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Research Note

Fractal basis functions and the CVBEM

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The Complex Variable Boundary Element Method or CVBEM has recently been applied to the use of new fractal basis functions for defining a global trial function on the problem domain. In that recent advance, a topic for future research was the need for development of an algorithm to construct a global trial function that converges to the true problem boundary conditions, assumed to be continuous on the boundary. In this paper, such an algorithm for constructing a sequence of fractal basis functions is presented.

1 INTRODUCTION

The Complex Variable Boundary Element Method (CVBEM) is a complex variable function approximation method that develops an approximator, $\hat{\omega}(z)$, that is composed of a pair of two-dimensional functions, $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$ where $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are both harmonic conjugate functions and hence exactly satisfy the Laplace equation. Several papers and books (e.g. Ref. [1]) develop the CVBEM in detail. The focus of this note is the development of a mathematical equation that links a function, $\omega(z)$, analytic in a simply connected domain, Ω, with a simple closed boundary contour, Γ, to a series expansion of $\omega(z)$ using fractal basis functions. The series expansion includes a direct representation of a discretization algorithm which is more explicit than that of Hromadka and Whitley.[2]

1.1 Complex function integrals

The boundary curve Γ, of length L, with parametric equation $\zeta = \zeta(t)$, $0 \leq t \leq \ell$, will be assumed to be smooth with $\zeta(t)$ having a continuous nonzero derivative $\zeta'(t)$ at every point $0 \leq t \leq \ell$.

Consider the partition, of a smooth curve Γ, by n points $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n$ positioned consecutively along Γ in the counterclockwise direction. Form the partial sum, S_n,

$$S_n = \sum_{j=1}^{n} f(\gamma_j)\Delta\zeta_j \tag{1}$$

where $f(z)$ is a complex function defined over a domain Ω containing Γ, $\Delta\zeta_j = \zeta_j - \zeta_{j-1}$; and γ_j is a point in Γ in the arc from ζ_{j-1} to ζ_j. For an integrable function f, for example f continuous on Γ, the following limit exists:

$$\lim_{k \to 0} \sum_{j=1}^{n} f(\gamma_j)\Delta\zeta_j = \alpha = \int_\Gamma f(z)\,\mathrm{d}z \tag{2}$$

independent of the choice of points ζ_j and γ_j, where k is the maximum value of the lengths $\|\Delta\zeta_j\|$, $j = 1, 2, \ldots, n$.

If Γ is smooth, being composed of m smooth curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, joined end to end (i.e. a piecewise smooth curve), then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{\Gamma_1} f(z)\,\mathrm{d}z + \int_{\Gamma_2} f(z)\,\mathrm{d}z + \ldots + \int_{\Gamma_m} f(z)\,\mathrm{d}z \tag{3}$$

1.2 Fractal basis functions and bisection partitioning of boundary Γ

Let z_r and z_s be two successive points, in the usual positive direction, on Γ. Then a linear fractal basis function is defined, on the arc C_{rs}, of Γ, bounded by z_r and z_s, by

$$\overset{1}{\underset{rs}{\Delta}}(\zeta) = \begin{cases} 0; & \zeta \notin C_{rs} \\ \dfrac{(\zeta - \zeta_r)}{\zeta_t - \zeta_r}; & \zeta \in C_{rt} \\ \dfrac{(\zeta_s - \zeta)}{\zeta_s - \zeta_t}; & \zeta \in C_{ts} \end{cases} \tag{4}$$

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The extension of a linear fractal basis function to a constant, parabolic, cubic, or other basis function is similar.

A sequence of successive bisection of arcs in Γ can be used to add partition points on Γ, and thereby define a set of fractal basis functions.

First, a reference point, ζ_1, is located on Γ. The arc C_{11} is all of Γ, beginning and ending at point ζ_1, transversed in the usual counterclockwise direction.

The first fractal basis function is denoted by

$$\underset{1\,1}{\overset{2}{\Delta}}(\zeta)$$

where point ζ_1 serves as both the beginning point and endpoint of arc C_{11}, with midpoint $\zeta_2 \in \Gamma$ (the midpoint of arc C_{11} is located a distance $\ell_{11}/2$ from ζ_1, as measured along C_{11}, where ℓ_{11} is the arc length of C_{11}). The second pass in partitioning Γ results in locating two more points ζ_3 and ζ_4 in Γ where ζ_3 is the midpoint of arc C_{12}, and ζ_4 is the midpoint of arc C_{21}. The corresponding fractal basis functions are

$$\underset{1\,2}{\overset{3}{\Delta}}(\zeta)$$

and

$$\underset{2\,1}{\overset{4}{\Delta}}(\zeta),$$

respectively. The third pass in partitioning Γ results in locating four more points, ζ_5, ζ_6, ζ_7, ζ_8 at the midpoints of arcs C_{13}, C_{32}, C_{24}, C_{41}, respectively.

