The Existence of Approximate Solutions to Mixed Boundary Value Problems

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Approximate solutions, similar to the type used in the Complex Variable Boundary Element Method, are shown to exist for two dimensional mixed boundary value potential problems on multiply connected domains. These approximate solutions can be used numerically to obtain least squares solutions or solutions which interpolate given boundary conditions. Areas of application include fluid flow around obstacles and heat flow in a domain with insulated boundary segments. Numer Methods Partial Differential Eq 15: 191–199, 1999

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I. INTRODUCTION

Most two dimensional steady-state potential engineering problems can be numerically solved by means of the Complex Variable Boundary Element Method (CVBEM) [1, 2]. While many of these problems are Dirichlet problems, where the potential is specified on the entire boundary of the domain, there is also an important class of mixed boundary value problems. Two typical examples of mixed problems are: steady state temperatures in a domain, where the temperature is specified on part of the boundary and the remainder of the boundary is insulated; and fluid flow

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in a domain with obstacles, represented by holes in the domain, where the velocity potential is specified on part of the boundary and on the remainder of the boundary the presence of walls and obstacles is indicated by having zero fluid flow through this part of the boundary.

The CVBEM methods use analytic functions of the form

$$h(z) = a_0 + a_0' z + \sum_{k=1}^{m} a_k (z - \beta_k) \log(z - \beta_k), \tag{1}$$

with nodes β_k sited on the boundary of the domain, together with various ways of selecting the coefficients to approximate the harmonic function, which is the exact solution of the given problem by means of the real part of h. A central theoretical issue is to establish that the solutions of these boundary value problems can indeed be approximated by the functions given in (1). This was done in [3, 4] for Dirichlet problems on a simply connected domain. The purpose of this article is to establish similar approximation results for mixed boundary value problems on multiply connected domains. In mixed boundary value problems, it is necessary to approximate the gradient of the potential, as well as the potential. Since the derivative of the function in (1) is not bounded at the points β_k, a modification is necessary. Let

$$f_\beta(z) = (z - \beta) \log(z - \beta), \tag{2}$$

$$F_\beta(z) = \frac{(z-\beta)^2}{2} \log(z - \beta) - \frac{(z-\beta)^2}{4}, \tag{3}$$

and note that the derivative of F_β is f_β. It will turn out that the appropriate function to use in approximating the potential on a simply connected domain will be

$$H(z) = a_0 + a_0' z + a_0'' z^2 + \sum_{k=1}^{m} a_k F_{\beta_k}(z). \tag{4}$$

II. SIMPLY CONNECTED DOMAINS

Let Ω be a simply connected domain in the complex plane with a piecewise continuously differentiable boundary Γ, which is a simple closed curve of finite length, parameterized by

$$\gamma\colon [0, 1] \to \Gamma. \tag{5}$$

It is assumed that the map γ is continuous on [0, 1], one-to-one on [0, 1) with $\gamma(0) = \gamma(1)$, is continuously differentiable, with nonzero derivative, except at a finite number of parameter points $c_1, \ldots, c_r$ corresponding to corners that are not cusps; so that the right- and left-hand limits of the derivative exist at each corner, are not zero, and satisfy the condition that for each j, $\gamma'(c_j+) + \gamma'(c_j-)$ is not zero so that c_j is not a cusp.

In order to correctly define the functions (2) and (3), for each β_k on Γ, we need to specify a continuous non-self-intersecting path P_{β_k}, joining β_k to infinity, which lies in the complement of $\Omega \cup \Gamma$. Then $P_{\beta_k} - \beta_k$ can be used as a branch cut to define a branch of the logarithm, $\log_{\beta_k}(z - \beta_k)$, which is analytic for z not on the branch cut $P_{\beta_k} - \beta$. To make this dependence on the branch cuts clear, (2) and (3) will be written as

$$f_\beta(z) = (z - \beta) \log_\beta(z - \beta), \tag{6}$$

$$F_\beta(z) = \frac{(z-\beta)^2}{2} \log_\beta(z - \beta) - \frac{(z-\beta)^2}{4}. \tag{7}$$

These branch cuts and related matters are discussed in [5].

