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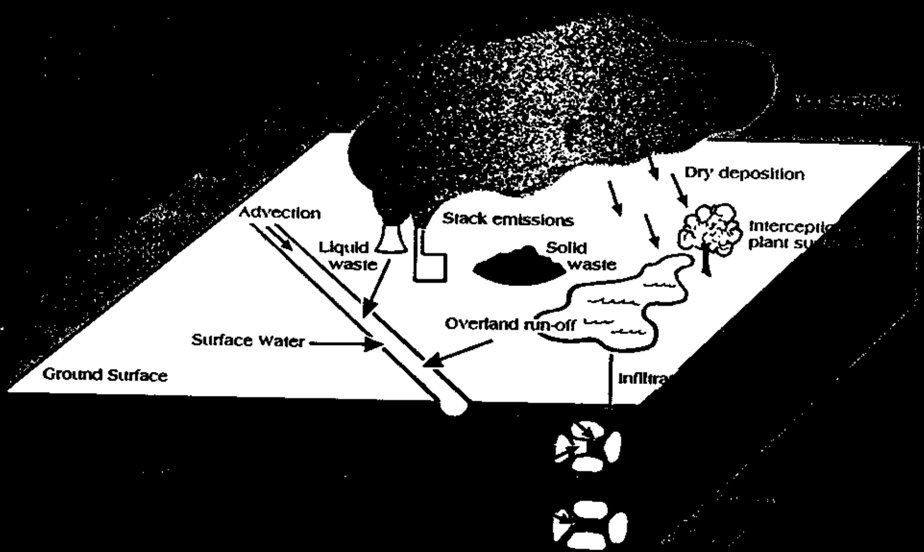

Computational Mechanics Publications

Chapter 8

Approximating rainfall-runoff modeling response using a stochastic integral equation.

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Abstract

An underlying reason for the proliferation of rainfall-runoff models, is the lack of success in any one rainfall-runoff model in estimating runoff from rainfall data. A principal reason for this lack of modeling success is due to the uncertainty in the effective rainfall over the catchment (Hromadka and McCuen, 1989). That is, not only is the rainfall highly variable with respect to both space and time (given a pint measurement of rainfall), but the rainfall less losses (i.e., "effective rainfall"; that is, the runoff after all losses due to soil infiltration, evaporation, etc., are subtracted) is highly variable over the catchment. Generally, the runoff estimate for a several square mile catchment is based upon a single rain gauge measurement that oftentimes is not even in the catchment. As a result, the runoff predictions obtained from any rainfall-runoff model are highly uncertain.

In this work, the rainfall-runoff modeling problem is reanalyzed with respect to the theory of stochastic integral equations. The results of the presented work have been adopted by two major flood control governmental agencies (Hromadka and McCuen, 1986a,b), and provide a good case study in the application of statistics, stochastics, and computational method for solving modern problems.

The main objective of this work is to present a method of statistical or stochastic prediction in rainfall-runoff analysis in contradistinction with the usual deterministic prediction techniques.

Introduction

The problem of predicting watershed runoff (i.e., flood flows) from rainfall data is of key importance to society, affecting the safety of millions of people who live in lowland areas, and costing hundreds of millions of dollars to provide flood protection. As an indication of the difficulty experienced by modelers in attempting to predict runoff from rainfall data, more than one-hundred rainfall-runoff models have been reported in the literature (Hromadka et al, 1987), resulting in a wide range in predicted values of runoff depending on the particular rainfall-runoff model chosen.

The state-of-the-art in rainfall-runoff models is to use computers to approximately solve the various complex mathematical partial differential equations (PDE) that describe the hydrologic cycle as distributed over the catchment (or watershed), and to approximately solve the flood flow timing PDE involved in conduit flow routing (i.e., time varying flow effects in streams, channels, pipes, or other structures). Empirical equations are used to describe the hydrologic cycle time-varying components of evaporation, plant transpiration, infiltration of rainfall into the soil, percolation of soil-moisture into deeper soils, and ponding of water, among other effects. The hydraulic effects of flood flow routing in streams and conduits are described by the nonlinear PDE known as the Navier-Stokes equations, but are approximated by simplified computational algorithms such as the kinematic wave, Muskingum, convex, or other flow routing techniques (see Hromadka et al, 1987). The main thrust in the computer modeling of the rainfall-runoff process is to subdivide the catchment into smaller subcatchments (or subareas) that are "linked" together by the hydraulic flow routing models used to represent flow routing effects in the streams and channels. Each subarea is assumed to have a representative rainfall-runoff response, described by a unique set of hydrologic cycle parameters and equations. The subarea runoff, which is assumed to depend only on the rainfall history (and subarea hydrologic cycle characteristics), concentrates at a "nodal point", which represents the time distribution of runoff for the subject subarea, for the subject storm. The assemblage of all these links and nodes forms the catchment "link-node" rainfall-runoff model.

An underlying reason for the proliferation of rainfall-runoff models is the lack of success in any one rainfall-runoff model in estimating runoff from rainfall data. A principal reason for this lack of modeling success is due to the uncertainty in the effective rainfall over the catchment (Hromadka and McCuen, 1989). That is, not only is the rainfall highly variable with respect to both space and time (given a point measurement of rainfall), but the rainfall less losses (i.e., "effective rainfall"; that is, the runoff after all losses due to soil infiltration, evaporation, etc., are subtracted) is highly variable over the catchment. Generally, the runoff estimate for a several square mile catchment is based upon a single rain gauge measurement that oftentimes is not even in the catchment. As a result, the runoff predictions obtained from any rainfall-runoff model are highly uncertain.

In this work, the rainfall-runoff modeling problem is reanalyzed with respect to the theory of stochastic integral equations. The results of the presented work have been adopted by two major flood control governmental agencies (Hromadka and McCuen, 1986a,b), and provide a good case study in the application of statistics, stochastics, and computational methods for solving modern problems. The main objective of this work is to present a method of statistical or stochastic prediction in rainfall-runoff analysis in contradistinction with the usual deterministic prediction techniques. The work effort is presented in four parts as follows:

A. Development of a generalized stochastic integral equation representation of rainfall-runoff modeling response. Although over one-hundred link-node modeling techniques are currently reported, almost the totality of these rainfall-runoff models can be shown to be composed of similar fundamental mathematical components. As a result, the rainfall-runoff modeling approach can be "unified" into a single mathematical expression as a convolution integral equation. Due to the noted variability in the effective rainfall over the catchment, the convolution integral equation is developed into a stochastic integral equation.

B. Correlation of the stochastic integral equation to catchment characteristics. In order to apply the stochastic integral equation at ungauged catchments, a method to correlate the identified distribution of transfer functions (used in the stochastic integral) is developed. Because only a few catchments have both rainfall and runoff data, there is a need to develop synthetic runoff data given only rainfall data (which is generally much more available). In this part of the analysis, synthetic runoff may be developed from rainfall data and catchment characteristics.

C. Estimating Uncertainty in Rainfall-Runoff Modeling Estimates. From the stochastic integral equation formulation, uncertainty estimates in prediction may be developed.

D. Application of the Stochastic Integral Equation. The stochastic formulation is applied to rainfall-runoff data from Los Angeles, California, United States.

The above provides a good example of the application of stochastic integral equations and computation to "real world" problems. With the ever-increasing inexpensive computational power afforded analysts, more attention is being paid towards developing computational estimates in terms of probabilistic distributions rather than in single-valued deterministic type answers. This work presents just such an application of probabilistic thinking to the important problem of predicting flood flows for flood protection.

A. DEVELOPMENT OF A GENERALIZED STOCHASTIC INTEGRAL EQUATION REPRESENTATION OF RAINFALL-RUNOFF MODELING RESPONSE

The study of Hjelmfelt and Burwell (1984) provides a good case study as to the spatial and temporal variation of effective rainfall over a catchment. In their study, 40 adjacent plots of land, each 27.5 m x 3.2m in size, were maintained to "simulate an intensively sampled watershed of field size," so that the results of measurements would describe the "spatial variation of runoff and water retention. Uniformity in applied management was maintained among the 40 plots, so that in the context of field-size watershed modeling even comprehensive mathematical models would treat the plots as identical." The runoff quantiiies were measured in tanks located at the outlet of each plot. A rain gauge was located within the 40-plot test site.

Based on the vicinity of these plots to each other and with respect to the rain gauge, one would expect similar runoffs produced from each plot. The measurements showed that there was a wide range in runoff measurements even though the plots could be considered identical.

The Hjelmfelt and Burwell experiment (1984) is useful in introducing some of the concepts to be used in later sections of this paper. Let's modify the field experiment by assuming that each plot drains into a small channel which ultimately joins at some central collection point. Each plot has a runoff measuring device that measures the flow rate versus time, and the central point measures the combined runoff from all 40 plots. The runoff from subarea R_j is noted as $q_j^i(t)$ for storm event number i, where $q_j^i(t)$ is the runoff flow rate as a function of time. The runoff flow rate measured at the central collection point is noted as $Q_g^i(t)$ where hereafter this collection point is called a stream gauge. We additionally assume that each of the 40 connecting channels deliver each $q_j^i(t)$ to the stream gauge in such a fashion such that the subarea R_j runoff, as measured at the stream gauge is simply $q_j^i(t - \tau_j)$; that is, each $q_j^i(t)$ is translated in time by a constant travel time of τ_j for subarea R_j, regardless of the storm event.

Based on the above assumptions, the stream gauge measured runoff, $Q_g^i(t)$, can be equated to the several $q_j^i(t)$ by

$$Q_g^i(t) \sum_{j=1}^{40} q_j^i(t - \tau_j) \tag{1}$$

where each $q_j^i(t - \tau_j) = 0$ for $t < \tau_j$.

In our thought experiment we actually know the $q_j^i(t)$ due to the subarea runoff measurement devices. We also know the precipitation measured at the rain gauge, $P_g^i(t)$. However, how do we relate the $P_g^i(t)$ to each $q_j^i(t)$, and the $Q_g^i(t)$, given the observed variability?

Consider another plot, R_{41}, which is identical to the other 40 plots, and place this 41st plot at the rain gauge location. The 41st plot is also measured for runoff flow rates, $q_{41}{}^i(t)$. It is assumed that due to the rain gauge and R_{41} being immediately adjacent to each other that the rainfall data, $P_g{}^i(t)$, is an accurate measure of rainfall over R_{41} and, therefore, enables a precise relationship to be developed between $P_g{}^i(t)$ and $q_{41}{}^i(t)$.

The relationship between $P^i(t)$ and $q_{41}{}^i(t)$ is very complex, however, involving many time varying components of the hydrologic cycle. Given this rainfall-runoff relationship for R_{41}, defined by the operator F where F transforms each $P_g{}^i(t)$ into $q_{41}{}^i(t)$, then F: $P_g{}^i(t) \rightarrow q_{41}{}^i(t)$, and we could then use either $F(P^i(t))$ or $q_{41}{}^i(t)$ to compute each $q_j{}^i(t)$, for $j = 1,2,\cdots,40$.

Due to the variability between the $q_j{}^i(t)$ measured by Hjelmfelt and Burwell, however, we must assume that the $q_j{}^i(t)$ are not equal, for a given storm i, but are probabilistically distributed with respect to each other.

Let each of the 40 plots' runoff be equated to $q_{41}{}^i(t)$ by use of the random variables $[\lambda_j]$ and $[\theta_j]$ for each R_j where for each storm event, i,

$$q_j{}^i(t) = \lambda_j{}^i\, q_{41}{}^i(t - \theta_j{}^i) \tag{2}$$

where $q_{41}{}^i(t - \theta_j{}^i) \equiv 0$ for $t < \theta_j{}^i$. The distributions of $[\lambda_j]$ and $[\theta_j]$ could be developed from the frequency distribution of values obtained by a large collection of values determined from Eq. (2). It is noted that the several random variables may be all mutually dependent.

Then by means of Eq. (2), the stream gauge runoff, $Q_g{}^i(t)$ can be written as

$$Q_g{}^i(t) \sum_{j=1}^{40} \lambda_j{}^i\, q_{41}{}^i(t - \tau_j - \theta_j{}^i) \tag{3}$$

where the set of values of $\{\lambda_j{}^i, \theta_j{}^i\}$ are samples of the corresponding random variables.

