UNIT HYDROGRAPH UNCERTAINTY ANALYSIS

T.V. Hromadka II
Professor of Mathematics and Environmental Studies, Department of Mathematics, California State University, Fullerton, California 92634

Abstract

The Stochastic Integral Equation Method (or SIEM) has been shown to be a useful tool in analyzing modeling error associated to rainfall-runoff hydrograph models. Not only does the SIEM approximately develop a distribution of modeling error realizations to be added to a modeling result as applied to a runoff forecast, but the SIEM also enables rainfall-runoff models to be compared on equal flooring as to which model is the "best estimator" of a selected criterion variable (e.g., peak flow rate, 1-hour volume, etc.).

In this paper, the SIEM is analyzed with respect to being applied as stochastic Toeplitz matrices. This application not only simplifies the SIEM approach, but also illuminates the theoretical underpinnings of the SIEM. A significant advantage obtained by use of the Toeplitz matrix approach over simply discretizing the SIEM equations, is that the SIEM can now be directly coupled to rainfall-runoff computer models by operating on the computer model's output of storm runoff estimates.

Introduction

In Hromadka (1997), the Stochastic Integral Equation Method (SIEM) is used to represent an arbitrary rainfall-runoff modeling structure (of a free-draining catchment such as described in detail in Hromadka and Whitley (1989) and Abbott (1978)) and its corresponding modeling error. Because the entire runoff hydrograph is mathematically analyzed, statistical analysis of the entire modeling error realization is possible, and a spectrum of different modeling structure types (e.g., HEC-1 Unit Hydrograph, MITCAT, etc.; see Abbott (1978)) can be compared as to bias and variance; that is, the modeling structures can be ranked with respect to being the "best estimator" (i.e., with respect to variance in modeling estimates of a chosen criterion variable such as peak flow rate, peak 1-hour runoff volume, etc.).

In this paper, the SIEM formulation is redeveloped by introducing stochastic Toeplitz matrix representations as a substitute for the stochastic integral equations of Hromadka and Whitley (1989). This work serves the role of establishing a convenient procedure for coupling an uncertainty estimation, based on the SIEM, to commonly used rainfall-runoff computer models. Because the procedure produces a statistical distribution of criterion variable values (e.g., peak flow rate, total volume, etc.), the computed results can be analyzed with respect to confidence intervals or other statistical measures.

Mathematical Development

Let $Q^i(t)$ be the runoff hydrograph, for storm event i, as a function of time, t. For a selected model structure type, let $M^i(t)$ be the model produced runoff hydrograph for storm event, i. In our analysis, we assume that $M^i(t)$ is of an event that is not an element of calibration data used to calibrate the model. Model error, for storm event i, is

$$E^i(t) = Q^i(t) - M^i(t) \tag{1}$$

Each of the components in Eq. (1) can be treated as a realization of a respective stochastic process. To proceed with our development, we discretize each realization with respect to a constant and uniformly spaced unit period of time, obtaining a histogram representation for each realization; the unit period of time is chosen to be "sufficiently small" such as to represent the characteristics of each realization, and to begin with a unit period at time t=0. The unit period values, for each realization, can be assembled into respective nx1 column vectors such that Eq. (1) is rewritten as

$$(\underset{\sim}{E^i}) = (\underset{\sim}{Q^i}) - (\underset{\sim}{M^i}) \tag{2}$$

where each component of Eq. (2) is a column vector corresponding to the respective component of Eq. (1). At this point, it is noted that Eq. (2) is fully defined, for a given modeling structure realization. In order to develop the SIEM for representing modeling error from an arbitrary runoff model structure we will first focus upon the well-known unit hydrograph technique in order to develop the mathematics, and then we will return to an arbitrary model structure of a free draining catchment.