Theorem 1. *Let ϕ be a function harmonic in Ω with a gradient $\nabla\phi$, which is continuous on $\Omega \cup \Gamma$. For any positive ϵ there is a function H, of the form given in (4), harmonic in Ω and continuous on $\Omega \cup \Gamma$, with*

$$|Re(H)(z) - \phi(z)| < \epsilon, \tag{8}$$

for z in $\Omega \cup \Gamma$. Further, the conjugate of the derivative of H is close to the gradient of ϕ :

$$|\overline{H'(z)} - \nabla\phi(z)| < \epsilon, \tag{9}$$

for z in $\Omega \cup \Gamma$.

Proof. Since the Cauchy–Riemann equations are satisfied, the conjugate of $\nabla\phi$ is a function $f(z)$ analytic in Ω and continuous on $\Omega \cup \Gamma$, so Mergelyan's Theorem [6, p. 270, 7, p. 271] asserts the existence of a polynomial $P(z)$ with

$$|P(z) - f(z)| < \epsilon \tag{10}$$

for z in $\Omega \cup \Gamma$. By Theorem 1 of [4], there is a CVBEM function $h(z)$ as in (1), with

$$|h(z) - P(z)| < \epsilon \tag{11}$$

also holding on $\Omega \cup \Gamma$. Thus,

$$|h(z) - f(z)| < 2\epsilon \tag{12}$$

for z belonging to $\Omega \cup \Gamma$.

Chose a fixed point z_o in Ω and define F on Ω by

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta, \tag{13}$$

the integral being independent of the path in Ω joining z_0 to z, because Ω is simply connected. Since $f(z)$ is continuous on $\Omega \cup \Gamma$, $|f(z)|$ is bounded by M there. If z_1 and z_2 in Ω are joined by a curve lying in Ω, then

$$|F(z_2) - F(z_1)| \leq Marc(z_1, z_2), \tag{14}$$

$arc(z_1, z_2)$ denoting the arc length of the curve. Since Γ has a piece-wise smooth parameterization γ, given a point w_0 on Γ, there is a neighborhood U of w_0 and a constant M_0 so that, given two points z_1 and z_2 in $U \cap \Omega$, a curve joining them and lying in $U \cap \Omega$ can be chosen with $arc(z_1, z_2) \leq M_0|z_1 - z_2|$. Hence, $F(z)$ is locally uniformly continuous and so extends continuously to $\Omega \cup \Gamma$. Therefore, $\phi(z)$, which is equal to $ReF(z)$ to within a constant, also extends continuously to $\Omega \cup \Gamma$.

There is a bound B such that any two points in Ω can be joined by a curve of length less than or equal to B. To see this, let z_1 and z_2 in Ω be given, and let w_1 and w_2 be the closest points on Γ to these two points. The curve C consisting of the straight line from z_1 to w_1, then along the shorter of the two arcs from w_1 to w_2, and then along a straight line from w_2 to z_2 has length less than the diameter of $\Omega \cup \Gamma$ plus one-half the arc length of Γ. The curve C, containing an arc of Γ, does not lie entirely in Ω, but can be moved slightly inside Ω, as in the proof of Theorem 1 of [4], thereby obtaining a curve in Ω joining z_1 to z_2, with length as close as desired to the length of the curve C.

Integrating Eq. (12),

$$|H(z) - F(z)| \leq 2B\epsilon \quad (15)$$

holds for z in Ω, and, therefore, for z in $\Omega \cup \Gamma$ since both $H(z)$ and $F(z)$ are continuous there, and it follows that, modifying H to include the constant of integration,

$$|ReH(z) - \phi(z)| \leq 2B\epsilon, \quad (16)$$

for z in $\Omega \cup \Gamma$. Equations (8) and (9) follow from (12) and (16). Q.E.D.

Corollary 1. *The nodes $\{\beta_k\}$ in the function $H(z)$ of Theorem 1 can be chosen to lie outside of $\Omega \cup \Gamma$.*

Proof. Immediately after Eq. (10) in the proof of Theorem 1, chose a piecewise smooth curve Γ' containing $\Omega \cup \Gamma$, and then proceed as in the proof to obtain a CVBEM function h, with nodes on Γ', which approximates the polynomial P to within ϵ on and inside Γ'. The function h then approximates P to within ϵ on $\Omega \cup \Gamma$ and has nodes lying on the arbitrary piecewise smooth curve Γ' containing the closure of the given domain. Now continue as in the proof of Theorem 1. Q.E.D.

The result of Corollary 1 is surprising from the point of view of [3], where the proof given of the existence of a CVBEM approximate solution to any Dirichlet problem relies on the properties of singular integrals with nodes on Γ.

