Comparison of the Toeplitz matrix formulation to the stochastic integral equation method

The Stochastic Integral Equation Method or SIEM

(Hromadka and Whitley, 1989) provides a mathematical representation of rainfall–runoff models by use of stochastic processes. The Toeplitz matrix systems used in this paper are discretized versions of the stochastic integral equations used in the SIEM, and hence the theoretical underpinnings of the SIEM applies also to the Toeplitz matrix formulation.

For a matrix $[A] \in T(n)$, it is seen that $[A]$ is fully defined by an $n \times 1$ column vector $\underset{\sim}{a}$ where

$$\underset{\sim}{a} = \begin{vmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix}; \quad [A] = \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & \cdots & a_1 \end{vmatrix}. \tag{34}$$

The vector $\underset{\sim}{a}$ is a discretization in time of a typical realization of a stochastic process in the SIEM. Also, due to the need to solve the SIEM systems computationally, methods are required that provide numerical integration of the stochastic integrals (Hromadka and Whitley, 1989), resulting in the Toeplitz matrix systems derived above. Consequently, the matrix systems involved in the numerical integration of the stochastic integral equations, and the Toeplitz matrix systems developed in the current paper, are seen in practice to be the same. This observation provides a direct link between the computer program HEC-1 and related systems, and the SIEM.

3.7. Including rainfall variations across the catchment

In this section, a Toeplitz matrix is constructed by considering the sum of proportions of translates of a reference vector. Given the reference rainfall data, for storm i, at a rain gauge, $P_g^i(t)$, the rainfall at another location, $P_j^i(t)$, can be modeled as the sum of products of proportion factors (which are variable between storms) with the $P_g^i(t)$, as follows:

$$P_j^i(t) = \lambda_{j1}^i P_g^i(t) + \lambda_{j2}^i P_g^i(t-\Delta t) + \lambda_{j3}^i P_g^i(t-2\Delta t) + \cdots$$

$$= \sum_{k=1}^{n_p} \lambda_{jk}^i P_g^i(t-(k-1)\Delta t) \tag{35}$$

where $P_g^i(\tau) = 0$ for $(\tau) < 0$; Δt is the model time step; and n_p is sufficiently large to achieve the selected accuracy (issues regarding n_p, Δt, and analogs to the generalized Fourier series can be found in Hromadka and Whitley, 1989).

In vector form, $P_j^i(t)$ can be represented as $\underset{\sim}{P}^{\mathrm{T}} = (p_1, p_2, \ldots, p_n)$, and the various translates of Eq. (35) follows. For example

$$P_g^i(t-2\Delta t) \Rightarrow \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{vmatrix}. \tag{36}$$

Thus, Eq. (35) can be written as another Toeplitz matrix system composed of n_p Toeplitz matrices,

$$\underset{\sim}{P}_j^i = \lambda_{j1}^i \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \underset{\sim}{P}_g^i + \lambda_{j2}^i \begin{vmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} P_g^i$$

$$+ \lambda_{j3}^i \begin{vmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} P_g^i + \cdots$$

$$= \begin{vmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_2 & \lambda_1 & 0 & \cdots & 0 \\ \lambda_3 & \lambda_2 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_n & \lambda_{n-1} & \lambda_{n-2} & \cdots & \lambda_1 \end{vmatrix}_j \underset{\sim}{P}_g^i \tag{37}$$

where the Toeplitz matrix on the right-hand side of Eq. (37), is referenced to storm i, where point j is usually the centroid of subarea Ω_j. Denoting this final Toeplitz matrix of Eq. (37) as $[V_j^i]$, then

$$\underset{\sim}{P}_j^i = [V_j^i]\underset{\sim}{P}_g^i \tag{38}$$

which is a relation analogous to Eq. (31).

Example 6. *Effective rainfall variabilities for a constant fraction loss rate model:*

If the loss function is a constant proportion loss fraction $(1 - c_j)$, for subarea Ω_j, where $0 < c_j < 1$, then

$$\underset{\sim}{e}_j^i = c_j[V_j^i]\underset{\sim}{P}_g^i = [T_j^i]\underset{\sim}{e}_g^i \tag{39}$$

defining, for this loss function the relation between $[V_j^i]$ and $[T_j^i]$.

4. Some additional topics

In the following, a few topics of possible interest are focused upon.

4.1. Topic 1: derivation of the UH method as an approximation

The UH method can be derived as an approximation from a certain natural assumption. To begin, let the effective rainfall on a catchment or subarea, at times $t_j = j(\Delta t)$, be $e_1, \ldots, e_n$, with discharges $q_1, \ldots, q_n$ at the same times. Consider the increments

$$\Delta e_k = e_k - e_{k-1}, \tag{40}$$

$$\Delta q_k = q_k - q_{k-1} \tag{41}$$

for $k = 1, 2, \ldots, n$, where $e_0 = 0 = q_0$.

It is plausible to assume that there is an unknown function f of M variables with

$$\Delta q_k = f(\Delta e_k, \Delta e_{k-1}, \ldots, \Delta e_1, 0, 0, \ldots, 0). \tag{42}$$

The assumption that such a function exists is really the assumption that the hydrologic process which produces incremental discharge from incremental effective rainfall, via Eq. (42), is stationary; i.e. starting at any base time t_0, the same sequence of increments in effective rainfall will produce the same sequence of incremental discharges.The interpretation of f requires

$$f(0, \ldots, 0) = 0. \tag{43}$$

To simplify the notation, write $f(x_1, \ldots, x_m)$ for the right-hand side of Eq. (42), and $f(0)$ for $f(0, \ldots, 0)$.