The above method of partitioning Γ results in a set of partition points, each adjacent pair being at the same arclength distance from each other, for each iteration of the algorithm. Practically speaking, it is more convenient to partition $[0,1]$ by $0 = t_0 < t_l < t_N < 1$ and use $\zeta_j = \zeta(t_j)$ as partition points on Γ.

1.3 Global fractal function, $G(\zeta)$

Consider a continuous function, $\omega(z)$, defined on a domain Ω that contains a piecewise smooth curve, Γ. For any partition of Γ, by n points $\zeta_1, \zeta_2, \ldots, \zeta_n$ placed successively in the positive direction along Γ, a global trial function, $G(\zeta)$, can be defined for $\zeta \in \Gamma$ by a sum of fractal basis functions weighted by point values of $\omega(z)$:

$$G(\zeta) = \left[\underset{k-1\,k+1}{\overset{k}{\Delta}}(\zeta)\right]\omega_k \tag{5}$$

where $\omega_k = \omega(\zeta_k)$, with $\omega_0 = \omega_n$. This fractal basis function is analogous to the usual linear polynomial basis function approximation, but in the usual approach, all points ζ_{k-1}, ζ_k, ζ_{k+1} are defined initially.

Another approach is to use an algorithm that describes a sequence of successively finer partitioning of Γ, where the fractal basis function is weighted by the difference between the newly added partition point value, $\omega(\zeta_t)$, and, for the linear basis function, the linear approximation $(\omega(\zeta_r) + (\omega(\zeta_s))/2$.

For the bisection partitioning algorithm, the sequence of successive partitioning passes result in the global fractal function. Given the initial point $\zeta_1 \in \Gamma$ with value $\omega_1 = \omega(\zeta_1)$,

$$G(\zeta) = \omega_1 + \sum_{k=1}^{\infty} \sum_{u=1}^{2^{k-1}} \delta_{rs}^{t} \underset{r\,s}{\overset{t}{\Delta}}(\zeta) \tag{6}$$

where k is the bisection algorithm partition pass number; for this k-th pass, 2^{k-1} points are added, namely all the points halfway between the points already in the partition; d_r, d_s, d_t are the arc distances, along Γ, in the positive direction, from initial point $\zeta_1 \in \Gamma$ to partition points ζ_r, ζ_s, ζ_t, given by $d_r = (u-1)L/2^{k-1}$, $d_s = uL/2^{k-1}$, $d_t = (2u-1)L/2^k$, respectively; and δ_{rs}^t is the weighting defined by

$$\delta_{rs}^{t} = \omega(\zeta_t) - \frac{\omega(\zeta_r) + (\zeta_s)}{2} \tag{7}$$

After k_0 passes the remainder of the series, $R_{k_0}(\zeta)$, is given by

$$R_{k_0}(\zeta) = \sum_{k=k_0+1}^{\infty} \sum_{u=1}^{2^{k-1}} \delta_{rs}^{t} \underset{r\,s}{\overset{t}{\Delta}}(\zeta), \tag{8}$$

with the truncated global fractal function

$$G_{k_0}(\zeta) = \omega_1 + \sum_{k=1}^{k_0} \sum_{u=1}^{2^{k-1}} \delta_{rs}^{t} \underset{r\,s}{\overset{t}{\Delta}}(\zeta). \tag{9}$$

Suppose that ω satisfies a Lipschitz condition on Γ:

$$|\omega(z_1) - \omega(z_2)| \le M|z_1 - z_2| \tag{10}$$

for some constant M and all z_1 and z_2 on Γ. This will be true, for example, if the derivative ω' exists and is continuous, and therefore bounded on Γ.

In this case the sum

$$S = \sum_{u=1}^{2^{k-1}} \delta_{rs}^{t} \underset{r\,s}{\overset{t}{\Delta}}(\zeta) \tag{11}$$

has only two non-zero terms; the factor

$$\underset{r\,s}{\overset{t}{\Delta}}(\zeta)$$

is bounded in absolute value because the curve is smooth

$$|\underset{r\,s}{\overset{t}{\Delta}}(\zeta)| \le B \tag{12}$$

and

$$|\delta_{rs}^{t}| \le \left|\frac{\omega(\zeta_t) - \omega(\zeta_r)}{2}\right| + \left|\frac{\omega(\zeta_t) - \omega(\zeta_s)}{2}\right| \le \frac{M}{2^{k-1}} \tag{13}$$

hence

$$|S| \le \frac{BM}{2^{k-1}} \tag{14}$$

and so the series for $G_{k0}(\zeta)$ converges absolutely and uniformly on Γ.

