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PII: S0955-7997(96)00027-6

Engineering Analysis with Boundary Elements **18** (1996) 39–41

0955-7997/96/$15.00

Research Note

A new formulation for developing CVBEM approximation functions

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(Received 10 January 1996, accepted 10 May 1996)

The complex variable boundary element method, or CVBEM, is a numerical technique that exactly solves the Laplace or many forms of the Poisson equation, over a two-dimensional domain. A considerable effort is expended, however, in developing formulations involving higher order trial function CVBEM approximations. In this work, a new expansion of the CVBEM is developed that significantly mitigates the complexity involved in preparing such higher order trial function approximations. Copyright © 1996 Elsevier Science Ltd

Key words: Complex variable boundary element method, Taylor series, error analysis, potential flow problems.

INTRODUCTION

The objective in using the complex variable boundary element method (CVBEM) is to approximate analytic complex functions. Given that ω is a complex function which is analytic over a simply connected domain Ω with boundary values $\omega(\zeta)$ for $\zeta \in \Gamma$ (Γ is a simple closed contour), then both the real (ϕ) and imaginary (ψ) parts of $\omega = \phi + i\psi$ satisfy the Laplace equation over Ω. Thus two-dimensional potential flow problems can be approximated by the CVBEM, including steady-state heat transport, soil water flow, plane stress and elasticity, among other topics.

The development of the CVBEM for engineering and applied mathematics applications is detailed in several publications (see Refs 2, 4–7). In general, a CVBEM approximation is developed using a set of polynomial basis functions which are subsequently integrated in the Cauchy integral formula (see Ref. 1). This integration process is lengthy and, for basis functions beyond the second order, quite formidable. In this work, an alternative to the usual integration is presented.

In the alternative approach the CVBEM is expanded by use of Taylor series defined on each boundary element, expanded with respect to each nodal point. By introducing another complex polynomial that is simply the trial function evaluated at the independent variable point, z_0, a significant new simplification of the CVBEM is constructed. This new construction significantly simplifies the development of higher order trial function CVBEM approximations.

CVBEM Taylor series expansion for each node

Let Ω be a simply connected domain enclosed by a simple closed contour, Γ, which is the boundary of Ω. Let $\omega(z)$ be analytic on $\Omega \cup \Gamma$. Then by Cauchy's theorem (please see references for full details)

$$\omega(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)d\zeta}{\zeta - z_0}, z_0 \in \Omega \tag{1}$$

Suppose Γ is a piecewise linear boundary, composed of v straight line segments and with v vertices. The usual numerical analog is to minimally place nodal points at each vertex. Then the boundary elements result, identified as Γ_j, which is the straight line segment joining nodes j and $(j+1)$, with coordinates z_j and z_{j+1}, respectively.

Because, by assumption, $\omega(z)$ is analytic on Γ, then a Taylor series expansion exists for any node j, expanded about $z = z_j$, with radius of convergence, $R_j > 0$, such that for $|z - z_j| < R_j$

$$\omega(z) = \omega(z_j) + \omega^{(1)}(z_j)(z - z_j) + \omega^{(2)}(z_j)\frac{(z-z_j)^2}{2!} + \ldots \quad (2)$$

where again $|z - z_j| < R_j$; and $\omega^{(n)}(z_j)$ is the complex nth order derivative of $\omega(z)$ evaluated at $z = z_j$. Hereafter, the notation $T_j(z)$ is used to represent the Taylor series of eqn (2), and $T_j^n(z)$ is notation for the partial sum of the first $n+1$ terms (which is an order n complex polynomial).

In our model development, additional nodal points are added to Γ such that a finite cover of Γ results from the collection of m disks from the set $\{|z - z_j| < R_j;\ j = 1, 2, \ldots, m\}$.

CVBEM polynomial integration

Let Γ_j be a boundary element from the CVBEM model. The contribution of eqn (1) from Γ_j is computed by noting

$$\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\mathrm{d}\zeta}{\zeta - z_o} = \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\Gamma_j} \frac{\omega(\zeta)\mathrm{d}\zeta}{\zeta - z} \quad (3)$$

where m nodes are defined on Γ according to the above development. The usual procedure is to now insert the CVBEM basis functions into the numerator of eqn (3) and integrate the resulting rational function on each boundary element.