Applying a Taylor series expansion yields

$$f(x_1, \ldots, x_m) = \sum_i^M \frac{\partial f}{\partial x_j}(0)x_j + \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}(0)x_i x_j + \cdots \tag{44}$$

For small Δe_j, neglecting the higher order terms results in

$$\Delta q_k \cong a_1 \Delta e_k + a_2 \Delta e_{k-1} + \cdots + a_k e_1 \tag{45}$$

where

$$a_j = \frac{\partial f}{\partial x_j}(0). \tag{46}$$

Then

$$\begin{aligned} \Delta q_1 &= a_1(e_1 - 0) \\ \Delta q_2 &= a_1(e_2 - e_1) + a_2(e_1 - 0) \\ &\vdots \\ \Delta q_k &= a_1(e_k - e_{k-1}) + \cdots + a_k(e_1 - 0) \end{aligned} \tag{47}$$

Add by columns to obtain

$$q_k = a_1 e_k + a_2 e_{k-1} + \cdots + a_k e_1 \tag{48}$$

(keeping in mind that if the approximation (45) is good to within ε, (48) may only be accurate to within $k\varepsilon$).

Then

$$\begin{vmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 \end{vmatrix} \begin{vmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{vmatrix} = \begin{vmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{vmatrix} \tag{49}$$

and the Toeplitz matrix

$$[A] = \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_k & a_{k-1} & \cdots & a_1 \end{vmatrix} \tag{50}$$

is the UH producing $q_1,...,q_k$ from $e_1,...,e_k$. A similar derivation can be developed for various hydrograph-routing Toeplitz matrices.

4.2. Topic 2: approximating uncertainty in link-node UH models for risk analysis

Given the link-node UH model structure of Eq. (32), an approximation of modeling uncertainty due to variations in storms can be developed by treating the output of the link-node UH model structure as a single-area UH model (see Eq. (33)), and multiplying each link-node UH model output by the set of runoff hydrograph Toeplitz correction matrices (rescaled to the study catchment) developed for single-area UH models (Hromadka and Whitley, 1997).

For example, if $\{[\beta^i];\ i = 1,2,...,k\}$ are k storm Toeplitz correction matrices (properly scaled to the catchment under study), such as that developed in Eqs. (15)–(17), and $\underset{\sim}{Q}^j$ is a link-node UH model output vector for a storm j, then an estimate of the distribution of hydrograph predictions is the discrete set of outcomes:

$$\{[\beta^i]\underset{\sim}{Q}^j;\ i = 1,2,...,k\}. \tag{51}$$

In Hromadka and Whitley (1997), details and mathematical derivations are provided regarding the application of Eq. (51) to single-area UH model structures such as Eq. (33). It is noted that in the application of Eq. (51), the Toeplitz matrices $[\beta^i]$ are conditioned according to the loss function transform used to develop effective rainfall (rainfall less losses from rainfall). Additionally, one may further condition $[\beta^i]$ matrices of Eq. (51) according to classes of storm size and intensity such as that discussed in Hromadka and Whitley (1989).

4.3. Topic 3: linear-reservoir basin routing and natural valley storage effects

The level-pool reservoir or modified-Puls method for routing a hydrograph through a flood-control basin, or for modeling the effects of natural valley storage, is given by the relation

$$(I_1 + I_2)/2 = (O_1 + O_2)/2 + (S_2 - S_1)/\Delta t \tag{52}$$

where I_1 and O_1 are inflow and outflow at reference time 1; S_1 is the storage at time 1; subscript 2 refers to future time 2; and Δt is the time step size. In the case of a linear storage–discharge relation, where the storage versus depth, and discharge (outflow) versus depth relations are proportional, then $O = kS$, and the resulting linear-reservoir method may be rewritten as

$$(I_1 + I_2)/2 = (O_1 + O_2)/2 + (O_2 - O_1)k\Delta t. \tag{53}$$

Combining the terms, the following equation is obtained:

$$c_1 I_1 + c_2 I_2 + c_3 O_1 = O_2 \tag{54}$$

where $c_1 = c_2 = k\Delta t/(2 + k\Delta t)$, $c_3 = (2 - k\Delta t)/(2 + k\Delta t)$; which is analogous to the Muskingum routing technique, can be entirely expanded in terms of only inflow unit values, resulting in another Toeplitz matrix system

$$\underset{\sim}{Q} = \begin{vmatrix} c_1 & 0 & 0 & 0 & \cdots & 0 \\ c_1(1+c_3) & c_1 & 0 & 0 & \cdots & 0 \\ c_1c_3(1+c_3) & c_1(1+c_3) & c_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_1c_3^{n-2}(1+c_3) & c_1c_3^{n-3}(1+c_3) & \cdots & \cdots & \cdots & c_1 \end{vmatrix} \underset{\sim}{I} \tag{55}$$

or, letting $[LR]$ be the above Toeplitz matrix

$$\underset{\sim}{Q} = [LR]\underset{\sim}{I} \tag{56}$$

where $\underset{\sim}{Q}$ and $\underset{\sim}{I}$ are the outflow and inflow unit-period value vectors, respectively.

As before, any routing technique is only an approximation and hence there would be another associated Toeplitz correction matrix involved. The $[LR]$ Toeplitz matrices can be inserted and manipulated into the link-node UH model structure of Eq. (32) analogous to the other Toeplitz matrices in that equation.

Given the Toeplitz matrix systems developed for the Convex, Muskingum, translation, and linear-reservoir hydrologic-routing procedures, the issue arises as to which routing method is the "best"? All of these methods are simply different Toeplitz matrices; is there a "best" Toeplitz matrix? That is, what element $[H] \in T(n)$ best satisfies $\underset{\sim}{Q} = [H]\underset{\sim}{I}$? If data are available for $\underset{\sim}{Q}$ and $\underset{\sim}{I}$, then $[H]$ is uniquely solved by forward substitution, and is simply a convolution such as that studied by Doyle et al. (1983) and Becker and Kundzewicz (1987), among others. That is, only the convolution technique that solves $\underset{\sim}{Q} = [H]\underset{\sim}{I}$, determines the "correct" Toeplitz matrix $[H]$.

It is noted that the above Toeplitz matrix system can also be used when the subarea (or total catchment) exhibits storage effects analogous to the proportional outflow–storage relation used in the above derivation of the $[LR]$ matrix.