Our analysis now turns to the important problem of prediction. Assuming a hypothetical storm event to occur at the study site, resulting in the rainfall $P_g{}^D(t)$ and the R_{41} runoff, $q_{41}{}^D(t)$, what would be the estimate of runoff at the stream gauge? Because we are in a prediction mode, the values for each $\lambda_j{}^D$ and $\theta_j{}^D$ are unknown for $j = 1,2,\cdots,40$, which are samples of mutually dependent random variables distributed as $[\lambda_j]$ and $[\theta_j]$, respectively. Then our estimate for runoff at the stream gauge is the stochastic process $[Q_g{}^D(t)]$ where

$$[Q_g{}^D(t)] = \sum_{j=1}^{40} [\lambda_j]\, q_{41}{}^D(t - \tau_j - [\theta_j]) \tag{4}$$

In Eq. (4), we have used the "measured" $q_{41}{}^{D}(t)$ to develop $[Q_g{}^{D}(t)]$ in order to simplify the presentation; the $F(P_g{}^{D}(t))$ could also have been used to motivate the estimate given in Eq. (4).

The general problem is to be given only $P_g{}^{i}(t)$ data, requiring the additional problem to be solved of defining the previously discussed operator, F, used to develop and estimate $Q_{41}{}^{D}(t)$ or its equivalent, effective rainfall (i.e., rainfall less losses due to infiltration of rainfall, local ponding, evaporation, etc.). In the remainder of this paper, measured runoff, such as $q_{41}{}^{i}(t)$, is assumed to be available for model development purposes. The additional complexities involving the definition of the operator F are addressed in subsequent text.

Stochastic Integral Equations in Rainfall-Runoff Modeling

Our mathematical problem setting is to predict runoff quantities at a specified point, given a hypothetical rainfall event defined to occur at another point. In order to develop such a prediction, we construct a rainfall-runoff model which incorporates catchment hydrologic (i.e., rainfall-runoff relationships) and hydraulic (i.e., characteristics of water flow in channels such as flow speed and channel storage effects) information with the specified rainfall input to produce an estimate of runoff. We are also given historic runoff data where a stream gauge is located, and the corresponding rainfall data at the rain gauge site. This historic rainfall-runoff data is used to calibrate the various modeling parameters.

A review of the pertinent literature indicates that no rainfall-runoff modeling approach appears to have adequately solved the runoff prediction problem. That is, a purely deterministic rainfall-runoff model does not represent the uncertainty in the prediction, and hence does not provide a range of probable values in runoff estimates computed from the performance of the model in agreeing with the historic data. The use of a stochastic integral equation provides a means to include this rainfall-runoff modeling uncertainty with the runoff estimates.

In order to apply stochastic integral equations to rainfall-runoff modeling, a quasi-linear rainfall-runoff modeling structure is developed. The model structure represents a wide spectrum of modeling structures in common use today and, therefore, is useful in analyzing how almost all rainfall-runoff models operate.

To simplify the mathematical development, an additional source of data is assumed available, namely, effective rainfall data (i.e., which is measured at the rain gauge site). More specifically, the work of Hjelmfelt and Burwell (1984) is recast into an idealized situation where our study catchment, R, can be subdivided into m square, 1-acre size, subareas, R_j, $j = 1,2,\cdots,m$, with each subarea being nearly identical in its rainfall-runoff properties. Additionally, at the rain gauge site another small 1-acre subarea is specified and monitored so that for each storm event, i, the rainfall and runoff from the subarea are both measured (assume the rain gauge is placed in the center of the subarea). That is, for each storm event i, we obtain the data $P_g{}^i(t)$ and $e_g{}^i(t)$, which reflect the measured rainfall and effective rainfall data, respectively, from the rain gauge site (see Fig. 1.) We assume that all of the subareas, including our monitored subarea, are of such similarity that one would expect nearly identical rainfall-runoff responses. (This idealized problem setting is similar to the situation reported in the referenced Hjelmfelt and Burwell (1984) study.) Use of the $e_g{}^i(t)$ data greatly simplifies our problem in that we eliminate (for now) the need to construct a hydrologic cycle model to predict $e_g{}^i(t)$ given the $P_g{}^i(t)$ data. The $e_g{}^i(t)$ data assumed available to us accurately represents all of the hydrologic cycle components of infiltration, antecedent moisture, initial abstraction, evapotranspiration, and other factors. Consequently, the predictions of subarea R_j effective rainfall, noted as $e_j{}^i(t)$ for storm event i, can be directly associated to the $e_g{}^i(t)$ data, with variations between the effective rainfalls considered to be random fluctuations. By constructing our rainfall-runoff model structure to be a function of the $e_g{}^i(t)$ data and also the probabilistic variations between $e_g{}^i(t)$ and the subarea effective rainfalls $e_j{}^i(t)$, $j = 1,2,\cdots,m$, the model output will be found to be a stochastic process in that our estimates of runoff at the stream gauge is not a single outcome, but a distribution of outcomes. Any criterion variable of interest (such as peak flow rate, channel size, etc.) will then be concluded to be a random variable whose distribution of values reflects the random variations between the $e_j{}^i(t)$ and $e_g{}^i(t)$, among other factors.

For storm event i, the runoff hydrograph from subarea j is noted as $q_j{}^i(t)$. The effective rainfall distribution over subarea j, for storm i, is noted by $e_j{}^i(t)$. Assuming that for storm i there are characteristic travel times for translation channel routing, the runoff hydrograph at the stream gauge, $Q_g{}^i(t)$, equates to the m-subarea contributions by

$$Q_g{}^i(t) = Q_m{}^i(t) = \sum_{j=1}^{m} q_j{}^i\,(t - \tau_j{}^i) \tag{5}$$

where $\tau_j{}^i$ is the sum of characteristic travel times for all channel links which connect subarea j to the catchment R stream gauge; and $Q_m{}^i(t)$ is the m-subarea rainfall-runoff model estimate of runoff for storm event i. We now expand upon the elements used in Eq. (5).

Subarea Effective Rainfall, $e_j{}^i(t)$

In each subarea, R_j, the effective rainfall is $e_j{}^i(t)$. In our problem, however, $e_j{}^i(t)$ would be unknown because there is neither a stream gauge nor rain gauge in Rj. Assuming that $e_j{}^i(t)$ can be written as a linear combination of translates of the measured $e_g{}^i(t)$ gives

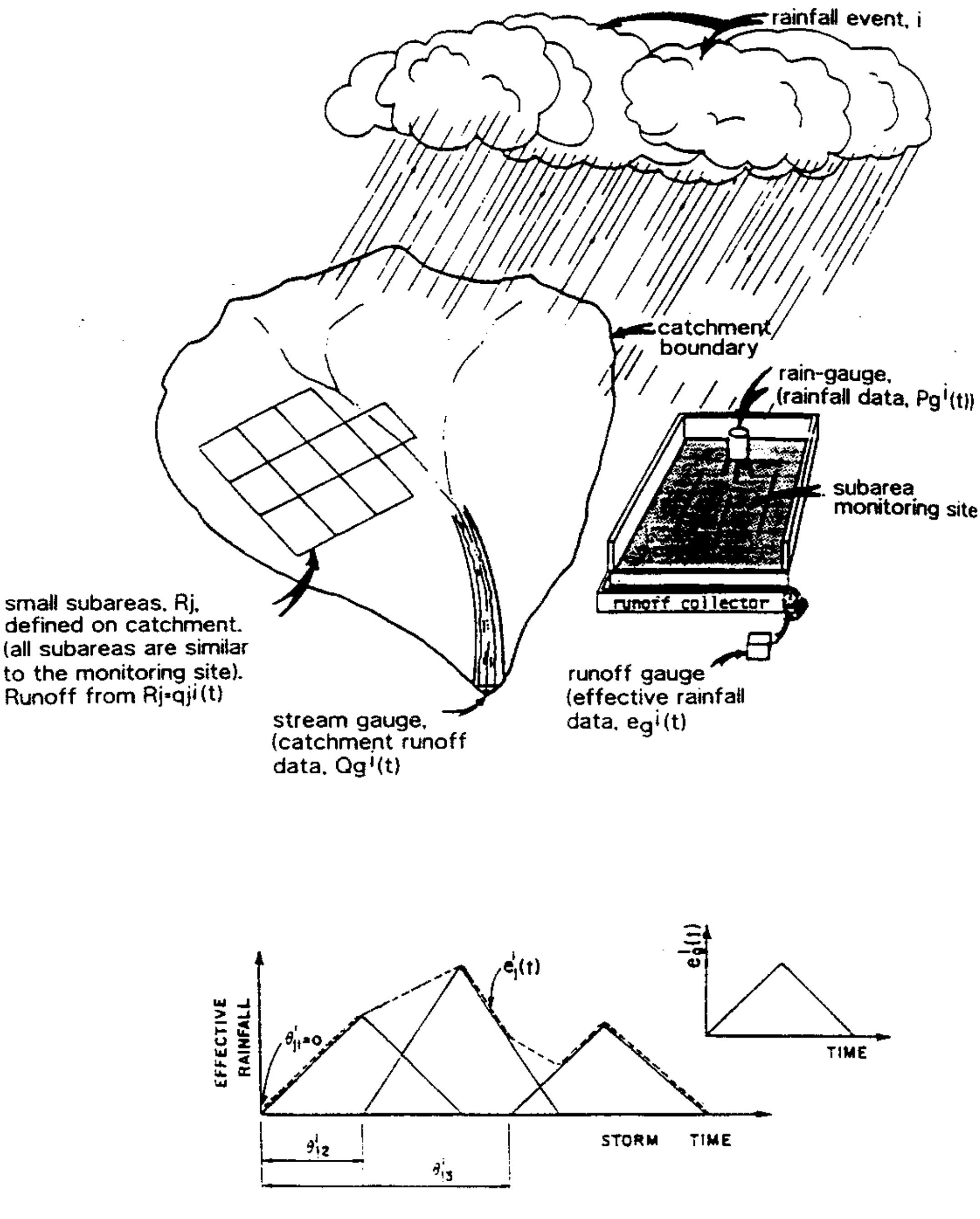

Figure 1. Subarea Effective Rainfall as a linear Combination of Translates of Measured Effective Rainfall

$$e_j^i(t) = \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, e_g^i(t - \theta_{jk}^i) \tag{6}$$

where

n_j^i = number of translates of $e_g^i(t)$ used to model $e_j^i(t)$, from storm i;

λ_{jk}^i = coefficients for subarea j and storm i; $k = 1,2,\cdots,n_j^i$;

θ_{jk}^i = constant timing offsets for subarea j, and storm i, where $e_g^i(t - \theta_{jk}^i) = 0$ for $t < \theta_{jk}^i$;

and all the n_j^i, λ_{jk}^i, θ_{jk}^i, vary on a storm basis, i.

The λ_{jk}^i and θ_{jk}^i are samples of random variables distributed as $[\lambda_{jk}]$ and $[\theta_{jk}]$, respectively. It is seen that Eq. (6) is a discrete approximation of a convolution.

Subarea Effective Hydrograph, $q_j^i(t)$

Given $e_j^i(t)$, the runoff hydrograph for storm i, $q_j^i(t)$, is given by the convolution.

$$q_j^i(t) = \int_{s=0}^{t} e_j^i(t - s) \, \phi_j^i(s) \, ds \tag{7}$$

where $\phi_j^i(s)$ is a transfer function (TF), for subarea j, and for storm i. It is noted that the $\phi_j^i(s)$, may differ between storms, i.

