Boundary Element Technology XIII

C.S. Chen, C.A. Brebbia and D.W. Pepper
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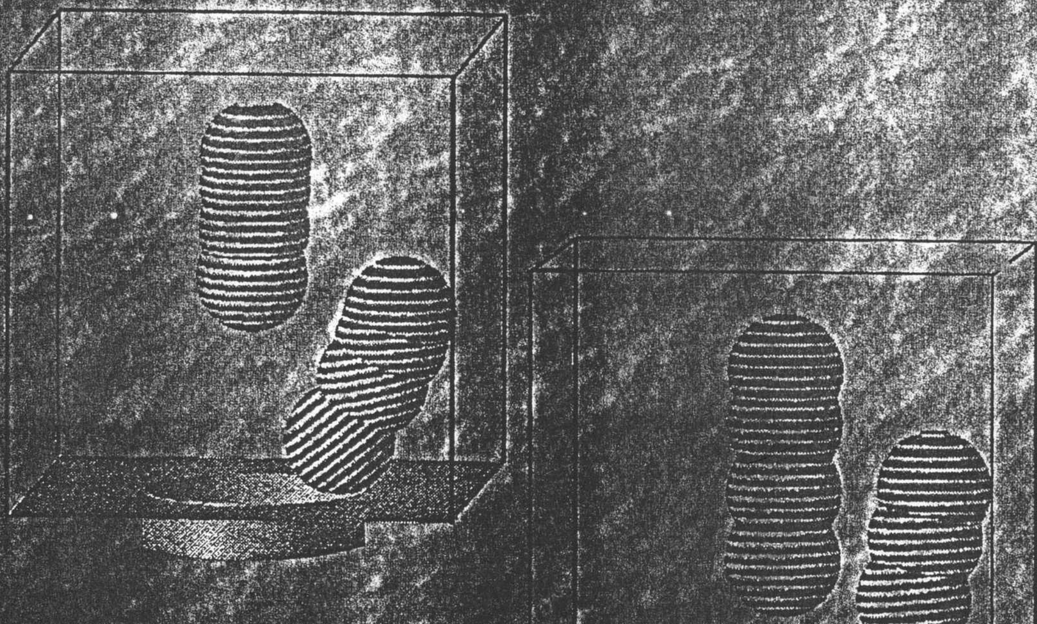

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Analysis of three-dimensional potential problems using the complex variable boundary element method

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Abstract

The Complex Variable Boundary Element Method, or CVBEM, is a two-dimensional (2D) potential problem numerical approximation technique. In this paper, the application of the CVBEM towards improving the numerical accuracy of three-dimensional (3D) numerical methods, in solving 3D potential problems, is introduced. The provided theoretical approach can be directly extended to other 2D numerical techniques.

I. Mathematical formulation

Let Ω be a three-dimensional (3D) domain, with boundary Γ, with coordinates given for an arbitrary point by (τ_1, τ_2, τ_3).

Consider the 3D potential problem $\nabla^2\phi = f(\tau_1, \tau_2, \tau_3)$ on $\Omega \cup \Gamma$ with boundary conditions of the Dirichlet type (BCs) defined on Γ by the function $\phi_b(\tau_1, \tau_2, \tau_3)$. For a selected coordinate value $\tau_3 = \tau'_3$, the corresponding boundary is the boundary of a 2D "slice", $\Gamma(\tau'_3)$, denoted hereafter as Γ', and the corresponding domain is the interior of the "slice", $\Omega(\tau'_3)$, denoted hereafter as Ω', with boundary conditions on Γ' given by $\phi_b(\tau_1, \tau_2)$. Similarly, for $\tau_3 = \tau'_3$, $f(\tau_1, \tau_2, \tau'_3)$; $f(\tau_1, \tau_2)$. On the domain Ω', which is subset of Ω, we have can be chosen in an infinite number of directions. Our goal is to approximate $\frac{\partial^2\phi}{\partial\tau_3^2}$ as closely as possible by $\frac{\partial^2\phi^P}{\partial\tau_3^2}$.