4.4. Topic 4: reducing a link-node UH model to a single-area UH model structure

Many types of loss functions can be specified in a rainfall–runoff model. Up to this point, no particular loss function has been specified to develop $\underset{\sim}{e}_j^i$ vectors. For the remainder of this paper, the distributed phi-index type of loss function is considered. Many other types of distributed loss functions can be formulated in terms of Toeplitz matrix structures.

For the example problem of a four-subarea link-node UH model (Fig. 2), the modeled runoff hydrograph, for storm i, is the approximation $\hat{\underset{\sim}{Q}}{}_{\mathrm{LN}}^i$, where

$$\hat{\underset{\sim}{Q}}{}_{\mathrm{LN}}^i = [\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}][U_1]\underset{\sim}{e}_1^i + [\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}][U_2]\underset{\sim}{e}_2^i + [\mathbb{R}_{\mathbb{BC}}][U_3]\underset{\sim}{e}_3^i + [U_4]\underset{\sim}{e}_4^i \quad (57)$$

where the subscript "LN" refers to the "link-node" model structure, and the hat indicates that the Toeplitz correction matrices are not included, and, hence, only an approximation is obtained. For the conditions that each $\underset{\sim}{e}_j^i = \underset{\sim}{P}_{\mathrm{g}}^i - \underset{\sim}{\phi}_j$ (i.e. the typical phi-index loss function is used for each subarea, and there is a single source of rainfall data for storm i, $\underset{\sim}{P}_{\mathrm{g}}^i$, with $P_{\mathrm{g}}^i(t) \geq \phi_j$ for each subarea j), then the above expansion for $\hat{\underset{\sim}{Q}}{}_{\mathrm{LN}}^i$ can be rewritten as

$$\hat{\underset{\sim}{Q}}{}_{\mathrm{LN}}^i = [U_{\mathrm{LN}}]\underset{\sim}{P}_{\mathrm{g}}^i - [F_{\mathrm{LN}}]\underset{\sim}{\delta}_k^i \quad (58)$$

where

$$[U_{\mathrm{LN}}] = [\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}]([U_1] + [U_2]) + [\mathbb{R}_{\mathbb{BC}}][U_3] + [U_4], \quad (59)$$

$$[F_{\mathrm{LN}}] = [\mathbb{R}_{\mathbb{BC}}][\mathbb{R}_{\mathbb{AB}}]([U_1]\phi_1 + [U_2]\phi_2) + [\mathbb{R}_{\mathbb{BC}}][U_3]\phi_3 + [U_4]\phi_4 \quad (60)$$

where $\underset{\sim}{\delta}_k^i$ is an $n \times 1$ column vector such that the values of rows 1 to k are 1, and the values of rows $k+1$ to n are zero, and subscript k refers to storm i with a duration of $k\Delta t$.

It is noted that Eq. (57) is a restatement of Eq. (30) without the Toeplitz correction matrices. From Eqs. (59) and (60), given the subarea UH unit values and the hydrograph-routing Toeplitz matrix unit values, an equivalent single-area UH is directly developed as $[U_{\mathrm{LN}}]$, and the distributed phi-index loss function is represented as a new vector, $\underset{\sim}{F}_{\mathrm{LN}}$, which is the first column of $[F_{\mathrm{LN}}]$ in Eq. (60).

If each subarea has identical phi-index values, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_0$, then Eq. (60) simplifies to

$$[F_{\mathrm{LN}}] = \phi_0[U_{\mathrm{LN}}] \quad (61)$$

and the equivalent single-area UH model of Eq. (58) simplifies to

$$\hat{\underset{\sim}{Q}}{}_{\mathrm{LN}}^i = [U_{\mathrm{LN}}]\underset{\sim}{e}_{\mathrm{g}}^i \quad (62)$$

where $\underset{\sim}{e}_{\mathrm{g}}^i$ is the discretized vector representation of $(\underset{\sim}{P}_{\mathrm{g}}^i - \underset{\sim}{\phi}_0^i)$, and $\underset{\sim}{\phi}_0^i = \phi_0\underset{\sim}{\delta}_k^i$.

A similar development of the above equations can be made for other loss rates, such as the constant fraction loss rate.

As in HEC-1 the phi-index method is a widely used loss function, hydrograph routing is typically modeled using a hydrologic technique such as the Muskingum method, and natural valley storage can also be modeled (for outflow being proportional to storage) as a Toeplitz matrix, the above mathematical relations provide a reduction of a complex link-node UH model structure to the single-area UH model structure of Eq. (58).

Example 7. *Reduction of a HEC-1 link-node UH model to a single-area UH model: San Sevaine Creek.*

A HEC-1 link-node UH model was prepared for the 137 km^2 San Sevaine watershed in the City of Fontana, CA. The HEC-1 model is composed of 24 Muskingum routing links and 19 subareas, each subarea with its own UH and phi-index loss-rate function (see Table 1).

Rather than directly constructing the link-node UH model $[U_{\mathrm{LN}}]$ and $[F_{\mathrm{LN}}]$ Toeplitz matrices using the equations developed previously, these two matrices will be developed indirectly by equating the