Combining Eqs. (6) and (7) relates $q_g^i(t)$ to the available data, $e_g^i(t)$, by

$$q_j^i(t) = \int_{s=0}^{t} \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, e_g^i(t - \theta_{jk}^i - s) \, \phi_j^i(s) \, ds \tag{8}$$

By a change of variables,

$$q_j^i(t) = \int_{s=0}^{t} e_g^i(t\text{-}s) \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i) \, ds \tag{9}$$

Combining Eqs. (5) and (9) gives the $Q_m{}^i(t)$ estimate for the runoff hydrograph at the stream gauge,

$$Q_m{}^i(t) = \sum_{j=1}^{m} \int_{s=0}^{t} e_g{}^i(t\text{-}s) \int_{k=1}^{\eta_j{}^i} \lambda_{jk}{}^i \, \phi_j{}^i(s - \theta_{jk}{}^i - \tau_j{}^i) \, ds \qquad (10)$$

which is rewritten as,

$$Q_m{}^i(t) = \int_{s=0}^{t} e_g{}^i(t\text{-}s) \sum_{j=1}^{m} \sum_{k=1}^{\eta_j{}^i} \lambda_{jk}{}^i \, \phi_j{}^i(s - \theta_{jk}{}^i - \tau_j{}^i) \, ds \qquad (11)$$

At this point of model development, it is recalled what data are available for storm event i. We only know $P_g{}^i(t)$, $e_g{}^i(t)$, $Q_g{}^i(t)$, and watershed characteristics. The information necessary to define the $\lambda_{jk}{}^i$, $\theta_{jk}{}^i(s)$, $\phi_j{}^i(s)$, $e_j{}^i$, are simply unknown. Given $e_g{}^i(t)$, however, we may suspect that the $e_j{}^i(t)$ are all different between each subarea and with respect to $e_g{}^i(t)$, and these differences vary on a storm by storm basis. Thus, the $\lambda_{jk}{}^i$ and $\theta_{jk}{}^i$ are all samples of random variables which may be strongly mutually dependent. Additionally, the values of $\phi_j{}^i(s)$ and $e_j{}^i$ may be random on a storm basis. If we knew the probability distributions of all of these random variables and processes, we could incorporate the distributions into Eq. (11) and our model estimate for a prediction of runoff would not be a single outcome, but a distribution of probable outcomes, each a possible candidate for being the runoff hydrograph at the stream gauge for the future storm event.

A Stochastic Integral Equation Formulation

Equation (11) can be written as a stochastic integral equation

$$Q_m{}^i(t) = Q_1{}^i(t) = \int_{s=0}^{t} e_j{}^i(t - s) \, \eta^i(s) \, ds \qquad (12a)$$

where from Eq. (11),

$$\eta^i(s) \sum_{j=1}^{m} \sum_{k=1}^{\eta_j{}^i} \lambda_{jk}{}^i \, \phi_j{}^i(s - \theta_{jk}{}^i - \tau_j{}^i) \qquad (12b)$$

In Eq. (12) $\eta^i(s)$ is a TF, for storm i, for the entire catchment. Consequently, given a set of storm effective rainfalls, $\{e_g{}^i(t)\}$, there is an associated set of realizations, $\{\eta^i(s)\}$, which not only represent the several unknown *variations in hydraulic response* in R (represented in Eq. (12b) by the parameters $\phi_j{}^i(s)$ and $\tau_j{}^i$), but also the several variations in the effective rainfall distribution (i.e., the hydrologic response) over R (represented in Eq. (12b) by the parameters $\lambda_{jk}{}^i$, $\theta_{jk}{}^i$, $\eta_j{}^i$). Because all of these uncertainties and variations cannot be evaluated without a supply of rainfall-runoff data for each subarea and channel hydraulic link used in $Q_m{}^i(t)$, the modeling output of $Q_1{}^i(t)$ must be, in a predictive mode, considered a stochastic process. Given a design (or predicted) effective rainfall distribution at the rain gauge site of $e_g{}^D(t)$, then the model output must be a stochastic process, $[Q_1{}^D(t)]$, where

$$[Q_1{}^D(t)] = \int_{s=0}^{t} e_g{}^D(t\text{-}s)\,[\eta(s)]\,ds \tag{13}$$

where $[\eta(s)]$ is the stochastic process with realizations developed from Eq. (12). In Eq. (13), the brackets are notation for a random or stochastic process. In Eq. (13), $[\eta(s)]$ equates to the distribution of TF's developed by inserting the mutually dependent distributions of $[\lambda_{jk}]$, $[\theta_{jk}]$, $[\phi_j(s)]$, $[\tau_j]$, and $[\eta_j]$, into Eq. (12b). The last result is important because even though the individual distributions used in Eq. (12b) cannot be evaluated (due to the lack of flow data), the effect of the several interdependent random processes are properly represented by the distribution of TF's, $[\eta(s)]$, used in Eq. (13).

Effects of Channel Routing

In this section, the development leading to the rainfall-runoff models of Eqs. (11) and (12) is extended to include the effects of unsteady flow routing due to channel storage effects. Channel routing effects are generally considered to be important, and have fueled the proliferation of rainfall-runoff models. Let $I_1(t)$ be the inflow hydrograph to a channel flow routing line (number 1), and $O_1(t)$ the outflow hydrograph. A linear routing model of the unsteady flow process is given by

$$O_1(t) = \sum_{\ell_1=1}^{n_1} a_{\ell_1}\, I_1(t - \alpha_{\ell_1}) \tag{14}$$

where the a_{ℓ_1} are coefficients which sum to unity; and the α_{ℓ_1} are timing offsets. Again, $I_1(t - \alpha_{\ell_1}) = 0$ for $t < \alpha_{\ell_1}$. Given stream gauge data for $I_1(t)$ and $O_1(t)$, the best fit values for the a_{ℓ_1} and α_{ℓ_1} can be determined by a least squares approximation. It is seen that the translation routing model of Eq. (1) can be written in terms of Eq. (14) with $I_1(t) = q_1{}^i(t)$; $n_1 = 1$; $a_{\ell_1} = 1$; and $\alpha_{\ell_1} = \tau_1$.

Should the above outflow hydrograph, $O_1(t)$, now be routed through another link (number 2), then $I_2(t) = O_1(t)$ and from the above

$$O_2(t) = \sum_{\ell_1=1}^{n_2} a_{\ell_2}\, I_2(t - \alpha_{\ell_2}) = \sum_{\ell_1=1}^{n_2} a_{\ell_2} \sum_{\ell_1=1}^{n_1} a_{\ell_1}\, I_1(t - \alpha_{\ell_1} - \alpha_{\ell_2}) \tag{15}$$

(It is understood that $a_{\ell_1} \neq \alpha_{\ell_1}$ for $\ell_1 = \ell_2$. The ℓ_1 and ℓ_2 notation also differentiates between the values of a (and α) used in Eq. (15).)

For L links, each with their own respective routing data, the above linear routing technique results in the outflow hydrograph for link number L, $O_L(t)$, being given by

$$O_L(t) = \sum_{\ell_{L-1}=1}^{n_{L-1}} a_{\ell_L} \sum_{\ell_{L-1}=1}^{n_{L-1}} a_{\ell_{L-1}} \cdots \sum_{\ell_2=1}^{n_2} a_{\ell_2} \sum_{\ell_1=1}^{n_1} a_{\ell_1} I_1(t - \alpha_{\ell_1} - \alpha_{\ell_2} \cdots - \alpha_{\ell_{L1}} - \alpha_{\ell_L}) \quad (16)$$

Using an index notation, $O_L(t)$ is written as

$$O_L(t) = \sum a_{<\ell>} I_1(t - \alpha_{<\ell>}) \quad (17)$$

It is seen that Eq. (17) is an approximation of another convolution. Hromadka and Whitley (1988) show that almost all flow routing techniques in use today can be represented by the discrete convolution approximation of Eq. (17), with constant parameters. The various routing methods differ in their respective empirical relationships in estimating these parameters; hence, the various routing methods result in different equations. Only by actual flow routing data can the true values of the $a_{<\ell>}$ and $\alpha_{<\ell>}$ be determined for each channel link used in the link-node model. Because such flow data is rarely available (for even one channel link), various empirical equations are used to estimate these parameters, resulting in a significant range in flow routing estimates.

For subarea R_j, the runoff hydrograph for storm i, $q_j^i(t)$, flows through L_j links before arriving at the stream gauge and contributing to the total measured runoff hydrograph, $Q_g^i(t)$. All of the constants $a_{<\ell>}$ and $\alpha_{<\ell>}$ are variable on a storm by storm basis. Consequently from the linearity of the routing technique, the m-subarea link node model is given by the sum of m contributions, $q_j^i(t)$, where

$$Q_m^i(t) = \sum_{j=1}^{m} \sum_{<\ell>_j} a^i_{<\ell>_j} q_j^i(t - \alpha^i_{<\ell>_j}) \quad (18)$$

where $<\ell>_j$ are associated to R_j, and all data are defined for storm i. It is noted that in order to preserve continuity of mass,

$$\sum_{<\ell>_j} a^i_{<\ell>_j} = 1 \quad (19)$$

For the above linear approximations for storm i, Eqs. (6), (9), and (18) can be combined to give the final form for our rainfall-runoff model,

$$Q_m^i(t) = \sum_{j=1}^{m} \sum_{<\ell>_j} a^i_{<\ell>_j} \int_{s=0}^{t} e_g^i(t-s) \sum_{k=1}^{n_j^i} \lambda_{jk}^i \phi_j^i(s - \theta_{jk}^i - \alpha^i_{<\ell>_j})\, ds \quad (20)$$

Because *the measured effective rainfall distribution,* $e_g^i(t)$, *is independent of* the model, Eq. (20) is rewritten as

$$Q_m^i(t) = \int_{s=0}^{t} e_g^i(t-s) \sum_{j=1}^{m} \sum_{\langle \ell \rangle} a^i_{\langle \ell \rangle j} \sum_{k=1}^{\eta^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i - \alpha^i_{\langle \ell \rangle j}) \, ds \tag{21}$$

where all parameters are evaluated on a storm by storm basis.

Equation (21) describes a model which represents the total catchment runoff response based on variable subarea TF's, $\phi_j^i(s)$; variable effective rainfall distributions on a subarea-by-subarea basis with differences in magnitude (λ_{jk}^i), timing (θ_{jk}^i), and pattern shape; and channel flow routing translation and storage effects (parameters $a^i_{\langle \ell \rangle j}$ and $\alpha^i_{\langle \ell \rangle j}$).

The major difference between the rainfall-runoff models of Eqs. (11) and (21) are the additional complexities introduced due to inclusion of a better approximation of unsteady flow routing effects in the channel links used in the link-node representation of the catchment. Almost all rainfall-runoff models in use today use a routing technique such as convex, Muskingum, *translation, kinematic wave, or the most recent diffusion (or zero-inertia)* form of the St. Venant equations. The convex, Muskingum, and translation models all frequently result in surprisingly similar routing effects when storms are modeled according to storm classes. That is, if one categorizes all storms into a few classifications such as severe, mild, minor, (or others, if data permits), then the various routing methods can have their respective parameters defined on a storm class basis. For example, if translation routing is used then one could specify for each channel link a certain flow velocity or translation time which depends upon the storm class the subject storm event is in. In this way low flows would be routed with slower flow velocities than would be used in high flow cases. The use of storm classes provides a form *of nonlinear response in the rainfall-runoff model which appears to be* generally considered as important among many hydrologists. Therefore, the model of Eq. (21) can be argued to reasonably represent a variety of unsteady flow routing methods when an equivalence or storm classification is properly set up. This last result is useful because we can now examine rainfall-runoff model structures in nearly total generality.

We now consider an important extension of Eq. (21). Suppose a simple storm classification system is developed where the *effective rainfall* distribution measured at the rain gauge, $e_g^i(t)$, can be classified as being in one of three categories: (1) severe; (2) moderate; or (3) minor. Thus, if $e_g^i(t)$ is a class 1 storm, we would expect all channel links to be flowing close to capacity due to high runoffs throughout the catchment. All routing parameters are defined as class 1 parameters and

$$Q_m{}^i(t) = \int_{s=0}^{t} e_g{}^i(t\text{-}s) \sum_{j=1}^{m} \sum_{\langle\rangle_j} a^1{}_{\langle\rangle j} \sum_{k=1}^{n_j^i} \lambda_{jk}{}^i\, \phi_j{}^i(s - \theta_{jk}{}^i - \alpha^1{}_{\langle\rangle j})\, ds \tag{22}$$

where the subarea TF's are similarly defined as being class 1 types. (It is noted that the use of superscript 1 indicates values dependent upon storm class, and not i=1.)