Let $T_j^n(z)$ be the $(n+1)$th partial sum (which is an order n complex polynomial) of the Taylor series expansion about $z - z_j$. The usual CVBEM integration of a boundary element trial function is simplified by writing

$$\int_{\Gamma_j} \frac{T_j^n(\zeta)\mathrm{d}\zeta}{\zeta - z_o} = \int_{z_j}^{z_{j+1}} \frac{T_j^n(\zeta) - T_j^n(z_o)}{\zeta - z_o}\mathrm{d}\zeta + T_j^n(z_o)\int_{z_j}^{z_{j+1}} \frac{\mathrm{d}(\zeta)}{\zeta - z_o} \quad (4)$$

where $z_o \in \Omega$; $Q(z) = T_j^n(z) - T_j^n(z_o)$ is a polynomial in z with $Q(z_o) = 0$, and thus $(z - z_o)$ is a factor of $Q(z)$. Then $Q(z)/(z - z_o)$ is a polynomial of degree $(n-1)$ in z, and by symmetry also in z_o, and so integrates to a polynomial of degree $(n-1)$ in z_o.

For each boundary element, Γ_j,

$$\frac{1}{2\pi i}\int_{z_j}^{z_{j+1}} \frac{T_j^n(\zeta)\mathrm{d}\zeta}{\zeta - z_o} = P_j^{n-1}(z_o) + T_j^n(z_o)H_j(z_o) \quad (5)$$

where $P_j^{n-1}(z_o)$ is an order $(n-1)$ complex polynomial in z_o; $T_j^n(z_o)$ is the $(n+1)$th partial sum of the Taylor series expansion about $z = z_j$, evaluated at z_o; and

$$H_j(z_o) = \frac{1}{2\pi i}\int_{z_j}^{z_{j+1}} \frac{\mathrm{d}\zeta}{\zeta - z_o} \quad (6)$$

Combining the results of eqns (3), (4) and (5) gives the CVBEM approximation, $\hat{\omega}(z_o)$,

$$\hat{\omega}(z_o) = \sum_{j=1}^{m} P_j^{n-1}(z_o) + \sum_{j=1}^{m} T_j^n(z_o)H_j(z_o) \quad (7)$$

Equivalently,

$$\hat{\omega}(z_o) = P^{n-1}(z_o) + \sum_{j=1}^{m} T_j^n(z_o)H_j(z_o) \quad (8)$$

where $P^{n-1}(z_o)$ is an order $(n-1)$ complex polynomial in $z_o \in \Omega$.

DISCUSSION

What is novel about the expansion in eqn (8) is that the integration of the trial function approximation (in this case, a partial sum of a Taylor series expansion), falls *outside* the Cauchy integral (compare with eqn (4)). This result leads to an any order polynomial trial function CVBEM approximation[1-3] much more conveniently due to the elimination of complex variable integration procedures. The $H_j(z_o)$ functions in eqn (8) are the usual complex logarithm components of the CVBEM (for each Γ_j).

What is also new in eqn (8) is the $(n-1)$ order complex polynomial $P^{n-1}(z_o)$. The $P^{n-1}(z_o)$ is the sum of m boundary elements $P_j^{n-1}(z_o)$ polynomials. Naturally, various order polynomial trial functions could be used over the boundary elements, but the highest order trial function used on any Γ_j dictates the order of $P^{n-1}(z_o)$.

Further research is needed regarding the character of $P^{n-1}(z_o)$, among other topics. Currently we apply the CVBEM by first finding the best fit (in the usual least squares error sense) complex polynomial to $\omega(z)$, on Γ, and then operate on the remaining summed components, in eqn (8), in fitting the new boundary value problem formed by $\omega(z) - P^{n-1}(z_o)$ on Γ. Whether use of a $P^{n-1}(z_o)$ is even necessary in the CVBEM is also under investigation; however, if $\omega(z)$ is an order $(n-1)$ complex polynomial, then obviously $P^{n-1}(z_o)$ will match $\omega(z)$.

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