Case Study: The Unit Hydrograph (UH) Method

Consider the unit hydrograph (UH) technique for estimating storm runoff. Given rainfall excess, for storm event i, noted as $e^i(t)$, and a corresponding transform realization for storm event i, $\psi^i(t)$, then runoff $Q^i(t)$ is given by (issues regarding existence and uniqueness of the realization $\psi^i(t)$, for storm event i, can be found in Hromadka and Whitley, (1989)),

$$Q^i(t) = \int_{s=0}^{t} e^i(s)\, \psi^i(t - s)\, ds \qquad (3)$$

Equation (3) is a convolution process which can be rewritten as (for example, see Tsokos and Padgett, 1974),

$$Q^i = e^i \otimes \psi^i \qquad (4)$$

where in Eq. (4) it is understood the correspondence to the integral of Eq. (3). Using the same unit time period discretization used for Eq. (2), we can rewrite Eq. (4) as

$$\underset{\sim}{Q}^i = \underset{\sim}{e}^i \otimes \underset{\sim}{\psi}^i \qquad (5)$$

or, in matrix form,

$$(Q^i) = [\psi^i](\underset{\sim}{e}^i) \qquad (6)$$

where (Q^i) and $(\underset{\sim}{e}^i)$ are nx1 column vectors, and $[\psi^i]$ is an nxn square stochastic Toeplitz matrix,

$$[\psi^i] = \begin{vmatrix} \psi_1^i & 0 & 0 & \cdots & 0 \\ \psi_2^i & \psi_1^i & 0 & \cdots & 0 \\ \psi_3^i & \psi_2^i & \psi_1^i & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \psi_n^i & \psi_{n-1}^i & \psi_{n-2}^i & \cdots & \psi_1^i \end{vmatrix} \qquad (7)$$

Note that $[\psi^i]$ is a square matrix in lower triangular form, with all matrix elements defined by the unit period values of the realization vector, ψ^i. Also note that the various vectors have the same dimension, n (which may be achieved by extending the vector dimension using zero entries). The matrix is Toeplitz because of its circulant structure, where diagonal and off-diagonal components are identical, respectively. A "stochastic" Toeplitz matrix includes the property that the matrix components may change for each particular event. The Toeplitz matrix structure occurs due to the convolution process of (5). Special properties of Toeplitz matrices included commutativity in multiplication, and the existence of inverses.

In forecast mode, the transform of Eq. (3) obviously cannot be known beforehand and so an estimate of the realization, $\psi^i(t)$, is used, namely the catchment unit hydrograph, u(t). Thus the UH model estimate of the runoff hydrograph for storm i is, in a matrix form consistent with the above,

$$(M^i_{UH}) = [u](\underset{\sim}{e^i}) \tag{8}$$

Thus, the UH modeling error for storm event i is, in matrix form, given by the vector, $\underset{\sim}{E^i}$, where with respect to Eqs. (2), (6), and (8),

$$\underset{\sim}{E^i} = Q^i - \underset{\sim}{M^i_{UH}} \tag{9}$$

Let $[\beta^i]$ be another nxn stochastic Toeplitz matrix (see the form of the matrix in Eq. (7)) defined by

$$[\psi^i] = [\beta^i]\,[u] \tag{10}$$

where

$$
[\beta^i] = \begin{vmatrix} \beta_1^i & 0 & \cdots & 0 \\ \beta_2^i & \beta_1^i & \cdots & 0 \\ \beta_3^i & \beta_2^i & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \beta_n^i & \beta_{n-1}^i & \cdots & \beta_1^i \end{vmatrix} \qquad (11)
$$

Then the matrix product of Eq. (10) is given by

$$
[\beta^i]\,[u] = \begin{vmatrix} \beta_1^i & 0 & 0 & \cdots & 0 \\ \beta_2^i & \beta_1^i & 0 & \cdots & 0 \\ \beta_3^i & \beta_2^i & \beta_1^i & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \beta_n^i & \beta_{n-1}^i & \beta_{n-2}^i & \cdots & \beta_1^i \end{vmatrix} \begin{vmatrix} u_1 & 0 & 0 & \cdots & 0 \\ u_2 & u_1 & 0 & \cdots & 0 \\ u_3 & u_2 & u_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ u_n & u_{n-1} & u_{n-2} & \cdots & u_1 \end{vmatrix}
$$

$$
= \begin{vmatrix} \beta_1^i u_1 & 0 & 0 & \cdots & 0 \\ (\beta_2^i u_1 + \beta_1^i u_2) & (\beta_1^i u_1) & 0 & \cdots & 0 \\ (\beta_3^i u_1 + \beta_2^i u_2 + \beta_1^i u_3) & (\beta_2^i u_1 + \beta_1^i u_2) & (\beta_1^i u_1) & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdots & \beta_1^i u_1 \end{vmatrix} \qquad (12)
$$

which is another nxn square matrix, in lower triangular form, with the same column vector relationship as the Toeplitz matrices of Eqs. (7) and (11).