Suppose in prediction, one is interested in the probable runoff at the stream gauge for a hypothetical storm event that is considered to be in storm class 1. Then the estimate of runoff is similar to the results of Eq. (22) except that now we have a distribution of outcomes, represented by the stochastic process

$$[Q_m{}^D(t)] = \int_{s=0}^{t} e_g{}^D(t\text{-}s) \sum_{j=1}^{m} \sum_{\langle\rangle_j} a_{\langle\rangle j} \sum_{k=1}^{n_j^i} [\lambda_{jk}{}^1]\, \phi_j{}^1(s - [\theta_{jk}{}^1] - \alpha^1{}_{\langle\rangle j})\, ds \tag{23}$$

where $[\lambda_{jk}{}^1]$ and $[\theta_{jk}{}^1]$ are distributions of mutually dependent random variables, which are now assumed to have a different probability distribution depending upon the storm class. The development of runoff equations for storm class 2 and 3 follow analogous to the development of Eq. (23), resulting in a set of stochastic prediction equations, dependent upon the storm class system.

Stochastic Integral Representation

The rainfall-runoff model of Eq. (23) can be written as a set of stochastic integral equations which provide a variation in prediction due to the storm class system,

$$[Q_1{}^D(t)]_q = \int_{s=0}^{t} e_g{}^D(t\text{-}s)\, [\eta(s)]_q\, ds \tag{24}$$

where $[\eta(s)]_q$ is the stochastic process of catchment TF's, associated to storm class q when $e_g{}^D(t)$ is in storm class q; and where $[\eta(s)]_q$ equates to the totality

$$[\eta(s)]_q = \sum_{j=1}^{m} \sum_{\langle\rangle_j} a^q{}_{\langle\rangle j} \sum_{k=1}^{n_j^q} [{}_{jk}{}^q]\, \phi_j{}^q\, (s - [\theta_{jk}{}^q] - \alpha^q{}_{\langle\rangle j}) \tag{25}$$

From Eq. (24), the effects of the uncertainty in the effective rainfall over R, and the randomness in the unsteady flow channel routing parameters, are all property integrated into the $[\eta(s)]_q$ realizations which reflect the combined distributions of all the considered hydrologic and hydraulic effects. This last result is important because almost all rainfall-runoff models in use today can be written in the form of Eqs. (24) and (25), but use a particular set of constants (or functions) given by the estimates $\hat{\phi}_j^q(s)$, $\hat{a}^q_{\langle A \rangle j}$, $\hat{\alpha}^q_{\langle A \rangle j}$, $\hat{\theta}_{jk}^q$, $\hat{\lambda}_{jk}^q$, and n_j^q, as defined on a storm class basis, q. The various rainfall-runoff modeling types all differ in their respective estimates of these various constants and functions, and hence the rainfall-runoff models differ in their respective deterministic estimates of runoff. All of the deterministic rainfall-runoff models neglect the randomness in $[\lambda_{jk}^q]$ and $[\theta_{jk}^q]$. Only when sufficient rainfall-runoff data and channel flow "link" data are available, can the various constants (and functions) used in Eq. (25) be accurately computed, and the distributions of $[\theta_{jk}]$ and $[\lambda_{jk}]$ be accurately determined. In the following section, the stochastic integral equation of Eq. (24) is used to sidestep the problem of insufficient data needed to determine the various components used in Eq. (25).

B. CORRELATION OF THE STOCHASTIC INTEGRAL EQUATION TO CATCHMENT CHARACTERISTICS

The previous sections develop a rainfall-runoff model structure which produces probabilistic estimates of runoff at a stream gauge given a single source of measured effective rainfall data. Almost all deterministic rainfall-runoff models in use today are seen to be special cases of Eq. (24) and (25), except that the various constants, functions, and distributions, are replaced with mean value estimates determined from particular empirical equations. The derived distribution of TF's from Eq. (24) and (25) are seen to represent the link-node model response, given all constants, functions, and distributions, are accurately defined. In this section we will apply our stochastic integral equations to actual rainfall-runoff data.

The $Q_1^i(t)$ model links the data pair $\{e_g^i(t), Q_g^i(t)\}$, for each storm i, for the given modeling assumptions. The sampling of all random processes is integrated into the time distribution of the realization, $\eta^i(s)$. Thus for each storm event i, there is an associated $\eta^i(s)$, just as there would be an associated set of samples $\{\lambda_{jk}^i, \theta_{jk}^i; j=1,2,\cdots,m; k=1,2,\cdots,100\}$ and other random variables.

To proceed with the analysis, it is assumed that there are "sufficient" effective rainfall data measured at the rain gauge site such as to develop equivalence classes of effective rainfall distributions. These storm classes are noted as $\langle\xi_q\rangle$ where q is the storm classification. Any two events in one class $\langle\xi_q\rangle$ would be considered to have identical associated parameters and random variable distributions such as used in Eqs. (24) and (25). It is assumed that there are sufficient effective rainfall data to develop such a set of storm classes $\langle\xi_q\rangle$ such that a reasonable statistical analysis can be made for each storm class individually.

Let $<\xi_o>$ be a class of storms. Let $e_o{}^i(t)$ be an element of $<\xi_o>$, for $i = 1,2,\cdots,n_o$, where n_o is the number of elements in $<\xi_o>$. To each $e_o{}^i(t)$ there is an associated $Q_o{}^i(t)$ measured at the stream gauge. Each pair of realizations $\{e_o{}^i(t), Q_o{}^i(t)\}$ can be linked by a stochastic integral equation, resulting in n_o realizations, $\{\eta_o{}^i(s); i = 1,2,\cdots,n_o\}$.

Each $\eta_o{}^i(s)$ can in turn be represented by a summation graph, $M_o{}^i(s)$, where

$$M_o{}^i(s) = \int_{\tau=0}^{s} \eta_o{}^i(\tau)\, d\tau \qquad (26)$$

Figure 2 shows a plot of $M_o{}^i(s)$ developed from storms considered to be of similar severity over a catchment in Los Angeles County, California. (In this case a lost function was used on measured rainfalls to estimate effective rainfalls, $e_g{}^i(t)$, for each storm. For our purposes, these $e_g{}^i(t)$ can be considered as "measured" for now.) By examining the plots, usually a normalization technique becomes apparent which aids in reducing the data into a form which is more easily analyzed by statistical techniques. In Fig. 2, plotting each $M_o{}^i(s)$ divided by its ultimate discharge, $U_o{}^i$, (i.e., $U_o{}^i = M_o{}^i(s = \infty)$), normalizes the vertical axis values to be between 0 and 100-percent. Defining lag^i to be the time that $M_o{}^i(s)$ reaches 50-percent of the ultimate discharge, $U_o{}^i$, normalizes the horizontal axis to be time in percent of lag. Figure 3 shows some resulting S-graphs, noted as $S_o{}^i(s)$ for storm class $<\xi_o>$. The several S-graphs can now be identified by characteristics parameters. A convenient parameter to use is the linear scaling Y between the enveloping curves of the S-graph data set. Usually, two of the S-graphs will bound the entire set (see Fig. 4). By identifying an average Y values to each S-graph, each S-graph is represented for storm i, by Y^i where

$$S_o{}^i(s) = Y^i S_o^A(s) + (1-Y^i)\, S_o^B(s) \qquad (27)$$

where $S_o^A(s)$ and $S_o^B(s)$ are the enveloping S-graphs; and Y^i is the scaling parameter with $0 \leq Y^i \leq 1$.

Based on the above normalization and parameterizations, each summation graph, $M_o{}^i(s)$, is identified by the parameter set $\mathbf{P}_o{}^i \equiv \{lag^i, U_o{}^i, Y^i\}$. Consequently, each realization, $\eta_o{}^i(s)$, is identified by the vector $\mathbf{P}_o{}^i$, for $i = 1,2,\cdots,n_o$.

The components of the parameter sets can be considered as random variables which are all mutually dependent. The marginal distributions are developed by plotting frequency-distributions of each component in the parameters set (see Fig. 5.).

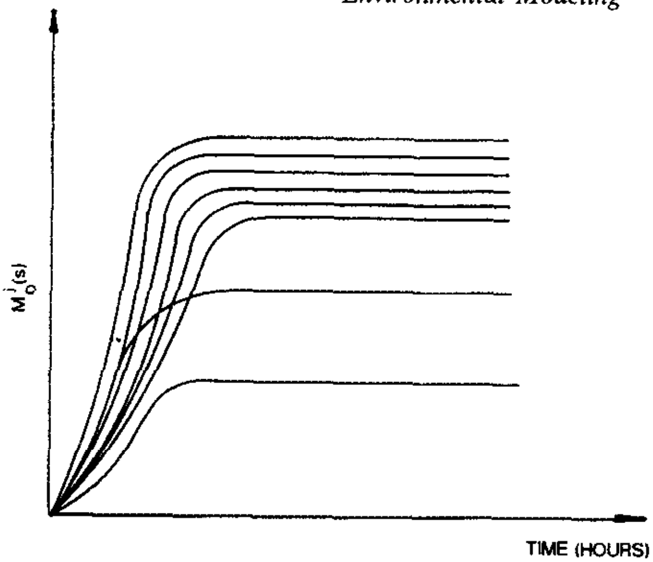

Figure 2. Example Summation Graphs, $M_o^j(s)$, for Storms in Class $<\xi_o>$.

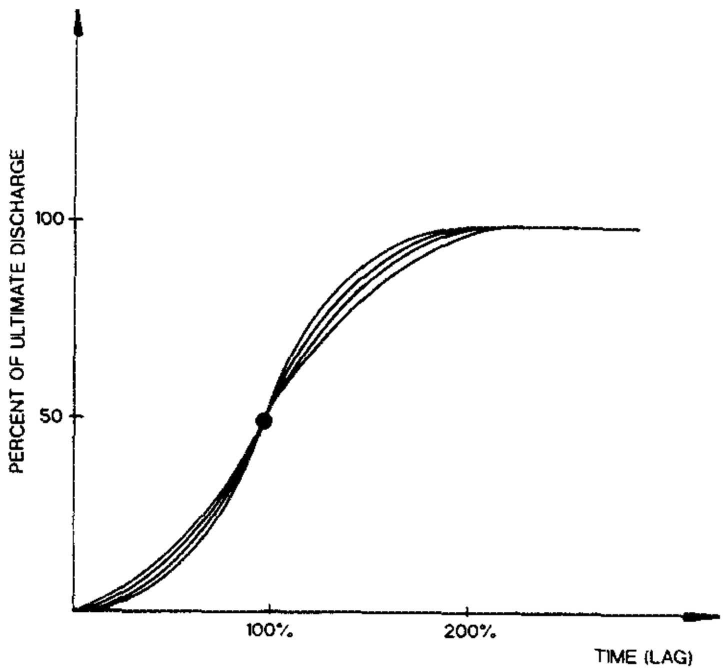

Figure 3. S-Graphs, $S_o^j(s)$, for Storm Class $<\xi_o>$.