If the curve Γ' arising in the proof of Corollary 1 is taken to be a large circle, the branch cuts for the logarithms in (6) and (7) can simply be taken to be the exterior perpendiculars to the circle at the nodes. Computational experiments need to be done to determine the advantages and disadvantages of various placements of nodes outside $\Omega \cup \Gamma$.

The appearance of the *conjugate* of the derivative of H in (9) is a commonplace in fluid flow, where $H(z) = \phi(z) + i\psi(z)$ is the complex potential for the flow, and $\phi(z)$ is the velocity potential. A simple consequence of the Cauchy– Riemann equations is

$$H'(z) = \phi_x(z) + i\psi_x(z) = \phi_x(z) - i\phi_y(z), \quad (17)$$

and the right-hand side of (17) is the conjugate of the velocity field $\nabla\phi$.

The usual form of an applied mixed boundary value problem is that the boundary Γ is divided into two disjoint pieces, C_1 and C_2, and it is desired to have the potential $\phi(z)$ equal to a given real valued function $g_1(z)$ on C_1 and the normal derivative equal to a given real valued function $g_2(z)$ on C_2:

$$\phi(z) = g_1(z), \text{ for } z \text{ on } C_1, \quad (18)$$

$$\frac{\partial\phi(z)}{\partial n} = g_2(z) \text{ for } z \text{ on } C_2. \quad (19)$$

There are two related but distinct ways of using Theorem 1 to approximate the solution of (18) and (19). The first method considers

$$\int_{C_1} (ReH(z) - g_1(z))^2|dz| + \int_{C_2} (\overline{H'(z)} \cdot n(z) - g_2(z))^2|dz|, \quad (20)$$

where, for $z = \gamma(t) = x(t) + iy(t)$ on C_2,

$$n(z) = \frac{(-y'(t), x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} \quad (21)$$

is the exterior normal to the curve at z. Write the coefficients of $H(z)$ given by (4)

$$a_0, a_0', a_0'', a_1, \ldots, a_m, \tag{22}$$

in terms of their real and imaginary parts

$$\alpha_0 + i\beta_0, \alpha_0' + i\beta_0', \alpha_0'' + i\beta_0'', \alpha_1 + i\beta_1, \ldots, \alpha_m + i\beta_m. \tag{23}$$

The customary least squares equations for minimizing (20) can be found by setting the partial derivatives of (20) with respect to the real and imaginary parts of the coefficients in (23) equal to zero, and solving the resulting set of linear equations for these real and imaginary coefficients, with coefficient matrix (nearly always nonzero in practice) having terms that are integrals of various products of real and imaginary parts of the functions $1, z, z^2, \{f_{\beta_k}(z)\}$, and $\{F_{\beta_k}(z)\}$ in the sums defining $H(z)$ and $\overline{H'(z)} \cdot n(z)$.

The other method of determining the coefficients of (23) is to chose

$$z_1, z_2, \ldots, z_{m_1} \text{ on } C_1, \tag{24}$$

$$z_1', z_2', \ldots, z_{m_2}' \text{ on } C_2, \tag{25}$$

and to consider the discrete versions of (18) and (19)

$$ReH(z_j) = g_1(z_j), \text{ for } z_j \text{ on } C_1 \tag{26}$$

$$\overline{H'(z_j')} \cdot n(z_j') = g_2(z_j') \text{ for } z_j' \text{ on } C_2. \tag{27}$$

If the number $m_1 + m_2$ of points on Γ is larger than the number of unknown real coefficients, the overdetermined Eqs. (26) and (27) can be solved in a least squares sense. The case in which the number of points equals the number of unknowns is particularly interesting, for then the solution interpolates to the correct values at the points (24) and (25). This is very useful when computing an approximate boundary, see [1, 2], which is an application of the idea of backward error analysis in which a domain close to Ω is found on which the computed solution is the exact solution.

Both these numerical methods can be applied to mixed boundary conditions that involve combinations of values of ϕ and its normal derivative on the boundary.

The hypotheses of Theorem 1 that $\phi(z)$ and $\nabla\phi(z)$ be continuous on $\Omega \cup \Gamma$ is physically clear for almost all applications. However, oversimplifications in modeling can create discontinuities.