Table 1
Example 7 link-node UH model parameters

Subarea data					Muskingum routing data		
Subarea number	Area (km²/mi²)	Phi-index ((mm/h)/(in/h))	S-graph[a]	Lag (h)	Link ID	X	K (h)
1	13.08/5.05	9.65/0.38	Fullerton–San Jose S-graph	0.32			
					1–2	0.1	0.047
2	19.63/7.58	9.91/0.39	Fullerton–San Jose S-graph	0.46			
3	6.58/2.45	10.41/0.41	LACDA[b] Urban S-graph	0.41			
					3(2)–3.1	0.1	0.047
					3.1–3.2	0.1	0.047
					3.2–4	0.1	0.047
4	9.32/3.6	9.14/0.36	LACDA Urban S-graph	0.53			
					4–4.1	0.1	0.047
					4.1–4.2	0.1	0.047
					4.2–5	0.1	0.047
5	9.53/3.69	8.38/0.36	LACDA Urban S-graph	0.46			
					5–5.1	0.1	0.047
					5.1–5.2	0.1	0.047
					5.2–6	0.1	0.047
6	5.41/2.09	6.35/0.25	LACDA Urban S-graph	0.38			
11	14.04/5.42	9.91/0.39	Fullerton–San Jose S-graph	0.33			
					11–12	0.1	0.047
12	0.75/0.29	13.97/0.55	Fullerton–San Jose S-graph	0.14			
					12–12.1	0.1	0.047
					12.1–12.2	0.1	0.047
					12.2–13	0.1	0.047
13	3.57/1.38	15.49/0.61	LACDA Urban S-graph	0.33			
					13–13.1	0.1	0.047
					13.1–13.2	0.1	0.047
					13.2–13.3	0.1	0.047
					13.3–14	0.1	0.047
14	1.11/0.43	7.62/0.30	LACDA Urban S-graph	0.19			
					14–16	0.1	0.047
15	3.76/1.45	9.14/0.36	LACDA Urban S-graph	0.38			
					15(6)–16	0.3	0.071
16	16.39/6.33	6.35/0.25	LACDA Urban S-graph	0.62			
					16–17	0.1	0.047
17	3.19/1.23	1.78/0.07	LACDA Urban S-graph	0.50			
					17–18	0.1	0.047
18	3.85/1.49	1.78/0.07	LACDA Urban S-graph	0.33			
					18–19	0.1	0.047
19	20.85/8.05	7.11/0.28	LACDA Urban S-graph	0.60			
					19–20	0.1	0.047
20	3.00/1.16	1.78/0.07	LACDA Urban S-graph	0.33			
21	1.19/0.46	2.29/0.09	LACDA Urban S-graph	0.26			
					21(20)–22	0.1	0.047
22	1.27/0.49	1.52/0.06	LACDA Urban S-graph	0.29			
					22–23	0.1	0.047
23	1.17/0.45	1.52/0.06	LACDA Urban S-graph	0.29			

[a] S-graphs are from HEC-1 (LAPRE1 version). LAPRE1, Los Angeles Preprocessor #1, for HEC-1.
[b] LACDA, Los Angeles County Drainage Area.

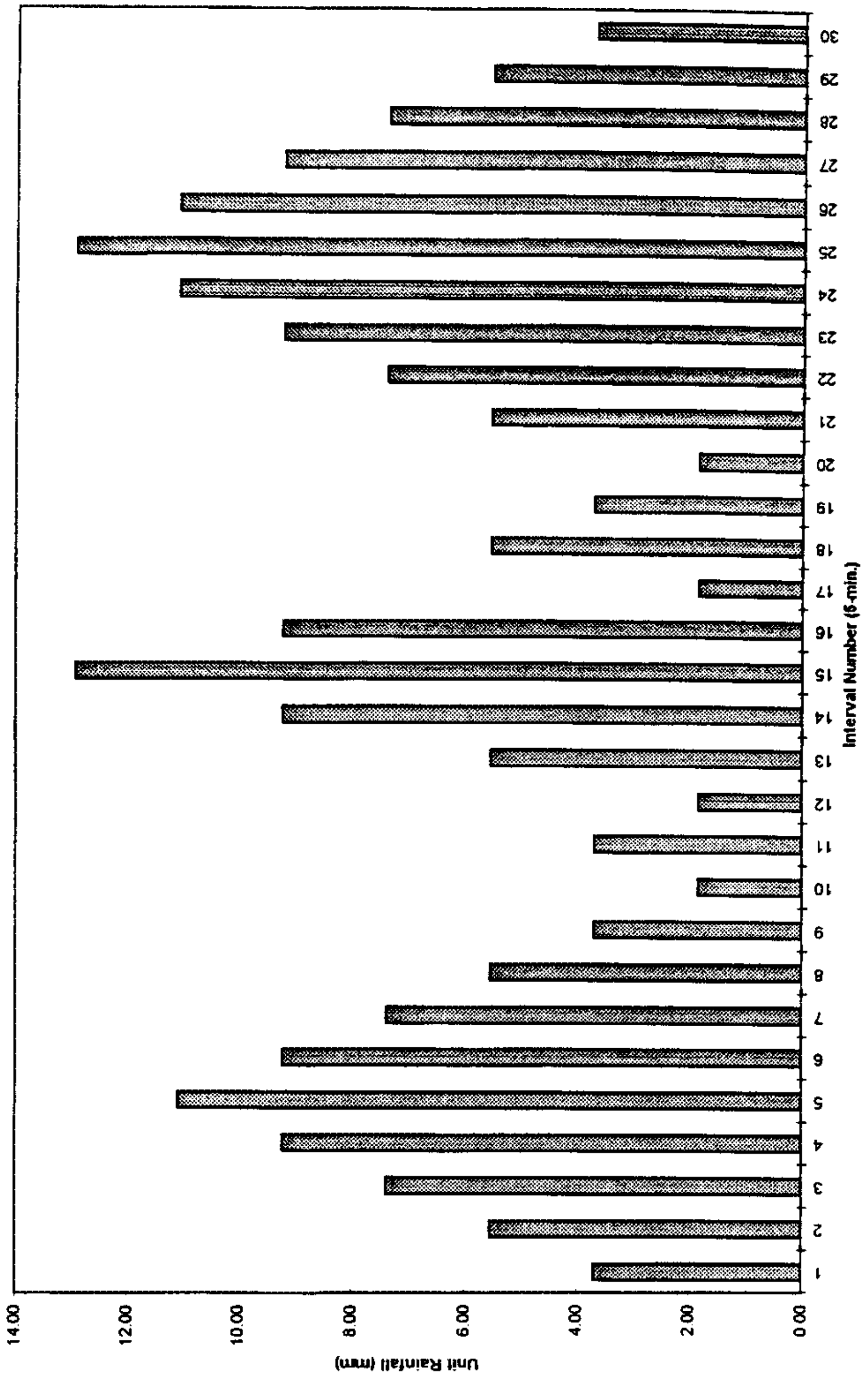

Fig. 3. Multi-peaked rainfall pattern used in Example 7.