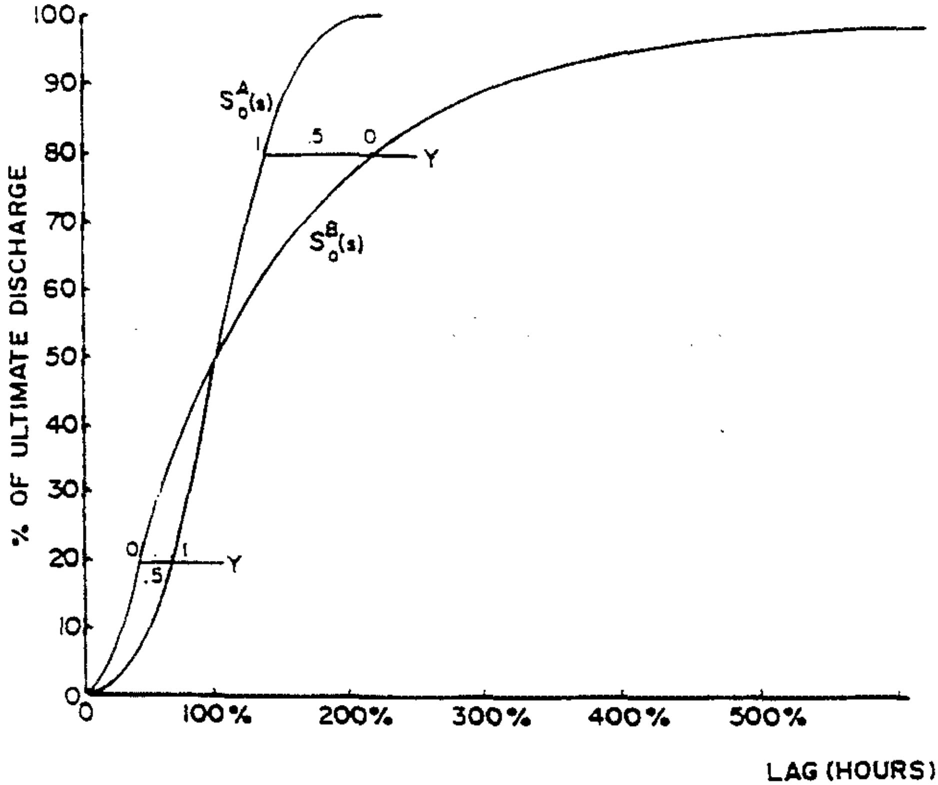

Figure 4 Definition of S-Graph Parameter Y Using S_0^A (s) and S_0^B (s)

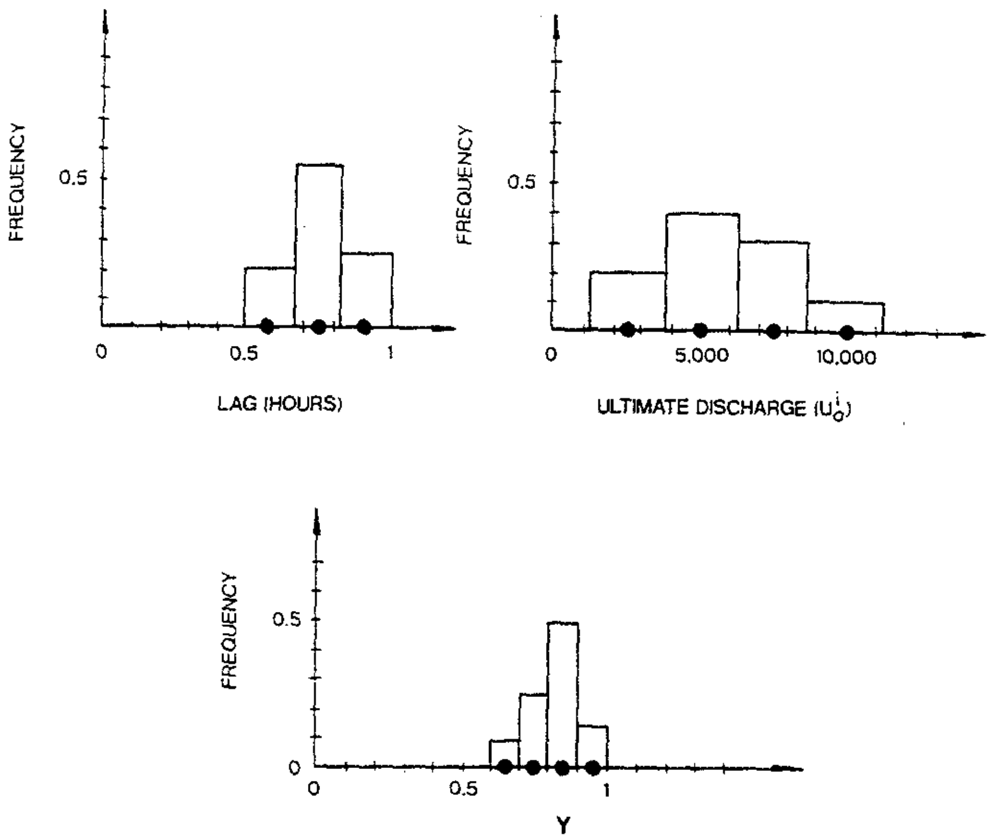

Figure 5. Marginal Distributions for Vector, P_o^i, Components.

Based on the marginal distributions, the frequency estimate associated to vector, P_o^i, is given by $Pr(\mathbf{P_o}^i)$ where

$$Pr(\mathbf{P_o}^i) = Pr(lag^i, U_o^i, Y^i) \tag{28}$$

Should more identifying characteristics be used to describe the $M_o^i(s)$, Eq. (27) is immediately extended. However, there should be sufficient storms in $<\xi_o>$ to develop a reliable frequency-distribution for each identifying characteristic. From the above, the distribution $[\eta(s)]_o$ of realizations, $\eta_o^i(s)$, is developed for storm class, $<\xi_o>$, using Eq. (28).

Each realization $\eta_q^i(s)$ associated to class $<\xi_q>$ represents the relationship between the effective rainfall data and the stream gauge data, using the stochastic integral equation, $Q_1^i(t)$.

Hydrologic Modeling Predictive Relationships

Given a design (i.e., future) storm effective rainfall distribution to be applied at the rain gauge site, $e_g^D(t)$, the hydrologic model is used to predict a runoff response from R, as could be measured at the stream gauge.

Let $e_g^D(t) \in <\xi_o>$.

Then the runoff response is the stochastic process, $[Q_1^D(t)]$, where

$$[Q_1^D(t)] = \int_{s=0}^{t} e_g^D(t-s)\,[\eta(s)]_o\,ds \tag{29}$$

where brackets indicate a random process. $[Q_1^D(t)]$ is the stochastic process of runoff hydrographs which are possible outcomes associated to the design storm effective rainfall, $e_g^D(t)$, measured at the rain gauge site. $[\eta(s)]_o$ is the stochastic process of realizations associated to storm class $<\xi_o>$ where $e_g^D(t)$ is considered to be sufficiently similar to the elements in $<\xi_o>$. Because Eq. (29) is a prediction, any of the realizations of $[\eta(s)]_o$, and hence $[Q_1^D(t)]$, are candidates as probable outcomes.

At the stream gauge, we are usually interested in the probable value of some criterion variable which would occur for the future storm event. For example the variation in flow rate estimates at storm time t_o is given from Eq. (29) by the random variable

$$[Q_1^D(t_o)] = \int_{s=0}^{t_o} e_g^D(t_o - s)\;[\eta(s)]_o\,ds \tag{30}$$

ɜtting t_p be the time of the peak flow rate (where t_p is a function of the random process, $[\eta(s)]_o$), the probability distribution of peak flow rate, $[q_p]$, is given by the random variable distribution

$$[q_p] = [Q_1{}^D(t_p)] = \int_{s=0}^{t_p} e_g{}^D(t_p - s)\ [\eta(s)]_o\, ds \tag{31}$$

Figure 6 shows the distributions of $[q_p]$ for a hypothetical storm event and an actual catchment. The frequency distribution of $[q_p]$ in Fig. 6 is determined by evaluating Eq. (31) using the marginal distributions shown in Fig. 5, according to the mutually dependent probability of occurrence given by Eq. (28). By scanning the entire set of $[\eta(s)]_o$ realizations developed for storm class $<\xi_o>$, the q_p frequency distribution is constructed, in histogram form.

C. <u>ESTIMATION UNCERTAINTY IN RAINFALL-RUNOFF MODELING ESTIMATES</u>

Due to the nondeterministic nature of the rainfall-runoff processes occurring over the catchment, the mathematical descriptions of these processes result in stochastic equations. Additionally, the so-called deterministic rainfall-runoff models used to describe the several physical processes contain parameters or coefficients which have well-defined physically-based meanings, but whose exact values are unknown. The boundary conditions of the problem itself are unknown (e.g., the temporal and spatial distribution of rainfall over the catchment for the storm event under study and also for all prior storm events) and often exhibit considerable variations with respect า the assumed boundary conditions and the measured rainfall at a single ,cation (e.g., Nash and Sutcliffe, 1970; Huff, 1970). Thus the physically-based parameters and coefficients, and also the problem boundary conditions, are not the assumed values but are instead random variables and stochastic processes whose process variations about the assumed values are governed by certain probability distributions.

In the following, the uncertainty problem is addressed by providing a methodology which can be incorporated into almost all rainfall-runoff models. The methodology is based upon the standard theory of stochastic integral equations which has been successfully applied to several problems in the life science and chemical engineering (e.g., Tsokos and Padgett, 1974, provide a thorough development). The stochastic integral formulation is used to represent the total error between a record of measured rainfall-runoff data and the model estimates, and provides an answer to the question: "based upon the historic rainfall-runoff data record and the model's accuracy in estimating the measured runoff, what is the distribution of probable values of the subject criterion variable given a hypothetical rainfall event?" Using the analysis results (Hromadka, 1989), we now extend our findings in order to generalize the analysis to arbitrary rainfall-runoff model structures.

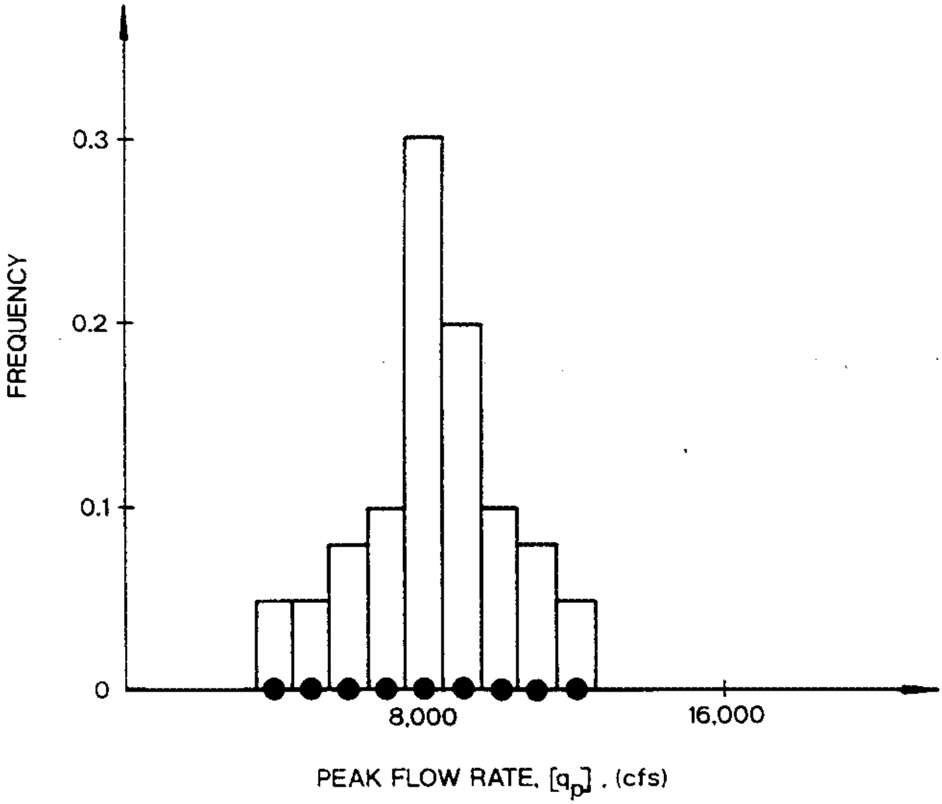

Figure 6. Frequency-Distribution for the Estimate of Peak Flow Rate $[q_p]$.

Almost all rainfall-runoff models in use today involve the subdivision of the catchment into smaller areas, linked together by a system of channel links. These "link-nodes" hydrologic models represent the flow processes within the channel links by a translation (moving in time) and an attenuation (reduction of the maximum or peak flow rate) of the inflow hydrograph. The runoff in each subarea is based upon the available rainfall data, modified according to an assumed "loss rate" due to soil-infiltration, ponding, evaporation, and other effects. The net effect of all these approximations is to result in a vast spectrum of possible modeling structures. Using the model structure presented in Hromadka, (1989), we can mathematically approximate may of these rainfall-runoff models with a single model structure, and can proceed to evaluate rainfall-runoff model uncertainty in overall generality. The approach used in this paper is to isolate the uncertainty in runoff predictions from the expected value of the model of the model runoff estimate, and then attempt to analyze the uncertainty as a separate form of information. In this way, the uncertainty may be analyzed as a stochastic process. Once the underlying distributions are identified, they can be normalized with respect to certain catchment characteristic variables, so that these distributions can be rescaled for application at arbitrary study sizes.

Rainfall-Runoff Model Errors

Let M be a deterministic rainfall-runoff model which transforms rainfall data for some storm event, i, noted by $P_g{}^i(t)$, into an estimate of runoff, $M^i(t)$,

$$M : P_g{}^i(t) \longrightarrow M^i(t) \tag{32}$$

where t is time. In our problem, rainfall data are obtained from a single rain gauge. The operator M may include loss rate and flow routing parameters, memory of prior storm event effects, and other factors. It is noted that precipitation data are now used in the current analysis rather than using a measured effective rainfall such as employed in Hromadka (1989).