A simple case of this is a domain with an insulated corner, for example the domain the unit square with the top and right-hand vertical side insulated; then for $\frac{\partial\phi(z)}{\partial n} = 0$ to hold around the upper right-hand corner, with of course $n(z)$ undefined at the corner, generally forces $\nabla\phi(z)$ to have a discontinuity at that corner. To avoid this, the corner can be rounded, or else the approximate boundary technique can be applied and the approximate domain on which the approximate CVBEM solution is exact will be found to have a rounded corner.

Another example is given by considering $F(z) = \sqrt{z}\log(z)$, with the branch cut for the square root and the logarithm being the negative x-axis, $\Omega = \{z\colon |z-1| < 1\}$, and $\phi(z) = ReF(z)$. Then $\phi(z)$ is continuous on $\Omega \cup \Gamma$, but $\frac{\partial\phi(z)}{\partial n}$ is unbounded in any neighborhood of 0. In this case, the boundary condition $\phi(z) = ReF(z)$ on Γ is not smooth enough at zero to have $\nabla\phi(z)$ extend continuously to the closure of Ω.

As a final example, consider $F(z) = -z\log(z)$ on a quarter circle $\Omega = \{z = re^{i\theta}\colon 0 < r < 1, 0 < \theta < \frac{\pi}{2}\}$, with $C_1 = \{(0, y)\colon 0 \le y \le 1\} \cup \{z = e^{i\theta}\colon 0 \le \theta < \frac{\pi}{2}\}$, and $C_2 = \{(x, 0)\colon$

$0 < x < 1\}$. Then $\phi(z) = ReF(z)$ is a solution to the following mixed boundary value problem. On the y-axis arc of $C_1, \phi(y) = \frac{\pi}{2}y$, while on the semicircular arc of $C_1, \phi(e^{i\theta}) = \theta \sin\theta$. On $C_2, \frac{\partial\phi(z)}{\partial n} = 0$. The corresponding boundary functions $g_1(z)$ and $g_2(z)$ of (18) and (19) are quite smooth. The fact that $\nabla\phi(z)$ is unbounded in any neighborhood of 0 is hidden by the discontinuous transition there from a boundary condition on $\phi(z)$ to a boundary condition on the normal derivative.

It is occasionally desirable to have a model in which there is a boundary discontinuity or singularity in the potential ϕ or in $\nabla\phi$. In this case, it is best to attempt to model this behavior from physical principles. Otherwise, a numerical solution using the CVBEM, or other techniques, perhaps on a domain of a slightly different shape as computed from approximate boundary methods or with smoother boundary conditions, will describe the physical situation better than a rigid adherence to a misleading simplified model.

III. MULTIPLY CONNECTED DOMAINS

Let Ω be a multiply connected domain in the complex plane with m holes; Γ_0 a simple closed curve of boundary points of Ω that includes Ω and all the holes, and Γ_j the simple closed curve consisting of boundary points of Ω that encloses the j-th hole, $j = 1, \ldots, m$. The curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ are assumed to each satisfy the smoothness conditions on the boundary of the domain in section I.

A specific example of a physical problem that uses such a domain [2, p. 110] is the problem of calculating the fluid flow through a section of a river bed around piers that support a bridge, the area under water being the domain Ω and the piers corresponding to holes in Ω.

Theorem 2. *Let Ω be a multiply connected domain, as described above, and let a_j be a point inside the hole surrounded by the curve $\Gamma_j, j = 1, \ldots, m$. Let ϕ be a function harmonic in Ω with a gradient $\nabla\phi$ that is continuous on $\Omega \cup \Gamma$, where $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \Gamma_m$.*

For any positive ϵ there are CVBEM functions $H_j(z), j = 0, 1, \ldots, m$, as given in (4) *with the nodes $\{\beta_k^j\}$ and branch cuts for the logarithms appearing in $H_j(z)$ chosen as follows: The nodes $\{\beta_k^0\}$ can be chosen on Γ_0 with branch cuts joining each node to infinity exactly as in the case where Ω is simply connected. For $j = 1, \ldots, m$, consider the function*

$$g_j(z) = \frac{1}{z - a_j}. \tag{28}$$

Chose nodes $\{\beta_k^j\}$ on Γ_j, and for each node let $\hat{L}_k^j$ be a simple curve joining $\hat{\beta}_k^j$ to a_j lying entirely in the hole except for the end point node. Set