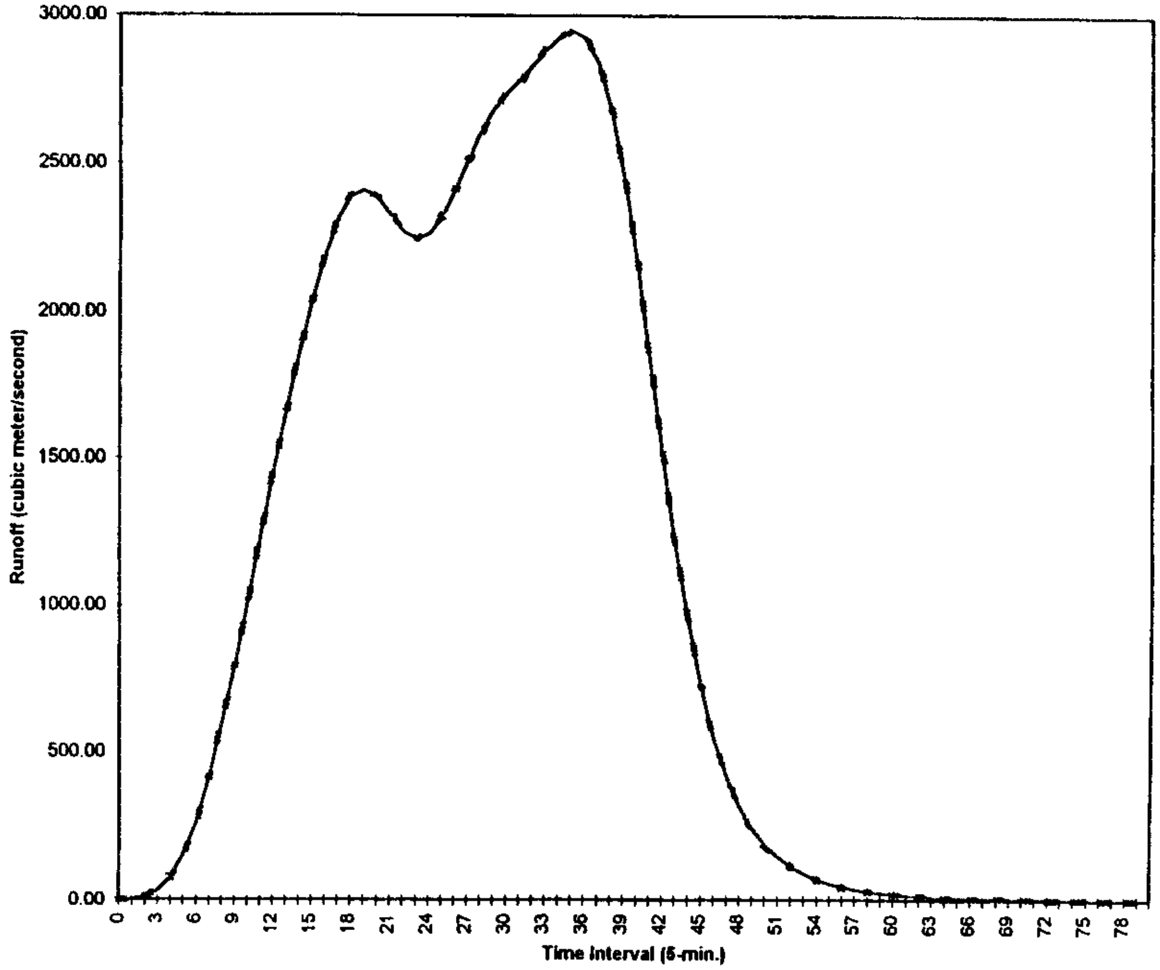

Fig. 4. Comparison plots of San Sevaine link-node UH HEC-1 (dashed line) versus the single-area UH representation (solid line).

single-area UH model structure representation to the HEC-1 link-node UH model output as described by the following procedure.

The approach used is to set both the single-area UH model and the link-node UH model using identical unit period values of rainfall. By choosing a unit period rainfall that exceeds all subarea loss rates (used in the link-node UH model), the effective rainfalls can be mathematically described by the previous equations, for each subarea. In the current case, a 5 min unit rainfall of 0.2 in. exceeds all subarea phi-index loss rate values. The link-node UH model, HEC-1, with the parameters given in Table 1 was used to generate runoff from two single unit-period storms, one with 0.2 in. of unit rainfall and the other with 1.2 in. of unit rainfall. By subtracting the ordinates of the simulated runoff for the first unit storm from those of the second unit storm, the UH (i.e. 1 in. of runoff) corresponding to the entire HEC-1 model is obtained. The link-node model and the synthesized single-area UH model are then applied to a multi-peaked rainfall time series (Fig. 3) and yielded identical results (Fig. 4). The mathematical underpinnings of this approach are described in the following text.