Let $P_g{}^i(t)$ be the rainfall measured from storm event i, and $Q_g{}^i(t)$ be the runoff measured at the stream gauge. Various error (or uncertainty) terms are now defined such that for arbitrary storm event i,

$$Q_g{}^i(t) = M^i(t) + E_m{}^i(t) + E_d{}^i(t) + E_r{}^i(t) \tag{33}$$

where

$E_m{}^i(t)$ is the modeling error due to inaccurate approximations of the physical processes (spatially and temporally);

$E_d{}^i(t)$ is the error in data measurements of $P_g{}^i(t)$ and $Q_g{}^i(t)$ (which is assumed hereafter to be of negligible significance in the analysis;

$E_r^i(t)$ is the remaining "inexplainable" error, such as due to the unknown variation of effective rainfall (i.e., rainfall less losses; rainfall excess) over the catchment, among other factors.

Let $E^i(t)$ be redefined to equal the total error

$$E^i(t) = E_m^i(t) + E_d^i(t) + E_r^i(t) \tag{34}$$

where $E^i(t)$ is necessarily highly correlated to $E_r^i(t)$ due to the given assumptions. Because $E^i(t)$ depends on the model M used in Eq. (32), then Eqs. (33) and (34) are combined as

$$Q_g^i(t) = M^i(t) + E_M^i(t) \tag{35}$$

where $E_M^i(t)$ is a conditional notation for $E^i(t)$, given model type M.

The several terms in Eq. (35) are each a realization of a stochastic process. And for a future storm event D, the $E_M^D(t)$ is not known precisely, but rather is an unknown realization of a stochastic process distributed as $[E_M^D(t)]$ where

$$[Q_M^D(t)] = M^D(t) + [E_M^D(t)] \tag{36}$$

In Eq. (36), $[Q_M^D(t)]$ and $[E_M^D(t)]$ are the stochastic processes associated to the catchment runoff and total modeling error, respectively, associated with model M, for hypothetical storm event D. Hence in prediction, the model output of Eq. (36) is not a single outcome, but instead is a stochastic distribution of outcomes, distributed as $[Q_M^D(t)]$. Should A be some functional operator on the possible outcome (e.g., detention basin volume; peak flow rate; median flow velocity, etc.) of storm event D, then the possible value of A for event D, noted as A_M^D, is a random variable distributed as $[A_M^D]$, where

$$[A_M^D] = A\,[Q_M^D(t)] \tag{37}$$

Developing Distributions for Model Estimates

The distribution for $[E_M^D(t)]$ may be estimated by using the available sampling of realizations of the various stochastic processes:

$$\{E_M^i(t)\} = \{Q_g^i(t) - M^i(t)\},\ i = 1,2,\cdots \tag{38}$$

Assuming elements in $\{E_M^i(t)\}$ to be dependent upon the "severity" of $Q_g^i(t)$, one may partition $\{E_M^i(t)\}$ into classes of storms such as mild, major, flood, or others, should ample rainfall-runoff data be available to develop significant distributions for the resulting subclasses. To simplify development purposes, $[E_M^D(t)]$ will be based on the entire set $\{E_M^i(t)\}$ with the underlying assumption that all storms are of "equivalent" error; storm classes will be used later.

The second assumption involved is to assume each $E_M{}^i(t)$ is strongly correlated to some function of precipitation, $F^i(t) = F(P_g{}^i(t))$, where F is an operator which includes parameters, memory of prior rainfall, and other factors. Assuming that $E_M{}^i(t_o)$ depends only on the values of $F^i(t)$ for time $t \leq t_o$, then $E_M{}^i(t)$ is expressed as a causal linear filter (for only mild conditions imposed on $F^i(t)$), given by the stochastic integral equation (see Tsokos and Padgett, 1974)

$$E_M{}^i(t_o) = \int_{s=0}^{t_o} F^i(t_o - s)\, h_M{}^i(s)\, ds \qquad (39)$$

where $h_M{}^i(t)$ is the transfer function between $(E_M{}^i(t), F^i(t))$. Other convenient candidates to be used in Eq. (39), instead of $F^i(t)$, are the storm rainfall, $P_g{}^i(t)$, and the model estimates itself, $M^i(t)$.

Given a significant set of storm data, an underlying distribution, $[h_M(t)]$ of the $\{h_M{}^i(t)\}$ may be identified, or the $\{h_M{}^i(t)\}$ may be used directly as in the case of having a discrete distribution of equally-likely realizations. Using $[h_M(t)]$ as notation for both cases of distribution stated above, the predicted response from M for future storm event D is estimated to be

$$[Q_M{}^D(t)] = M^D(t) + [E_M{}^D(t)] \qquad (40)$$

Combining Eqs. (39) and (40),

$$[Q_M{}^D(t)] = M^D(t) + \int_{s=0}^{t} F^D(t\text{-}s)\, [h_M(s)]\, ds \qquad (41)$$

and for the functional operator, A , Eq. (37) is rewritten as

$$[A_M{}^D] = A\, [Q_M{}^D(t)] = A \left(M^D(t) + \int_{s=0}^{t} F^D(t\text{-}s)\, [h_M(s)]\, ds \right) \qquad (42)$$

Confidence interval estimates for the chosen criterion variable can now be obtained from the frequency-distribution, $[A_M{}^D]$. It is noted that $[A_M{}^D]$ is necessarily a random variable distribution that depends on the model structure, M.

Development of Total Error Distribution

A translation unsteady flow routing rainfall-runoff model

The previous concepts are now utilized to directly develop the total error distributions, $[E_M(t)]$, for a set of three idealized catchment responses. Besides providing a set of applications, additional notation and concepts are introduced, leading to the introduction of storm classes.

Let F be a functional which operates on rainfall data, $P_g{}^i(t)$, to produce the realization, $F^i(t)$, for storm i by

$$F : P_g{}^i(t) \longrightarrow F^i(t) \tag{43}$$

The catchment R is subdivided into m homogeneous subareas, $R = \cup R_j$, (see Fig. 7; where, m = 9), such that in each R_j, the effective rainfall, $e_j{}^i(t)$, is assumed given by

$$e_j{}^i(t) = \lambda_j(1 + X_j{}^i)\, F^i(t) \tag{44}$$

where λ_j is a constant proportion factor; and where $X_j{}^i$ is a sample of a random variable, which is constant for storm event i. The parameter λ_j is defined for subarea R_j and represents the relative runoff response of R_j in comparison to $F^i(t)$, and is a constant for all storms, whereas $X_j{}^i$ is a sample of the random variable distributed as $[X_j]$, where the set of distributions, $\{[X_j];\ j=1,2,\cdots,m\}$ may be mutually dependent.

The subarea runoff is

$$q_j{}^i(t) = \int_{s=0}^{t} e_j{}^i(t-s)\, \phi_j{}^i(s)\, ds = \int_{s=0}^{t} \lambda_j(1 + X_j{}^i)\, F^i(t-s)\, \phi_j{}^i(s)\, ds \tag{45}$$

At this stage of development, unsteady flow routing along channel links (see Fig. 7) is assumed to be pure translation. Thus, each channel link, L_k, has the constant translation time, T_k. Hence from Fig. 7, the total runoff response at the stream gauge for storm event i, $Q_g{}^i(t)$, is the sum of associated link travel times:

$$Q_g{}^i(t) = \sum_{j=1}^{9} q_j{}^i(t - \tau_j) \tag{46}$$

where $q_j{}^i(t - \tau_j)$ is defined to be zero for negative arguments; and τ_j is the sum of link travel times (e.g., from Fig. 7, $\tau_1 = T_1 + T_2 + T_3$; $\tau_6 = T_5 + T_6$; $\tau_9 = 0$).

For the above particular assumptions,

$$Q_g{}^i(t) = \sum_{j=1}^{9} \int_{s=0}^{t} \lambda_j(1 + X_j{}^i)\, F^i(t - s)\, \phi_j{}^i(s - \tau_j)\, ds$$

$$= \int_{s=0}^{t} F^i(t - s) \left(\sum_{j=1}^{9} \lambda_j(1 + X_j{}^i)\, \phi_j{}^i(s - \tau_j) \right) ds \qquad (47)$$

In the final form, the runoff response for the given simplification is

$$Q_g{}^i(t) = \int_{s=0}^{t} F^i(t - s) \sum_{j=1}^{9} \lambda_j\, \phi_j{}^i(s - \tau_j)\, ds$$

$$+ \int_{s=0}^{t} F^i(t - s) \sum_{j=1}^{9} \lambda_j\, X_j{}^i \phi_j{}^i(s - \tau_j)\, ds \qquad (48)$$

In the above equations, the samples $\{X_j{}^i\}$ are unknown to the modeler for any storm event i. From Eq. (48), the model structure, M, used in design practice is

$$M^i(t) = \int_{s=0}^{t} F^i(t - s) \sum_{j=1}^{9} \lambda_j\, \phi_j{}^i(s - \tau_j)\, ds \qquad (49)$$

Then, $Q_g{}^i(t) = M^i(t) + E_M{}^i(t)$ where

$$E_M{}^i(t) = \int_{s=0}^{t} F^i(t - s)\, h_M{}^i(s)\, ds \qquad (50)$$

where $h_M{}^i(s)$ follows directly from Eqs. (48) and (49).

Should the subarea UH all be assumed fixed, (i.e., $\phi_j{}^i(t) = \phi_j(t)$, for all i), as is assumed in practice, then the above equations can be further simplified as

$$M^i(t) = \int_{s=0}^{t} F^i(t - s)\, \Phi(s)\, ds \qquad (51)$$

where $\Phi(s) = \sum_{j=1}^{9} \lambda_j\, \phi_j(s - \tau_j)$. Additionally, the distribution of the stochastic process $[h_M(t)]$ is readily determined for this simple example,

$$[h_M(t)] = \sum_{j=1}^{9} [X_j]\ \lambda_j\, \phi_j(t - \tau_j) \qquad (52)$$

where $[h_M(t)]$ is directly equated to the 9 random variables, $\{X_j, j = 1,2,\cdots,9\}$. It is again noted that the random variables, X_j, may be all mutually dependent.

In prediction, the estimated runoff hydrograph is the distribution $[Q_M{}^D(t)]$ where $[Q_M{}^D(t)] = M^D(t) + [E_M{}^D(t)]$, and M refers to the model structure of Eqs. (49) or (51).

For the example problem, the stochastic integral formulation is

$$[Q_M{}^D(t)] = \int_{s=0}^{t} F^D(t-s)\ \Phi(s)\, ds + \int_{s=0}^{t} F^D(t-s)\, [h_M(s)]\, ds \qquad (53)$$

where the error distribution, $[E_M{}^D(t)]$, is assumed to be correlated to the model input, $F^D(t)$, as provided in Eqs. (50) and (52).

Multilinear unsteady flow routing and storm classes

The above application is now extended to include the additional assumption that the channel link travel times are strongly correlated to some set of characteristic descriptions of the runoff hydrograph being routed, such as some weighted mean flow rate of the associated hydrograph. For example, the widely used Convex Routing technique (Mockus, 1972) often utilized the 85-percentile of all flows in excess of one-half of the peak flow rate as a statistic used to estimate the routing parameters. But by the previous development (i.e., definition of $e_j{}^i(t)$), all runoff hydrographs in the link-node channel system would be highly correlated to an equivalent weighting of the model input, $F^i(t)$. Hence, storm classes $[\xi_z]$, of "equivalent" $F^i(t)$ realizations could be defined where all elements of $[\xi_z]$ have the same characteristic parameter set, $C(F^i(t))$, by

$$[\xi_z] = \{F^i(t) \mid C(F^i(t) = z\} \qquad (54)$$

And for all $F^i(t) \in [\xi_z]$, each respective channel link travel time is identical, that is $T_k = T_{k_z}$ for all $F^i(t) \in [\xi_z]$. In the above definition of storm class, z is a characteristic parameter set in vector form. (An example of such a characteristic parameter set is given in a subsequent section.)