III. Approximation of $\frac{\partial^2\phi}{\partial\tau_3^2}$ by $\frac{\partial^2\phi^P}{\partial\tau_3^2}$

A second order finite difference approximation of both terms in the equality setting $\frac{\partial^2\phi}{\partial\tau_3^2} = \frac{\partial^2\phi^P}{\partial\tau_3^2}$ is

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta s^2} = \frac{\phi^P_{i+1} - 2\phi^P_i + \phi^P_{i-1}}{\Delta s^2} \qquad (9)$$

where s is a tangential coordinate along the 3D boundary, Γ, and is chosen to include a nonzero τ_3 component (so as to be outside of the selected 2D slice), and i is an evaluation point on Γ, where n evaluation points are used for numerical integration purposes.

Then, at evaluation point i, we set $\phi_{i+1} - 2\phi_i + \phi_{i+1} = \eta_i = \phi^P_{i+1} - 2\phi^P_i + \phi^P_{i-1}$, or in terms of matrices and vectors, for n evaluation points on Γ to analyze $\frac{\partial^2\phi}{\partial\tau_3^2}$,

$$<\eta>_{nx1} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{nxn} \begin{Bmatrix} \phi^P_1 \\ \phi^P_2 \\ \bullet \\ \bullet \\ \bullet \\ \phi^P_n \end{Bmatrix}_{nx1} = [A]<\phi^P> \qquad (10)$$

where A is a square nxn matrix; and $<\phi^P>$ is a nx1 column vector.

But $\phi^P_i = \sum_{j=1}^{N} jg_j(p_i)$, such that the coordinate of evaluation point p_i is

$$(\tau_1, \tau_2, \tau_3)_i. \qquad (11)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial\tau_1^2} + \frac{\partial^2\phi}{\partial\tau_2^2} + \frac{\partial^2\phi}{\partial\tau_3^2} \quad (1)$$

which on $\Omega'\cup\Gamma'$ reduces to,

$$\frac{\partial^2\phi}{\partial\tau_1^2} + \frac{\partial^2\phi}{\partial\tau_2^2} = f(\tau_1, \tau_2) - \frac{\partial^2\phi}{\partial\tau_3^2}, \text{ with BCs } \phi_b(\tau_1, \tau_2). \quad (2)$$

Let functions ϕ_f^P and ϕ^P satisfy

$$\frac{\partial^2\phi_f^P}{\partial\tau_1^2} + \frac{\partial^2\phi_f^P}{\partial\tau_2^2} = f(\tau_1, \tau_2) \quad (3)$$

and

$$\frac{\partial^2\phi^P}{\partial\tau_1^2} + \frac{\partial^2\phi^P}{\partial\tau_2^2} = -\frac{\partial^2\phi}{\partial\tau_3^2} \quad (4)$$

Then, on $\Omega'\cup\Gamma'$, we can solve the governing 3D partial differential equation (PDE) by first solving the 2D PDE,

$$\frac{\partial^2\hat{\phi}}{\partial\tau_1^2} + \frac{\partial^2\hat{\phi}}{\partial\tau_2^2} = 0, \text{ with BCs } \hat{\phi}_b = \phi_b - \phi_f^P - \phi^P \text{ on } \Gamma'. \quad (5)$$

Then, the 3D approximation in $\Omega'\cup\Gamma'$, is $\hat{\Phi} = \hat{\phi} + \phi_f^P + \phi^P$. (6)

II. Numerical modeling approach

This paper's approach to solving (2) is to use a 3D function ϕ^P of the form

$$\phi^P = \sum_{j=1}^{N3} c_j g_j \quad (7)$$

where the g_j are mutually independent functions such that the 3D Laplacian $\nabla^2 g_j = 0$. The notation, N3, is the number of 3D nodes that enclose $\Omega\cup\Gamma$. We select c_j so that ε_1 is minimized in the usual residual error sense, where

$$\varepsilon_1 = \| \phi_{ss}^P - \phi_{ss} \|_{\Gamma'} \quad (8)$$

where ϕ_{ss} is the tangential second partial derivative of ϕ along Ω. Note that s