Combining Eqs. (6) and (10),

$$(\underset{\sim}{Q}^{i}) = [\beta^{i}]\,[u]\,(\underset{\sim}{e}^{i}) \qquad (13)$$

Combining Eqs. (8) and 13),

$$(Q^{j}) = [\beta^{i}]\,(\underset{\sim}{M}^{i}_{UH}) \qquad (14)$$

We are interested in the modeling error from our selected UH model structure. For storm event i, Eqs. (9) and (14) are combined to give,

$$(\underset{\sim}{E}^{i}) = (\underset{\sim}{Q}^{i}) - (\underset{\sim}{M}^{i}_{UH}) = [\beta^{i}]\,(\underset{\sim}{M}^{i}_{UH}) - [I]\,(\underset{\sim}{M}^{i}_{UH}) \qquad (15)$$

where [I] is an nxn identity matrix; and modeling error for storm event i, $\underset{\sim}{E}^{i}$, depends on the particular rainfall-runoff computer model selected for use. Then another nxn Toeplitz matrix, $[\alpha^{i}]$; is defined by

$$(\underset{\sim}{E}^{i}) = [\alpha^{i}]\,(\underset{\sim}{M}^{i}_{UH}) \qquad (16)$$

where from Eqs. (14) to (16),

$$[\beta^{i}] = [I] + [\alpha^{i}] \qquad (17)$$

From Eqs. (14) and (16), the modeling error for storm event i is represented by the convolution matrix, $[\alpha^{i}]$, which is readily resolved by forward substitution applied to either Eqs. (14) or (16). It is noted from (15) that modeling error for storm event i, $\underset{\sim}{E}^{i}$, is given by the subtraction of the computed results from the runoff data as shown in the left side of (15). The matrix $[\alpha^{i}]$ is defined by (16) with the requirement that the computed runoff by (16) with the requirement that the computed runoff initiates with nonzero values at or prior (in model time) to the occurrence of nonzero values of measured runoff, $\underset{\sim}{Q}^i$.

In practice, given a statistically significant set of m stochastic Toeplitz matrices, $\{[\alpha^i];\ j=1,2,\cdots,m\}$, developed from m storm events that are not elements of the calibration events used to develop the UH model, an estimate of the modeling error stochastic distribution associated to our UH model, as applied to a forecast storm event (i.e., a future event), would be a set of m realizations, $\{(\underset{\sim}{E}^j);\ j=1,2,\cdots,m\}$ where

$$(\underset{\sim}{E}^j) = [\alpha^j]\,(\underset{\sim}{M}_{UH}^{P});\ j = 1,2,\cdots,m \qquad (18)$$

and $(\underset{\sim}{M}_{UH}^{P})$ is our predicted runoff, from our UH model, for forecast storm event, P.

Similarly, the stochastic distribution of runoff hydrographs (to be considered as an estimate of the stochastic distribution of realization outcomes from our UH model) is, in matrix form, approximately given by the set of m discrete outcomes, $\{(\underset{\sim}{Q}^j);\ j=1,2,\cdots,m\}$ where

$$(\underset{\sim}{Q}^j) = [\beta^j]\,(\underset{\sim}{M}_{UH}^{P});\ j = 1,2,\cdots,m \qquad (19)$$

where each $[\beta^j]$ is defined from Eqs. (11) through (14); and $(\underset{\sim}{M}_{UH}^{P})$ is, again, the predicted single outcome from the UH model for forecast storm event, P.

A stochastic distribution of convolution matrices $[\beta^j]$, and also $[\alpha^j]$, can be generated by noting Eq. (19) can be rewritten as

$$(Q^j) = [\beta^j] (\underset{\sim}{M}^{P}_{UH}) = [\beta^j] [u] (\underset{\sim}{e}^{P}); j = 1,2,\cdots, m \quad (20)$$

where $(\underset{\sim}{e}^{P})$ is the column vector representation of the forecast storm event, P, rainfall excess. But from Eq. (10), we can rewrite Eq. (20) as

$$(Q^j) = [\psi^j] (\underset{\sim}{e}^{P}); j = 1,2,\cdots, m \quad (21)$$

which is the stochastic matrix representation precisely equivalent to the SIEM formulation for the UH modeling approach (Hromadka and Whitley, 1989).