$$\beta_k^j = g_j(\hat{\beta}_k^j) \tag{29}$$

and

$$L_k^j = g_j(\hat{L}_k^j). \tag{30}$$

The points (29) *are the nodes to be used for the term $F_{\beta_k^j}(z)$ in the sum* (4) *defining $H_j(z)$ and the curves*

$$L_k^j - \beta_k^j \tag{31}$$

the branch cuts for the logarithm appearing in that function. Setting

$$H(z) = H_0(z) + H_1\left(\frac{1}{z - a_1}\right) + H_2\left(\frac{1}{z - a_2}\right) + \cdots + H_m\left(\frac{1}{z - a_m}\right), \tag{32}$$

$$|ReH(z) - \phi(z)| < \epsilon \textit{ for } z \textit{ in } \Omega \cup \Gamma. \tag{33}$$

Further, the conjugate of the derivative of H is close to the gradient of ϕ on the domain

$$|\overline{H'(z)} - \nabla\phi(z)| < \epsilon \textit{ for } z \textit{ in } \Omega \cup \Gamma. \tag{34}$$

Proof. Since the conjugate f of $\nabla\phi(z)$ is a function analytic in Ω and continuous on $\Omega \cup \Gamma$, Mergelyan's theorem [7, pp. 390, 394; 6, p. 307] asserts the existence of a rational function $R(z)$, analytic on $\Omega \cup \Gamma$ with

$$|f(z) - R(z)| < \epsilon \text{ for } z \text{ in } \Omega \cup \Gamma. \tag{35}$$

Since $R(z)$ is analytic on a neighborhood of $\Omega \cup \Gamma$, Runge's theorem [7, p. 270; 6, p. 271] supplies a rational function $Q(z)$, with poles only in the set $\{\infty, a_1, a_2, \ldots, a_m\}$ and

$$|R(z) - Q(z)| < \epsilon \text{ for } z \text{ in } \Omega \cup \Gamma. \tag{36}$$

Writing out $Q(z)$ in terms of its principal parts associated with the poles $a_1, \ldots, a_m$ gives polynomials $P_0(z), P_1(z), \ldots, P_m(z)$ with

$$Q(z) = P_0(z) + P_1\left(\frac{1}{z-a_1}\right) + \cdots + P_m\left(\frac{1}{z-a_m}\right). \tag{37}$$

Hence,

$$\left|f(z) - P_0(z) - P_1\left(\frac{1}{z-a_1}\right) - \cdots - P_m\left(\frac{1}{z-a_m}\right)\right| < 2\epsilon \text{ for } z \text{ in } \Omega \cup \Gamma. \tag{38}$$

By Theorem 1 of [4], there is a CVBEM function $h_0(z)$ of the form (1) with nodes $\{\beta_k^0\}$ on Γ_0 with

$$|h_0(z) - P_0(z)| < \epsilon. \tag{39}$$

Equation (39) holds on Γ_0 and in the interior of Γ_0, which consists of Ω as well as all the holes in Ω.

Consider the j-th hole, bounded by Γ_j and containing a_j. The function g_j of (28) is analytic on $\Omega \cup \Gamma$, and maps the boundary of the j-th hole to another simple closed curve

$$g_j(\Gamma_j) = \Gamma'_j. \tag{40}$$

The domain Ω, as well as all the holes other than the j-th hole, are mapped by $g_j(z)$ into the interior Ω'_j of Γ'_j. Let

$$w = \frac{1}{z - a_j}. \tag{41}$$

Since 0 does not belong to $\Omega'_j \cup \Gamma'_j$, $w^{-2}P_j(w)$ is analytic on a neighborhood of $\Omega'_j \cup \Gamma'_j$. By Theorem 1 of [4], there is a CVBEM function $h_j(w)$ of the form (1) with nodes $\{\beta_k^j\}$ on Γ'_j and corresponding branch cuts and

$$\left|h_j(w) + \frac{1}{w^2}P_j(w)\right| < \epsilon \text{ for } z \text{ inside and on } \Gamma'_j. \tag{42}$$

Set

$$d = \min\{dist(a_j, \Gamma_j) \colon j = 1, \ldots, m\}. \tag{43}$$

For z belonging to $\Omega \cup \Gamma$ and not in the j-th hole,

$$\left| -\frac{1}{(z-a_j)^2} h_j\left(\frac{1}{z-a_j}\right) - P_j\left(\frac{1}{z-a_j}\right)\right| < \frac{\epsilon}{d}. \tag{44}$$