1.4 Complex Variable Boundary Element Method (CVBEM)

The CVBEM approximator, $\hat{\omega}(z)$, defined inside the problem domain, Ω, with boundary, Γ, is

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\mathrm{d}\zeta}{\zeta - z}, \; z \in \Omega \tag{15}$$

If $\omega(z)$ is analytic on $\Omega \cup \Gamma$, then for $z \in \Omega$,

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G_{k_0}(\zeta)\mathrm{d}\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_\Gamma \frac{R_{k_0}(\zeta)\mathrm{d}\zeta}{\zeta - z}, \; z \in \Omega \tag{16}$$

The typical CVBEM application involves a problem domain where increased accuracy can be obtained by irregular placement of nodes, an aspect which we will now neglect.

The CVBEM approximation can be expanded for $z \in \Omega$ by

$$\frac{1}{2\pi i}\int_\Gamma \frac{G_{k_0}(\zeta)\mathrm{d}(\zeta)}{\zeta - z} = \omega_1 \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - z}$$
$$+ \int_\Gamma \sum_{k=1}^{k_0}\sum_{u=1}^{2^{k-1}} \delta^{\mathrm{t}}_{\mathrm{rs}} \frac{\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta)\mathrm{d}\zeta}{\zeta - z} = \omega_1\left(\frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - z}\right)$$
$$+ \sum_{k=1}^{k_0}\sum_{u=1}^{2^{k-1}} \delta^{\mathrm{t}}_{\mathrm{rs}} \int_{C_{\mathrm{rs}}} \frac{\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta)\mathrm{d}\zeta}{\zeta - z} \tag{17}$$

due to

$$\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta) = 0 \text{ for } \zeta \notin C_{\mathrm{rs}}$$

The first integral in the eqn (17) expansion is $2\pi i$.

To integrate

$$\int_{C_{\mathrm{rs}}} \frac{\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta)\mathrm{d}\zeta}{\zeta - z},$$

consider the typical situation that C_{rs} is a line segment connecting ζ_{r} and ζ_{s}. Then, $\zeta_{\mathrm{t}} = (\zeta_{\mathrm{r}} + \zeta_{\mathrm{s}})/2$, and

$$\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta)$$

given by eqn (1) is

$$\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta) = \begin{cases} 0; & \zeta \notin C_{\mathrm{rs}} \\ \dfrac{(\zeta - \zeta_{\mathrm{r}})}{(\zeta_{\mathrm{t}} - \zeta_{\mathrm{r}})}; & \zeta \in C_{\mathrm{rt}} \\ \dfrac{(\zeta_{\mathrm{s}} - \zeta)}{(\zeta_{\mathrm{s}} - \zeta_{\mathrm{t}})}; & \zeta \in C_{\mathrm{ts}} \end{cases}$$

where $C_{\mathrm{rs}} = C_{\mathrm{rt}} \cup C_{\mathrm{ts}}$, and $C_{\mathrm{rt}} \cap C_{\mathrm{ts}} = \zeta_{\mathrm{t}}$.

Consequently, for $z \in \Omega$,

$$\int_{C_{\mathrm{rs}}} \frac{\underset{\mathrm{r\,s}}{\overset{\mathrm{t}}{\Delta}}(\zeta)\mathrm{d}\zeta}{\zeta - z} = \left(\frac{z - \zeta_{\mathrm{r}}}{\zeta_{\mathrm{t}} - \zeta_{\mathrm{r}}}\right)(\ln(\zeta_{\mathrm{t}} - z) - \ln(\zeta_{\mathrm{r}} - z))$$
$$- \left(\frac{z - \zeta_{\mathrm{s}}}{\zeta_{\mathrm{s}} - \zeta_{\mathrm{t}}}\right)(\ln(\zeta_{\mathrm{s}} - z) - \ln(\zeta_{\mathrm{t}} - z))$$

where the logarithm used must have its branch cut prescribed as described in Ref. [2].

2 CONCLUSIONS

A new algorithm for constructing a global trial function by a sequence of fractal basis functions is presented. The series converges to the true problem boundary conditions, under mild conditions of continuity. Because of the series representation structure, error evaluation of the approximation is readily achieved by using standard series evaluation techniques, given that the solution to the boundary value problem satisfies a Lipschitz condition on the problem boundary (see eqns (8) and (14)).

REFERENCES

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