A unit pulse vector, $\underset{\sim}{\delta}_k$, is a $n \times 1$ column vector where the values of rows 1 to k are 1, and the values of rows $k+1$ to n are zero. Choose any unit-period precipitation value, p^a, such that $p^a > \phi_j$, for each subarea j (recall that in this development, unit-period

Table 2
Equivalent single-area UH model Toeplitz matrix terms for Example 7 (column 4 is the single-area model UH equivalent to the 19 subarea HEC-1 model; column 5 is the loss vector equivalent to the distributed phi-index function) (maximum phi-index unit period value, 1.30 mm (0.051 in.); the numbers in the table are rounded off)

(1) Unit #	(2) Q ($P = 0.2$)[a]	(3) Q ($P = 1.2$)	(4) $[U]$: (3) − (2)	(5) $[F]$: $[U] * (0.2) - Q$ ($P = 0.2$)
1	77	482	405	4.00
2	371	2327	1956	20.20
3	862	5431	4569	51.80
4	1549	9821	8272	105.40
5	2264	14 486	12 222	180.40
6	2906	18 788	15 882	270.40
7	3455	22 581	19 126	370.20
8	3991	26 364	22 373	483.60
9	4550	30 356	25 806	611.20
10	5128	34 508	29 380	748.00
11	5600	37 972	32 372	874.40
12	5856	39 971	34 115	967.00
13	5919	40 604	34 685	1018.00
14	5699	39 218	33 519	1004.80
15	5121	35 316	30 195	918.00
16	4286	29 584	25 298	773.60
17	3352	23 149	19 797	607.40
18	2500	17 266	14 766	453.20
19	1826	12 614	10 788	331.60
20	1335	9226	7891	243.20
21	1001	6921	5920	183.00
22	774	5350	4576	141.20
23	609	4207	3598	110.60
24	481	3327	2846	88.20
25	376	2598	2222	68.40
26	288	1993	1705	53.00
27	219	1521	1302	41.40
28	166	1154	988	31.60
29	126	877	751	24.20
30	99	688	589	18.80
31	80	559	479	15.80
32	66	462	396	13.20
33	55	383	328	10.60
34	46	319	273	8.60
35	39	269	230	7.00
36	33	229	196	6.20
37	28	194	166	5.20
38	23	161	138	4.60
39	19	131	112	3.40
40	16	109	93	2.60
41	14	96	82	2.40
42	13	89	76	2.20
43	12	86	74	2.80
44	12	84	72	2.40
45	12	81	69	1.80
46	11	75	64	1.80
47	9	65	56	2.20
48	7	50	43	1.60
49	5	34	29	0.80
50	3	20	17	0.40
51	1	9	8	0.60
52	1	4	3	− 0.40

[a] $Q(P = 0.2)$ is the runoff hydrograph from 27-link and 19-subarea link-node UH HEC-1 model, given a single unit period storm rainfall of 0.58 mm (0.2 in).

rainfall is assumed to exceed unit-period losses) and define the unit-period precipitation value $p^b = p^a + 1$. Then, precipitation vectors, $\underset{\sim}{p}^a$ and $\underset{\sim}{p}^b$, are defined by

$$\underset{\sim}{p}^a = p^a \underset{\sim}{\delta}_1; \qquad \underset{\sim}{p}^b = p^b \underset{\sim}{\delta}_1 \tag{63}$$

where $\underset{\sim}{p}^a$ and $\underset{\sim}{p}^b$ are $n \times 1$ column vectors (consistent with all other matrices and vectors) composed of zeroes except at row 1, where unit-period values p^a and p^b are defined, respectively.

Let $\underset{\sim}{Q}^a_{\mathrm{LN}}$ and $\underset{\sim}{Q}^b_{\mathrm{LN}}$ be the runoff hydrographs from the link-node UH model operating on precipitation vectors $\underset{\sim}{p}^a$ and $\underset{\sim}{p}^b$, respectively. Equating the vectors $\underset{\sim}{Q}^a_{\mathrm{LN}}$ and $\underset{\sim}{Q}^b_{\mathrm{LN}}$ to the single-area UH model structure gives, from Eq. (58),

$$\underset{\sim}{Q}^b_{\mathrm{LN}} - \underset{\sim}{Q}^a_{\mathrm{LN}} = [U_{\mathrm{LN}}](\underset{\sim}{p}^b - \underset{\sim}{p}^a) = [U_{\mathrm{LN}}]\underset{\sim}{\delta}_1 \tag{64}$$

where $(\underset{\sim}{p}^b - \underset{\sim}{p}^a) = \underset{\sim}{\delta}_1$. But $[U_{\mathrm{LN}}]\underset{\sim}{\delta}_1$ is simply the first column of $[U_{\mathrm{LN}}]$

$$[U_{\mathrm{LN}}]\underset{\sim}{\delta}_1 = \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{vmatrix} \tag{65}$$

and, the single-area UH Toeplitz matrix, $[U_{\mathrm{LN}}]$, is completely specified by knowing $u_1, u_2, \ldots, u_n$.

In Eq. (60), we can rewrite each vector $\underset{\sim}{\phi}_j$ with respect to storm i by

$$\underset{\sim}{\phi}_j = \underset{\sim}{\phi}^i_j = \phi_j \underset{\sim}{\delta}^i_k \tag{66}$$

where $k\Delta t$ is the rainfall duration of storm i; $\underset{\sim}{\delta}^i_k$ is a unit-pulse vector as defined previously; and ϕ_j is the subarea j phi-index value. Let ϕ_0 be a reference phi-index value, then each $\phi_j = k_j\phi_0$ where k_j are constants. Then, Eq. (60) can be rewritten as

$$[F_{\mathrm{LN}}] = \phi_0([\mathbb{R}_{\mathrm{BC}}][\mathbb{R}_{\mathrm{AB}}]([U_1]k_1 + [U_2]k_2) + [\mathbb{R}_{\mathrm{BC}}] \times [U_3]k_3 + [U_4]k_4)\underset{\sim}{\delta}^i_k = \phi_0[L_{\mathrm{LN}}]. \tag{67}$$

To compute the parameter ϕ_0, a mass balance is applied based on the runoff vector $\underset{\sim}{Q}^a_{\mathrm{LN}}$ where total rainfall (i.e. $p^a A_{\mathrm{T}}$) less total runoff (i.e. the sum of the unit runoff values of $\underset{\sim}{Q}^a_{\mathrm{LN}}$) equals total losses (i.e. $\phi_0 A_{\mathrm{T}}$), where A_{T} is the total catchment area, and appropriate units are used (including the timestep chosen, Δt).