This extension of the translation unsteady flow routing algorithm to a multilinear formulation (involving a set of link translation times) modifies the previous runoff equations (51) and (52) to be, in general

$$M^i(t) = \int_{s=0}^{t} F^i(t-s) \sum_j \lambda_j\, \phi_j(s-\tau_j{}^z)\, ds = \int_{s=0}^{t} F^i(t-s)\ \Phi_z(s)\, ds;\ F^i(t) \in [\xi_z] \qquad (55)$$

where $\Phi_z(s) = \sum_j \lambda_j \phi_j(s-\tau_j^z)$, and

$$E_M^i(t) = \int_{s=0}^{t} F^i(t-s)\, h_{M_z}{}^i(s)\, ds;\; F^i(t) \in [\xi_z] \qquad (56)$$

The structure of the new set of equations motivates an obvious extension of the definition of the subarea UH, the subarea λ_j proportion factor, and the subarea random variable distribution $[X_j]$, to all be also defined on the storm class basis of $[\xi_z]$. Thus, Eq. (55) is extended as

$$M^i(t) = \int_{s=0}^{t} F^i(t-s) \sum_j \lambda_j^z\, \phi_j^z(s-\tau_j^z)\, ds$$

$$= \int_{s=0}^{t} F^i(t-s)\, \Phi_z(s)\, ds;\; F^i(t) \in [\xi_z] \qquad (57)$$

The stochastic process $[h_{M_z}(t)]$ is distributed as

$$[h_{M_z}(t)] = \sum_j [X_j^z]\, \lambda_j^z\, \phi_j^z(s - \tau_j^z);\; F^i(t) \in [\xi_z] \qquad (58)$$

And in prediction,

$$[Q_M{}^D(t)] = M^D(t) + [E_M{}^D(t)];\; F^D(t) \in [\xi_D] \qquad (59)$$

where

$$[E_M{}^D(t)] = \int_{s=0}^{t} F^D(t-s)\, [h_M(s)]\, ds;\; F^D(t) \in [\xi_D] \qquad (60)$$

A Multilinear Rainfall-Runoff Model

The previous two model derivations resulted in the development of the total error distribution, $[E_M(t)]$, for some particular model structures. In this section, the above results are generalized to include a wide range of possibilities.

As before, Let F be a functional defined on the assumed rainfall data, $F : P_g{}^i(t) \longrightarrow F^i(t)$. The catchment R is subdivided into m subareas, $\{R_j;\; j = 1,2,\cdots,m\}$ linked together by unsteady flow routing models. The link-node model drains freely to the single stream gauge where the data, $Q_g{}^i(t)$, is measured. The problem is to predict the runoff response at the stream gauge corresponding to a hypothetical storm event rainfall, $P_g{}^D(t)$.

Each subarea's effective rainfall, $e_j^i(t)$, is now defined to be the sum of proportions of $F^i(t)$ translates by

$$e_j^i(t) = \sum_k \lambda_{jk}(1 + X_{jk}^i)\, F^i(t - \theta_{jk}^i);\ F^i(t) \in [\xi_z] \tag{61}$$

where X_{jk}^i and θ_{jk}^i are samples of the random variables distributed as $[X_{jk}^i]$ and $[\theta_{jk}^i]$, respectively. In the above equation and all equations that follow, it is assumed that a storm class system is defined, $[\xi_z]$, such that $F^i(t) \in [\xi_z]$, all parameters and probabilistic distributions are uniquely defined, and there is no loss in understanding by omitting the additional notation needed to indicate the storm class.

The subarea runoff is

$$q_j^i(t) = \int_{s=0}^{t} \sum_k \lambda_{jk}(1 + X_{jk}^i)\, F^i(t - \theta_{jk}^i - s);\ \phi_j(s)\, ds \tag{62}$$

or in a simpler form,

$$q_j^i(t) = \int_{s=0}^{t} F^i(t - s) \sum_k \lambda_{jk}\, (1 + X_{jk}^i)\, \phi_j(s - \theta_{jk}^i)\, ds \tag{63}$$

It is assumed that the unsteady flow channel routing effects are highly correlated to the magnitude of runoff in each channel link, which is additionally correlated to the magnitude of the model input realization, $F^i(t)$. On a storm class basis, each channel link is assumed to respond linearly in that (e.g., Doyle et al 1983)

$$O_1^i(t) = \sum_l a_l I_1^i(t - \alpha_l) \tag{64}$$

where $O_1^i(t)$ and $I_1^i(t)$ are the outflow and inflow hydrographs for link 1, and storm event i; and $\{a_l\}$ and $\{\alpha_l\}$ are constants which are defined on a storm class basis which is also used for the model input, $F^i(t)$. Thus, the channel link flow routing algorithm is multilinear with routing parameters defined according to the storm class, $[\xi_z]$ (see Becker and Kundezewicz, 1987, for an analogy based on multilinear approximation of nonlinear routing).

For L links, each with their own respective stream gauge routing data, the above linear routing techniques result in the outflow hydrograph for link number L, $O_L(t)$, being given by

$$O_L(t) = \sum_{\ell_L=1}^{n_L} a_{\ell_L} \sum_{\ell_{L-1}=1}^{n_{L-1}} a_{\ell_{L-1}} \cdots \sum_{\ell_2=1}^{n_2} a_{\ell_2} \sum_{\ell_1=1}^{n_1} a_{\ell_1} I_1(t - \alpha_{\ell_1} - \alpha_{\ell_2} - \cdots - \alpha_{\ell_{L-1}} - \alpha_{\ell_L}) \quad (65)$$

Using an index notation, the above $O_L(t)$ is written as

$$O_L(t) = \sum_{<\ell>} a_{<\ell>} I_1(t - \alpha_{<\ell>}) \quad (66)$$

For subarea R_j, the runoff hydrograph for storm i, $q_j^i(t)$, flows through L_j links before arriving at the stream gauge and contributing to the total modeled runoff hydrograph, $M^i(t)$. All of the parameters $a^i_{<\ell>}$ and $\alpha^i_{<\ell>}$ are constants on a storm class basis. Consequently from the linearity of the routing technique, the m-subarea link node model is given by the sum of the m, $q_j^i(t)$ contributions,

$$M^i(t) = \sum_{j=1}^{m} \sum_{<\ell>_j} a^i_{<\ell>_j} q_j^i(t - \alpha^i_{<\ell>_j}) \quad (67)$$

Finally, the predicted runoff response for storm event D is the stochastic integral formulation

$$[Q_M{}^D(t)] = \int_{s=0}^{t} F^D(t-s) \left(\sum_{j=1}^{m} \sum_{<\ell>_j} a^i_{<\ell>_j} \sum_k \lambda_{jk}(1 + [X_{jk}])\, \phi_j(s - [\theta_{jk}] - \alpha^i_{<\ell>_j}) \right) ds;\ F^D(t) \in [\xi_D]$$

Given $F^i(t) \in [\xi_z]$, all subarea runoff parameters $\{\lambda_{jk}, \phi_j(t)\}$ and distributions $\{[X_{jk}], [\theta_{jk}]\}$ are uniquely defined for $j = 1,2,\cdots,m$; and all link routing parameters $\{a_\ell, \alpha_\ell\}$ are also uniquely defined. Then the entire link-node model is linear on a storm class basis and once more Eqs . (57)-(60) apply without modification.

The above multilinear rainfall-runoff model structure represents a highly detailed and distributed parameter model of the rainfall-runoff process which not only can be used to represent the catchment runoff response itself, but also can be used to appropriate the response of other hydrologic modeling structures.

Consequently, our final model structure can be used to study the effect on the runoff prediction (at the stream gauge) from arbitrary model M, due to the randomness exhibited by the mutually dependent set of random variables, $\{X_{jk}, \theta_{jk}\}$. Hence for any operator, A, on the predicted runoff response of Eq. (68), the outcome of A for storm event $P_g{}^D(t)$ is the distribution $[A_M{}^D]$, where for all model parameters defined,

$$[A_M{}^D] = A\,[A_M{}^D(t)] = A\,(\{[X_{jk}], [\theta_{jk}]\}) \quad (69)$$

Stochastic Integral Equations and Uncertainty Estimates

The distributed parameter rainfall-runoff model of Eq. (68) provides a useful approximation of almost any rainfall-runoff model in use today. A stochastic integral equation that is equivalent to Eq. (68) is

$$[Q_M^D(t)] = \int_{s=0}^{t} F^D(t-s)\,[\eta(s)]\,ds;\ F^D(t) \in [\xi_D] \tag{70}$$

where now $[\eta(s)]$ is the distribution of the stochastic process representing the random variations from the set of mutually dependent random variables, $\{X_{jk}, \theta_{jk}\}$, defined on a storm class basis. (It is recalled that on a storm class basis, the hydraulic parameters of $a_{<b>j}$ and $\alpha_{<b>j}$, and the $\phi_j(s)$, do not vary.) In prediction, the expected runoff estimate for storm events that are elements of $[\xi_D]$ is

$$E[Q_M^D(t)] = \int_{s=0}^{t} F^D(t-s)\ E[\eta(s)]\,ds;\ F^D(t) \in [\xi_D] \tag{71}$$

which is a multilinear version of the well-known unit hydrograph method (e.g., Hromadka et al, 1987), which is perhaps the most widely used rainfall-runoff modeling approach in use today.

Then the model M structure of Eq. (68), when unbiased, is given from Eq. (71), by

$$M^D(t) = E[Q_M^D(t)] \tag{72}$$

The total error distribution (for the subject model M) can be developed by

$$[E_M^D(t)] = [Q_M^D(t)] - E[Q_M^D(t)] \tag{73}$$

where all equations are defined on the storm class basis used in the previous equations. Given sufficient rainfall-runoff data, the total error distribution can be approximately developed by use of Eq. (73). Should another rainfall-runoff model structure be used, then $E[Q_M^D(t)]$ is replaced by the alternative model, and another set of realizations of $[E_M^D(t)]$ is obtained from (73). Equation (73) is important in that given a specified model, the total error in model estimation is approximately given by a stochastic process. And similar to any sampling process, the modeling total error distribution becomes better defined as the sampling population increases. Through the equivalence between Eqs. (68) and (70), the uncertainty of the rainfall-runoff model of Eq. (68) can be evaluated by use of Eq. (70). That is, due to the limited data available, one cannot evaluate each of the random variables and processes utilized in Eq. (68), but one can evaluate the total model error, as developable from Eq. (73).

D. APPLICATION OF THE STOCHASTIC INTEGRAL EQUATION

In our application problem, the model input functional F : $P_g^i(t) \longrightarrow F^i(t)$ is specified as

$$F: P_g^i(t) \longrightarrow \lambda P_g^i(t) \tag{74}$$

where λ is a constant runoff coefficient. The corresponding stochastic integral equation used to related rainfall-runoff data is

$$Q_g^i(t) = \lambda \int_{s=0}^{t} P_g^i(t-s)\, \eta^i(s)\, ds \tag{75}$$

In this application, storm classes are defined (z) according to the 85-percentile value of rainfall intensity in excess of one half of the maximum 5-minute mean intensity, and also according to the total rainfall mass which occurs within 3 days prior to the subject storm event. Storm classes are then assembled according to the characteristic z-value, at 0.5-inch increments.

For the study location of Southern California, Table 1 summarizes the study catchment characteristics. Table 2 lists the available rain gauge sites and the storm dates of events used in the rainfall-runoff data analysis. Because of the sparcity of rainfall-runoff data, several catchments are considered in order to regionalize the statistical results. All storms considered in Table 2 are assumed to be elements of the same storm class considered important for flood control. That is, it is hypothesized that the variations in the various random variables and processes identified in Eq. (68), can be considered samples from distributions that apply for each of the considered storm events of Table 2.

For each storm event and catchment, the rainfall-runoff data is used to directly develop the $\{\eta^i(s)\}$ by use of Eq. (75). On a catchment basis, the several resulting $\eta^i(s)$ are pointwise averaged together to determine an estimate for $E[\eta(s)]$ for the prescribed storm class, for the considered catchment.