But from (11), the column vector of values ϕ_i^P is given by,

$$<\phi^P>_{n\times 1} = \begin{bmatrix} g_1(p_1) & g_2(p_1) & \bullet\bullet\bullet & g_N(p_1) \\ g_1(p_2) & g_2(p_2) & \bullet\bullet\bullet & g_N(p_2) \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ g_1(p_n) & g_2(p_n) & \bullet\bullet\bullet & g_N(p_n) \end{bmatrix}_{n\times N} \begin{bmatrix} c_1 \\ c_2 \\ \bullet \\ \bullet \\ \bullet \\ c_N \end{bmatrix}_{N\times 1} \tag{12}$$

$$= [g(p)] <c>$$

where $n \geq N$.

Then, combining (11) and (12),

$$<\eta>_{n\times 1} = [A]_{n\times n} [g(p)]_{n\times N} <c>_{N\times 1} \tag{13}$$

where matrices [A] and [g(p)] are completely determined by the above definitions, and <c> is to be determined by a least-squares type solution to minimize the usual residual error norm. Note that $<\eta> = [A]<\phi>$ where $<\phi> = (\phi_1, \phi_2, \bullet\bullet\bullet, \phi_n)$. Then, equating matrix relationships,

$$[A]<\phi> = [A]\,[g(p)]<c> \tag{14}$$

or, for [A] nonsingular,

$$<\phi> = [g(p)]\,<c> \tag{15}$$

which is the typical direct formulation for developing a 3D approximator. Thus, the standard 3D numerical technique also is a model for estimating $\frac{f^2\phi}{f\tau_3^2}$.

Thus, a typical 3D solution of the LaPlace equation can be directly used as ϕ^P.

IV. The complex variable boundary element method (CVBEM)

The CVBEM will be used to develop the 2D approximation, on $\Omega'\cup\Gamma'$ according to (5). Background on the CVBEM is provided in numerous texts and papers (see references).

V. The 3D model particular solution, ϕ^P

Given ϕ^P, and ϕ_f^P, we now develop new BCs along Γ' to be used in solving for the 2D approximator $\hat{\phi}$ by the CVBEM. Note that these new BC values, on Γ', is the discrepancy between $(\phi^P + \phi_f^P)$ and ϕ along Γ'.

VI. Vector space considerations

The ϕ^P approximator utilizes 3D basis functions of the form $g_j(\tau_1, \tau_2, \tau_3)$. The CVBEM utilizes 2D basis functions of the form $h_j(\tau_1, \tau_2)$. The linear operator is $L = \nabla^2$. Then, for any j, and independently,

$$Lg_j = 0\ ;\ Lh_j = 0. (16)$$

Recall, $L\phi_f^P = f$. Then the basis developed in this numerical approach is $B = \{g_j; h_j; \phi_f^P\}$.

VII. Discussion

In general, accuracy in 3D problems is increased by adding 3D basis functions, h increases modeling complexity. In our new case, we get another chance towards reducing modeling error by, in effect, adding additional basis functions in a 2D subspace of the 3D domain. These additional 2D basis functions can only help in reducing the overall 3D modeling error, in a least squares sense, although such improvements are seen on a 2D "slice" by "slice" basis.

VII. Conclusions

The Complex Variable Boundary Element Method, or CVBEM, is a two-dimensional (2D) potential problem numerical approximation technique. In this paper, the application of the CVBEM towards improving the numerical accuracy of three-dimensional (3D) numerical methods, in solving 3D potential problems, is introduced. The theoretical approach provided can be directly extended to other 2D numerical techniques.

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