Finally, the Toeplitz matrix $[F_{\mathrm{LN}}]$ is determined by solving

$$\underset{\sim}{Q}^a_{\mathrm{LN}} = [U_{\mathrm{LN}}]\underset{\sim}{p}^a - [F_{\mathrm{LN}}]\underset{\sim}{\delta}_1 = (p^a[U_{\mathrm{LN}}] - [F_{\mathrm{LN}}])\underset{\sim}{\delta}_1 \tag{68}$$

or,

$$[F_{\mathrm{LN}}]\underset{\sim}{\delta}_1 = \begin{vmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{vmatrix} = \left\| p^a \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{vmatrix} - \begin{vmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{vmatrix} \right\|. \tag{69}$$

For the San Sevaine watershed HEC-1 link-node UH model, the various matrices are shown in Table 2. The equality of the single-area UH model with the 24 link and 19 subarea HEC-1 model is demonstrated by the equal model outputs for various storm events (where unit rainfalls exceed unit losses), an example of which is shown in the multipeaked storm of Fig. 3 and the corresponding runoff hydrographs of Fig. 4.

5. Discussion

From the previous derivations, a HEC-1 type distributed phi-index loss rate link-node model is mathematically equivalent to a single-area UH model structure

$$\underset{\sim}{Q}^i_{\mathrm{LN}} = [U_{\mathrm{LN}}]\underset{\sim}{P}^i - \phi_0[L_{\mathrm{LN}}]\underset{\sim}{\delta}^i_k \tag{70}$$

where $\phi_0[L_{\mathrm{LN}}] = [F_{\mathrm{LN}}]$ is a Toeplitz matrix; the duration of storm i is $k\Delta t$; $[U_{\mathrm{LN}}]$ and $[L_{\mathrm{LN}}]$ are Toeplitz matrices constructed from combining the subarea UH and loss rate Toeplitz matrices and link hydrograph-routing Toeplitz matrices. The Toeplitz matrices $[U_{\mathrm{LN}}]$ and $[L_{\mathrm{LN}}]$ are defined by the previously developed mathematical relations, and fully represent the link-node UH model structure, for the conditions assumed. If the subarea ϕ_j are all equal to ϕ_0, then $[F_{\mathrm{LN}}] = \phi_0[U_{\mathrm{LN}}]$ and $\underset{\sim}{Q}^i_{\mathrm{LN}} = [U_{\mathrm{LN}}](\underset{\sim}{P}^i - \phi_0\underset{\sim}{\delta}^i_k)$, which is the Toeplitz matrix form of the more traditional single-area UH model structure. Many other types of distributed loss functions defined in subareas will result in a Toeplitz matrix structure.

If one chooses parameters for each component of the link-node UH model, then the model structure produces runoff hydrographs $\underset{\sim}{Q}^i_{\mathrm{LN}}$ (again "LN" refers to a "link-node" model) as

$$\underset{\sim}{Q}^i_{\mathrm{LN}} = [U_{\mathrm{LN}}]\underset{\sim}{P}^i - [F_{\mathrm{LN}}]\underset{\sim}{\delta}^i_k \quad (71)$$

where $[U_{\mathrm{LN}}]$ and $[F_{\mathrm{LN}}]$ are the link-node model Toeplitz matrices constructed by the assemblage of the numerous link and subarea modeling element Toeplitz matrices.

In comparison, the traditional single-area UH model structure produces runoff hydrographs $\underset{\sim}{Q}^i_{\mathrm{SA}}$ (where "SA" refers to a "single-area" UH model) as

$$\underset{\sim}{Q}^i_{\mathrm{SA}} = [U_{\mathrm{SA}}]\underset{\sim}{P}^i - [F_{\mathrm{SA}}]\underset{\sim}{\delta}^i_k \quad (72)$$

where $[U_{\mathrm{SA}}]$ and $[F_{\mathrm{SA}}]$ refer to the single-area UH model Toeplitz matrices which are developed by direct calibration to gauged data (or from statistical evaluation of regional data).

Both the LN and SA models are calibrated to the same rainfall–runoff data. The SA model calibration process can be interpreted as the calibration of the numerous mutually dependent relations interior to the watershed such as the subarea UHs and phi-index values, the choice of link routing algorithm and parameters, and natural valley storage effects. Meanwhile, the LN model contains numerous constraints and errors imposed by the prescription and selection of algorithms, processes, and relations, leaving only a handful of parameters to be variable (and, furthermore, whose values are typically constrained to lie within "rational" limits).

Both the LN and SA models are seen to have identical mathematical structures; they simply differ in their Toeplitz matrix components. The question then arises as to which Toeplitz matrix set is "best"; is modeling error reduced by using the Toeplitz matrix set $\{[U_{\mathrm{LN}}],[F_{\mathrm{LN}}]\}$ instead of $\{[U_{\mathrm{SA}}],[F_{\mathrm{SA}}]\}$? Given the rainfall–runoff data for several storms, one calibrates the LN matrices by varying modeling algorithm parameters within prescribed domains; the resulting calibrated Toeplitz matrices are denoted as $\{[U^c_{\mathrm{LN}}],[F^c_{\mathrm{LN}}]\}$. The SA matrices can be calibrated by solving a least-squares error minimization problem simultaneously with respect to all storms and with respect to the n components of $[U_{\mathrm{SA}}]$ and the n components of $[F_{\mathrm{SA}}]$, resulting in the unique calibrated $n \times n$ matrices $\{[U^c_{\mathrm{SA}}],[F^c_{\mathrm{SA}}]\}$. As $\{[U^c_{\mathrm{SA}}],[F^c_{\mathrm{SA}}]\}$ achieves the minimum variance between the model structure and all the data, then the calibrated LN matrix set can perform no better. Although link-node models do not necessarily result in improved fits to measured hydrographs relative to single-area UH models; they have advantages for planning and design because changes in the watershed or channel can be considered. For example, if one subarea is urbanized, new loss rates and subarea UHs can be developed and the changes in runoff for the watershed can be examined.