In summary, what is important in the above case study is that rather than dealing directly with the UH model ψ^j realizations, we can introduce another convolution to represent the modeling error trends. That is, this SIEM formulation deals with a particular model's modeling runoff hydrographs (see Eq. (20)) rather than dealing with the model's modeling input of rainfall excess (see Eq. (21)). Consequently, UH model computed results of runoff can be directly reformulated into a stochastic distribution of runoff hydrograph realizations (see (20)), rather than generating a stochastic distribution of runoff hydrograph realizations by formulating a stochastic distribution of UH realizations (see (21)). This useful result can be applied to other rainfall-runoff computer models such as demonstrated in Hromadka (1997), and also briefly described in the following section.

Arbitrary Model Structure

Similar to the previous case study of a UH model, we will now relate an arbitrary model's history of runoff hydrograph modeling errors to the modeled runoff hydrograph by a convolution, given

$$(E_M^j) = (Q^j) - (\underset{\sim}{M}^j); \quad j = 1,2,\cdots, m \qquad (22)$$

where all terms are as defined and conditioned previously; and M refers to a particular rainfall-runoff model structure (for a free-draining catchment per Hromadka and Whitley, 1989). Note that in Eq. (22), the term ($\underset{\sim}{E}_M^j$) is not the same as used in Eqs. (2) and (15), as they are model dependent.

Then, convolution matrices are defined by (see Eq. (16)), solving for $[\alpha_M^j]$ in the relationship

$$(E_M^j) = [\alpha_M^j]\,(\underset{\sim}{M}^j) \qquad (23)$$

where the subscript M indicates modeling structure dependency; and i is a given storm event not used in a calibration of the selected model. For m such storm events (typically, such storm events are selected to be "similar" or of the same "class" as the forecast storm event to be analyzed), and a set of stochastic Toeplitz matrices result, $\{[\alpha_M^j]; j=1,2,\cdots,m\}$.

In forecast mode, an approximate distribution of model structure M outcomes is given by the discrete set of runoff hydrographs,

$$\{(Q_M^j) = (\underset{\sim}{M}^P) + (E_M^j)\,;\; j = 1,2,\cdots, m\} \qquad (24)$$

where $\underset{\sim}{M}^P$ is the model structure M runoff hydrograph estimate (in vector form) for future storm event, P; and the other terms are defined previously.

Given this last result, a statistical analysis can be conducted on a particular criterion variable, such as peak flow rate, to determine the "best estimator".

Accompanying Eq. (24) are other topics for future research, such as filtering the stochastic estimates, or whether constraints on the Toeplitz matrix components improves forecasting results in estimating storm runoff.

Application of the Stochastic Toeplitz Matrix Formulation

In order to demonstrate the presented matrix formulation of the SIEM, three example problems are presented below.

Example 1.

Suppose that for historic storm event i, and arbitrary model structure M, the error vector (E^i_M) is a zero vector. Then $[\alpha^i_M]$ = the zero square matrix satisfies this condition. Thus, from Eq. (17), $[\beta^i_M] = [I]$ for this single event. Assuming further that the above zero error vector occurred k times in m storm events analyzed, then from Eq. (24), the distribution of discrete stochastic outcomes would demonstrate a k/m occurrence of the zero modeling error realization when the model M is applied in forecast mode.

Example 2.

Suppose that for storm event i and model structure M, the error vector (E^i_M) is a simple proportion of the model estimated runoff hydrograph vector, $(\underset{\sim}{M}^i)$; that is, $(E^i_M) = \lambda^i(\underset{\sim}{M}^i)$, where λ^i is a constant real number for historic storm event i. Then $[\alpha^i_M] = \lambda^i[I]$ satisfies this condition, and $[\beta^i_M] = (1+ \lambda^i)[I]$. That is, for storm event i, $(Q^i) = (1 + \lambda^i)[I](\underset{\sim}{M}^i)$. Assuming further that, for m storm events, the error vectors continue to be simple proportions (but random and mutually independent between events) of the model runoff estimates, for each event i, then the distribution of stochastic outcomes, associated to a forecast storm event P, is approximately given, from Eq. (24), by the set of discrete realizations $\{(Q_M^j) = (\underset{\sim}{M}^P) + (E_M^j) = (1+\lambda^j)(\underset{\sim}{M}^P);\ j=1,2,\cdots,m\}$. Note that in this example, the values λ^j; j=1,2,...,m are a simple random sample (of size m) of a random variable, λ, and hence standard statistical analysis techniques can be directly applied to the analysis of the distribution of modeling error realizations.