Note that under the mapping g_j, nodes on Γ_j correspond to nodes on Γ_j' and branch cuts in the w-plane correspond to simple curves in the z-plane, lying in the j-hole and joining the nodes to a_j. Let

$$h(z) = h_0(z) - \sum_1^m \frac{1}{(z-a_j)^2} h_j\left(\frac{1}{z-a_j}\right), \tag{45}$$

and note

$$|h(z) - f(z)| < 2\epsilon + \frac{m\epsilon}{d}, \tag{46}$$

on $\Omega \cup \Gamma$. Then, letting H_j denote the function of the form given in (4) with derivative h_j,

$$H(z) = H_0(z) + \sum_1^m H_j\left(\frac{1}{z-a_j}\right) \tag{47}$$

is analytic in Ω, continuous on $\Omega \cup \Gamma$, and has

$$H'(z) = h(z). \tag{48}$$

Equation (34) follows from (38), (44), (46), and (48).

For any point of Γ, there is a sufficiently small neighborhood U, which intersects Ω in a simply connected set. Since $f(z)$ is analytic on this simply connected set, there is a function $\hat{F}(z)$, analytic on U, with $\hat{F}'(z) = f(z)$ holding in U. (Note that this is indeed a local phenomenon; the function $\frac{1}{z-a_j}$ is analytic in Ω and continuous on $\Omega \cup \Gamma$, but there is no analytic function defined on all of Ω with derivative $\frac{1}{z-a_j}$.) As in the proof of Theorem 1, $\hat{F}(z)$ is seen to be uniformly continuous on U, ϕ is seen to extend continuously to the part of the boundary of Ω contained in U, and thus extend continuously to all of Γ.

To circumvent the problems caused by the local integrability of $f(z)$, the domain will be temporarily be altered. A cross-cut is a non-self-intersecting curve joining two boundary points of Ω and lying entirely in Ω otherwise. Chose a point on the boundary Γ_1 of the first hole and join it, via a cross-cut, to a point on the boundary Γ_0. (It is simplest to let Γ_1 be the hole boundary that is closest to Γ_0 whereby this cross-cut can be taken to be a straight line.) Removing this cross-cut from Ω gives a domain with $m-1$ holes. Proceeding inductively, after removing $m-1$ cross-cuts from the domain, the remaining domain, Ω_c, is a simply connected subset of Ω [8, p. 3]. The function $f(z)$ is analytic on Ω_c and so can be integrated on this simply connected domain to give a function $F(z)$, also analytic on Ω_c, with

$$F'(z) = f(z) = \overline{\nabla\phi(z)} \text{ for } z \text{ in } \Omega_c. \tag{49}$$

As in the proof of Theorem 1,

$$|H(z) + F(z_0) - H(z_0) - F(z)| \le \epsilon B \text{ on } \Omega_c. \tag{50}$$

Incorporating the constants in (50) into $H(z)$,

$$|Re H(z) - \phi(z)| \le \epsilon B \text{ on } \Omega_c. \tag{51}$$

Since both $ReH(z)$ and $\phi(z)$ are continuous on $\Omega \cup \Gamma$, Eq. (51) holds on the closure of Ω_c, which is $\Omega \cup \Gamma$, and (33) follows. Q.E.D.

The point of the choice of nodes and branch cuts in (29) and (30) is simply that, for z in Ω, the values $\frac{1}{z-a_j}$ need to belong to a domain in which H_j is analytic so that $H(z)$ in (47) is analytic.

As in the first section, the nodes $\{\beta_k^0\}$ can be chosen to lie in the exterior of Γ_0, and the nodes $\{\hat{\beta}_k^j\}$ can be chosen to lie inside the hole bounded by Γ_j. If, for example, the nodes $\{\hat{\beta}_k^j\}$, for $j \neq 0$, are chosen on a circle of small radius r, centered at a_j, and the curves $\hat{L}_k^j$ to be radii joining a_j to these nodes, then $\{\beta_k^j\}$ lie on a circle with large radius $\frac{1}{r}$, centered at 0, and the curves L_k^j are the external normals to this circle at those points.

If the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ of Ω is divided into two disjoint subsets C_1 and C_2, then the mixed boundary value problem of (18) and (19) can be approached numerically exactly as in the first section, by minimizing Eq. (20) or finding the least-squares solution of Eqs. (26)–(27).

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