Other topics that are readily developed include baseflow effects, multiple rain gauge data sources, and the conditioning of the Toeplitz matrix system representations with respect to storm classes (and hence including the effects of quasi-linearity). These topics follow directly from the Stochastic Integral Equation Method formulation of Hromadka and Whitley (1989).

5.1. Topic 6: calibrating the single-area UH model structure

The mathematical structure under study is, with respect to any storm i,

$$\underset{\sim}{Q}^i_{\mathrm{SA}} = [U_{\mathrm{SA}}]\underset{\sim}{P}^i - [F_{\mathrm{SA}}]\underset{\sim}{\delta}^i_k + \underset{\sim}{B} \quad (73)$$

where $\underset{\sim}{B} = b_0\underset{\sim}{\delta}_n$, $[F_{\mathrm{SA}}] = \phi_0[L_{\mathrm{SA}}]$, b_0 is a selected constant baseflow, ϕ_0 is a selected loss rate reference phi-index value. Although the mathematical development has not considered in detail the constants b_0 and ϕ_0, these are likely to vary on a storm-class basis (Hromadka and Whitley, 1989); for example, different antecedent soil-moisture conditions tend to correlate with different values of b_0 and ϕ_0. The SA model is a function of $(2n + 1)$ parameters:

$$\langle \mathbb{P} \rangle = \langle u_1, u_2, \ldots, u_n;\ f_1, f_2, \ldots, f_n;\ b_0 \rangle \quad (74)$$

where $\langle \mathbb{P} \rangle$ is the SA model parameter set, $u_1, u_2, \ldots, u_n$ are the $[U_0]$ Toeplitz matrix components, $f_1, f_2, \ldots, f_n$ are the $[F]$ Toeplitz matrix components (because ϕ_0 is typically chosen (i.e. weighted) to "balance" runoff volumes), and b_0 may also be chosen based on assumed conditions, resulting in $2n + 1$ parameters for $\langle \mathbb{P} \rangle$.

Model error, for storm i, noted as E^i, is

$$\underset{\sim}{E}^i = \underset{\sim}{Q}^i - \underset{\sim}{Q}^i_{\mathrm{SA}} \tag{75}$$

and a ℓ_2 (i.e. the usual least-squares residual minimization) measure of error is E_2^i where

$$E_2^i = \sum_{j=1}^{n} (q_j^i - q_{j\mathrm{SA}}^i)^2 \tag{76}$$

where q_j^i and $q_{j\mathrm{SA}}^i$ are the unit-period j values from $\underset{\sim}{Q}^i$ and $\underset{\sim}{Q}^i_{\mathrm{SA}}$, respectively.

The total ℓ_2 error, for m storm events, is E_m, where

$$E_m = \sum_{j=1}^{m} E_2^j. \tag{77}$$

In the above, weightings may be imposed to focus a better fit towards peak flow rate, or other attributes (such as the parameters b_0 and ϕ_0). Obviously, the choice of weightings affects the SA model parameter set, $\langle \mathbb{P} \rangle$.

The $\langle \mathbb{P} \rangle$ set is determined using Gramm–Schmidt vector orthogonalization techniques such as Hromadka and Whitley (1993, Chapt. 5). The resulting calibrated set is $\langle \mathbb{P}^c \rangle$, which minimizes the total ℓ_2 error according to the weightings prescribed (if any). In this case, ℓ_2 error minimization is across all storms, rather than on a storm by storm basis. Once a $\langle \mathbb{P}^c \rangle$ is determined, it is suitable for statistical regionalization analysis with other catchment $\langle \mathbb{P}^c \rangle$ sets, analogous to the procedures described in DeVries (1982).

In comparing calibrated the LN and SA models, for the type of forecasting problem posed, for the same weightings of the ℓ_2 residual error, the SA model achieves the minimum ℓ_2 error and hence the minimum modeling error variance; thus, the SA model is the best estimator by this measure. When runoff data are not available, then the comparison is between an uncalibrated LN model and a SA model based on regionalized trends (for SA model $\langle \mathbb{P} \rangle$ parameter sets); and as of this paper's writing, there is no proof that either model is the best estimator.

In the San Sevaine catchment HEC-1 example problem, an LN model Toeplitz matrix set was developed using various submodel parameters and algorithms. Other choices of submodel parameters and algorithms would result in a different set of LN model Toeplitz matrices. Thus, a large number of plausible LN model Toeplitz matrix sets is possible, each set being a possible candidate as a runoff hydrograph estimator. The problem is selecting the Toeplitz matrix set that is the best estimator according to the error measure selected. The above ℓ_2 minimization procedure provides the best estimator, and this best estimator provides the minimum variance in modeling error. Because the LN models do not achieve the minimum residual error, it is possible to have "model response surfaces" in which "parameter contours" can be described that result in identical total ℓ_2 residual error values, and hence exhibit "optimized" parameter sets that are not unique.

6. Conclusions

The UH rainfall–runoff method is mathematically formalized in the setting of multiple subareas linked together by a network of hydrologic routing links (such as used in the computer program HEC-1 and related computer programs), with UHs developed in each subarea, a distributed phi-index loss rate defined on a subarea basis, and a hydrologic routing method used for each link such as the Muskingum technique. Toeplitz matrices are developed for several of the algorithms available in computer program HEC-1 and related systems, including subarea runoff hydrographs using a subarea UH and a phi-index loss function; modified-Puls hydrograph routing with a linear storage versus discharge relation: the linear reservoir method; hydrologic routing using Muskingum, Convex, convolution, or pure translation methods; combining watercourse runoff hydrographs at confluence points; linking subareas; mixing routing methods; adding hydrographs, and inclusion of rainfall variations across a catchment. These Toeplitz matrices mathematically describe the various hydrologic model processes and provide a firm mathematical formalization for rainfall–runoff computer models such as HEC-1. The Toeplitz matrices are easy to manipulate algebraically, and reduce the effort needed to investigate various tasks concerning the performance and accuracy of rainfall–runoff models such as the computer program HEC-1.

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