Example 3.

In this example, we will demonstrate how a UH model output runoff hydrograph can be convoluted with a $[\beta^i]$ Toeplitz matrix (see Eq. (14)) and the resulting runoff hydrograph is identical to the runoff hydrograph had a storm dependent unit hydrograph $[\psi^i]$ been used (see Eqs. (3) to (8)).

Consider the vectors $\underset{\sim}{u} = (1,2,0,0,0)$, $\underset{\sim}{\psi}^i = (2,3,0,0,0)$, $\underset{\sim}{e}^i = (1,1,2,1,0)$, $\underset{\sim}{Q}^i = (2,5,7,8,3)$, $M_{\sim UH}^{\ i} = (1,3,4,5,2)$, where all vectors are as defined in the previous text.

Here, the "correct UH", for storm i, is $\underset{\sim}{\psi}^i$, whereas our UH model is using $\underset{\sim}{u}$. Then the UH model error, for storm event i, is $\underset{\sim}{E}^i = \underset{\sim}{Q}^i - M_{\sim UH}^{\ i} = (1,2,3,3,1)$. From Eq. (16) we set

$\underset{\sim}{E}^i = [\alpha^i]\ M_{\sim UH}^{\ i}$, or

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 1 \end{pmatrix} = \begin{vmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 0 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{vmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix} \quad (25)$$

giving, by forward substitution, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, -1, 2, -4, 8)$. From Eq. (17), $[\beta^i] = [I] + [\alpha^i]$, and $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (2, -1, 2, -4, 8)$. From Eqs. (10) to (12), $[\beta^i]\,[u]$ is given by

$$[\beta^i]\,[u] = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 \\ -4 & 2 & -1 & 2 & 0 \\ 8 & -4 & 2 & -1 & 2 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & 2 \end{vmatrix} = [\underset{\sim}{\psi}^i] \quad (26)$$

This example demonstrates how the model output runoff hydrograph can be used directly to evaluate modeling error rather than determining the storm event $\underset{\sim}{\psi}^i$ vector.

Conclusions

The Stochastic Integral Equation Method (or SIEM) has been shown to be a useful tool in analyzing modeling error associated to rainfall-runoff hydrograph models. Not only does the SIEM approximately develop a distribution of modeling error to be added to a modeling result as applied to a forecast, but the SIEM also enables rainfall-runoff models to be compared as to which model is the "best estimator" of a selected criterion variable.

In Hromadka (1997), the SIEM is used to represent an arbitrary rainfall-runoff modeling structure (of a free-draining catchment such as described in Hromadka and Whitley (1989)) and its corresponding modeling error. Because the entire runoff hydrograph is mathematically analyzed, statistical analysis of the entire modeling error is possible, and a spectrum of different modeling structure types (e.g., HEC-1 Unit Hydrograph, MITCAT, etc.) can be compared as to bias and variance; that is, the modeling structures can be ranked with respect to being the "best estimator" (i.e., with respect to minimum variance in modeling estimates of a chosen criterion variable such as peak flow rate, peak 1-hour runoff volume, etc.).

In this paper, the SIEM is analyzed with respect to being applied as stochastic Toeplitz matrices. This approach not only simplifies the SIEM approach, but also illuminates the theoretical underpinnings of the SIEM. Current research underway includes using the stochastic Toeplitz matrix setting in mathematically representing link-node model structures, and examining uniqueness and existence properties of unit hydrographs given rainfall-runoff data. Also being studied are regionalization procedures to normalize the error transforms so that the modeling error distributions can be applied at ungauged catchment locations.

References

1. Hromadka II, T.V., and Whitley, R.J., Stochastic Integral Equations in Rainfall-Runoff Modeling, Springer-Verlag, New York, 1989.
2. Hromadka II, T.V., Stochastic Evaluation of Rainfall-Runoff Prediction Performance, ASCE Journal of Hydrologic Engineering, Vol. 2, No. 4, October, 1997.
3. Abbott, J., Testing of Several Runoff Models on An Urban Watershed, Tech. Paper No. 59, U.S. Army Corps of Engineers, Hydro. Engrg. Ctr., Vicksburg, Mississippi, 1978.
4. Tsokos, C.P. and Padgett, W.J. Random Integral Equations with Applications to Life Science and Engineering, Academic Press, Inc., San Diego, 1974.