TABLE 1. WATERSHED CHARACTERISTICS

Watershed Name	Watershed Geometry						Storm Date	Calibration Results		
	Area (mi^2)	Length (mi)	Length of Centroid (mi)	Slope (ft/mi)	Percent Impervious (%)	Tc (hrs)		Peak F_p (in/hr)	Lag (hrs)	Basin Factor
Alhambra Wash[1]	13.67	8.62	4.17	82.4	45	0.89	Feb. 78 Mar. 78 Feb. 80	0.59, 0.24 0.35, 0.29 0.24	0.62	0.015
Compton2[1]	24.66	12.69	6.63	13.8	55	2.22	Feb. 78 Mar. 78 Feb. 80	0.36 0.29 0.44	0.94	0.015
Verdugo Wash[1]	26.8	10.98	5.49	316.9	20	—	Feb. 78	0.65	0.64	0.016
Limeklin[1]	10.3	7.77	3.41	295.7	25	—	Feb. 78 Feb. 80	0.27 0.27	0.73	0.026
San Jose[2]	83.4	23.00	8.5	60.0	18		Feb. 78 Feb. 80	0.20 0.39	1.66	0.20
Sepulveda[3]	152.0	19.0	9.0	143.0	24	—	Feb. 78 Mar. 78 Feb. 80	0.22, 0.21 0.32 0.42	1.12	0.017
Eaton Wash[1]	11.02[4] (57%)	8.14	3.41	90.9	40	1.05	—	—	—	0.015[7]
Rubio Wash[1]	12.20[5] (3%)	9.47	5.11	125.7	40	0.68	—	—	—	0.015[7]
Arcadia Wash[1]	7.70[6] 14%	5.87	3.03	156.7	45	0.60	—	—	—	0.015[8]
Compton1[1]	15.08	9.47	3.79	14.3	55	1.92	—	—	—	0.015[8]
Dominquez[1]	37.30	11.36	4.92	7.9	60	2.08	—	—	—	0.015[8]
Santa Ana Delhi[3]	17.6	8.71	4.17	16.0	.40	1.73	—	—	—	0.053[9] 0.040[10]
Westminster[3]	6.7	5.65	1.39	13	40		—	—	—	0.079[9] 0.040[10]
El Modena-Irvine[3]	11.9	6.34	2.69	52	40	0.78	—	—	—	0.028[9]
Garden Grove-Wintersburg[1]	20.8	11.74	4.73	10.6	64	1.98	—	—	—	—
San Diego Creek[1]	36.8	14.2	8.52	95.0	20	1.39	—	—	—	—

Notes
1: Watershed Geometry based on review of quadrangle maps and LACFCD storm drain maps.
2: Watershed Geometry based on COE LACDA Study.
3: Watershed Geometry based on COE Reconstitution Study for Santa Ana Delhi and Westminster Channels (June, 1983).
4: Area reduced 57% due to several debris basins and Eaton Wash Dam reservoir, and groundwater recharge ponds.
5: Area reduced 3% due to debris basin.
6: Area reduced 14% due to several debris basins.
7: 0.013 basin factor reported by COE (subarea characteristics, June, 1984).
8: 0.015 basin factor assumed due to similar watershed values of 0.015.
9: *Average basin factor computed from reconstitution studies.*
10: COE recommended basin factor for flood flows.
11: COE = U.S. Army Corps of Engineers
12: LACDA = Los Angeles County Drainage Area Study by COE.
13: LACFCD = Los Angeles County Flood Control District.

TABLE 2. PRECIPITATION GAUGES USED IN LOS ANGELES COUNTY FLOOD RECONSTITUTIONS

Stream Gauge Location	Storm Reconstitution	LACFCD Rain Gauge No.
Alhanmbra Wash near Klingerman Street	Feb 78	191, 303, 1114B
	Mar 78	191, 303, 1114
	Feb 80	191B, 235, 280C, 1014
Compton Creek near Greenleaf Drive	Feb 78	116, 291
	Mar 78	116, 291
	Feb 80	116, 291, 716
Limekiln Creek above Aliso Creek	Feb 78	57A, 446
	Feb 80	259, 446
San Jose Creek Channel above Workman Mill Road	Feb 78	92, 1078, 1088X
	Feb 80	96CE, 347E, 1088
Sepulveda Dam (inflow)	Feb 78	57A, 292DE, 446, 735H
	Mar 78	57A, 435, 762
	Feb 80	292, 446, 735
Verdugo Wash at Estelle Avenue	Feb 78	280C, 373C, 498, 758

No.	Station Name	Lat.	Long.	Elev.	Type
L057A	Camp Hi Hill (OPIDS)	34-15-18	118-05-41	4240	SR
L0092	Claremont-Pomona College	34-05-48	117-42-33	1185	SR
L0096CE	Puddingstone Dam	34-05-31	117-48-24	1030	SR
L0116	Inglewood Fire Station	33-47-53	118-21-22	153	SR
L0191(B)	Los Angeles-Alcazar	34-03-46	118-11-54	400	SR
L0235	Henninger Flats	43-11-38	118-05-17	2550	SR
L0259	Chatworth-Twin Lakes	34-16-43	118-35-41	1275	SR
L0280C	Sacred Heart Academy	34-10-54	118-11-08	1600	R
L0291	Los Angeles-96th & Central	33-56-56	118-15-17	121	R
L0292(DE)	Encino Reservoir	34-08-56	118-30-57	1075	SR
L0303	Pasadena-Cal Tech	34-08-14	118-07-25	800	SR
L0347E	Baldwin Park-Exp. Station	34-05-56	118-57-40	384	SR
L0373C	Briggs Terrace	34-14-17	118-13-27	2200	SR
L0435	Monte Nido	34-04-41	118-41-35	600	SR
L0446	Aliso Canyon-Oat Canyon	34-18-53	118-33-25	2367	SR
L0498	Angeles Coast Hwy-Drk Cny Tr	34-15-21	118-11-45	2800	R
L0716	Los Angeles-Ducommun Street	34-03-09	118-14-13	306	SR
L0735(H)	Bell Canyon	34-11-40	118-39-23	895	R
L0758	Griffith Park-Lower Spr Cyn	34-08-02	118-17-27	600	R
L0762	Upperstone Canyon	34-07-27	118-27-15	943	R
L1014	Rio Hondo Spreading	33-59-57	118-06-04	170	SR
L1078	Covina-Griffith	34-04-10	117-50-47	975	SR
L088(X)	LaHabra Hts-Mut Water Co	33-56-55	117-57-51	445	SR
L1114(B)	Whittier Narrows Dam	34-01-29	118-05-02	239	SR

S = Standard 8" raingauge (non-recording).
R = Recording raingauge.

Summation (or distribution) graphs of the $\{\eta^i(s)\}$ indicate that normalizing could be performed by plotting mass along the y-axis from 0 to 100-percent of mass, and the x-axis as time with respect to the parameter "lag" where 100 percent of the lag equals the time at 50 percent of total mass. Plots of normalized summation graphs of the $\eta^i(s)$ realizations for Alhambra Wash, for several storms, are shown in Fig. 7, and plots of summation graphs of the estimates of $E[\eta(s)]$ for the several catchments are shown in Fig. 8. From the data used in Fig. 7, the expected value (for the Alhambra Wash stream gauge) of the characteristic parameters lag and ultimate discharge, U, are obtained.

A comparison between Figs. 7 and 8 shows that the variation in the summation graphs of the $E[\eta(s)]$ among the several considered catchments is of a magnitude similar to the variation between the summation graphs of $\eta^i(s)$ for Alhambra Wash alone. Therefore in order to regionalize the total error distributions, and to increase the population of the random process sampling, the variations among all the catchment, $\eta^i(s)$ summation graph realizations are normalized and assembled together to form one regionalized distribution of summation graph realizations.

To describe the data, a "shape" scaling parameter, Y, is introduced by plotting each summation graph realization on Fig. 9 and averaging the upper and lower reading for Y. The regionalized marginal distribution for the parameter Y is shown in Fig. 10. With the normalization process, the variations in the timing parameter, lag^i, and the summation graph total mass (i.e., ultimate discharge, U^i), must be also accounted, and were assumed to be distributed according to the sampled frequency-distributions. From these descriptor variables, each $\eta^i(s)$ is represented, in summation graph form, by the parameter values of $\{lag^i, U^i, Y^i,\}$.

Based upon the model M defined by Eqs. (72)-(75), a severe storm, of March 1, 1983 (which was not used in the development of $[\eta(s)]$ is analyzed for the Alhambra Wash stream gauge. The outcomes of $[Q_M{}^D(t)]$ are plotted along with the recorded stream gauge data in Fig. 11. From the figure, the uncertainty in the model prediction of $[Q_M{}^D(t)]$ is significant, and should be included when analyzing an operator *A* on the runoff predictions.

If the underlying distribution between the variables in {Y, lag, mass} could be ascertained, the distribution of $[\eta(s)]$ could be identified (associated to F). One simple approach is to assume each of the variables to be independent, and to generate $[Q_M{}^D(t)]$ by probabilistic modeling.

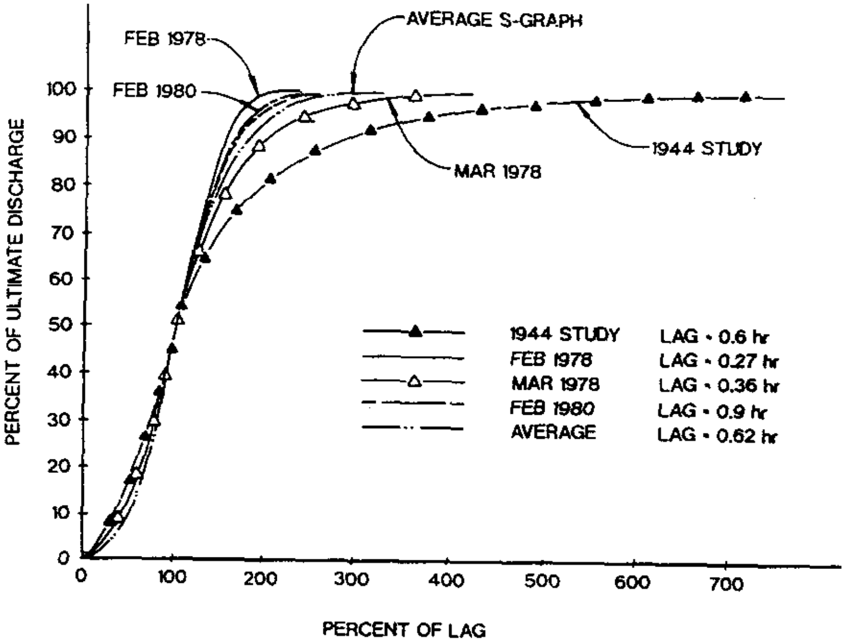

Figure 7 Distribution of Normalized Summation Graphs for Alhambra Wash.

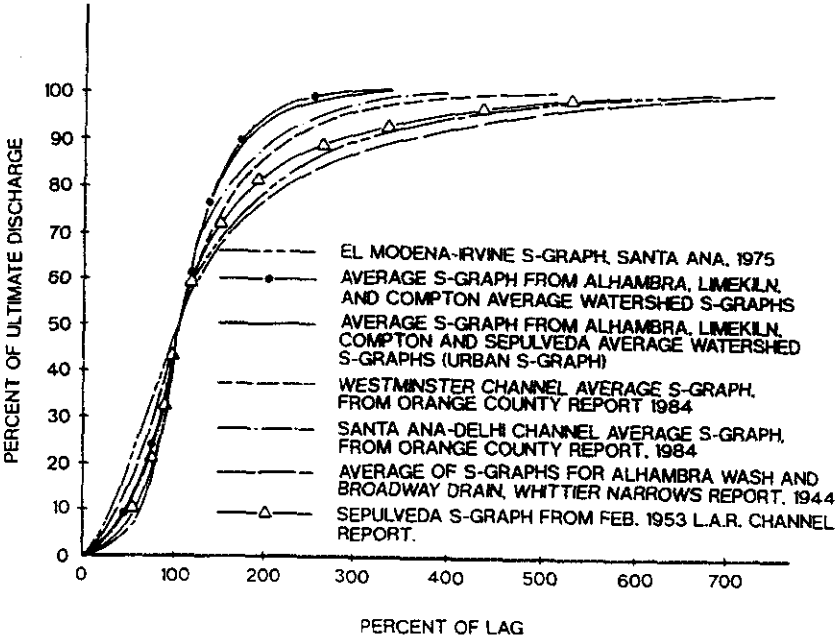

Figure 8. Distribution of Expected Normalized Summation Graphs for Several Catchments.

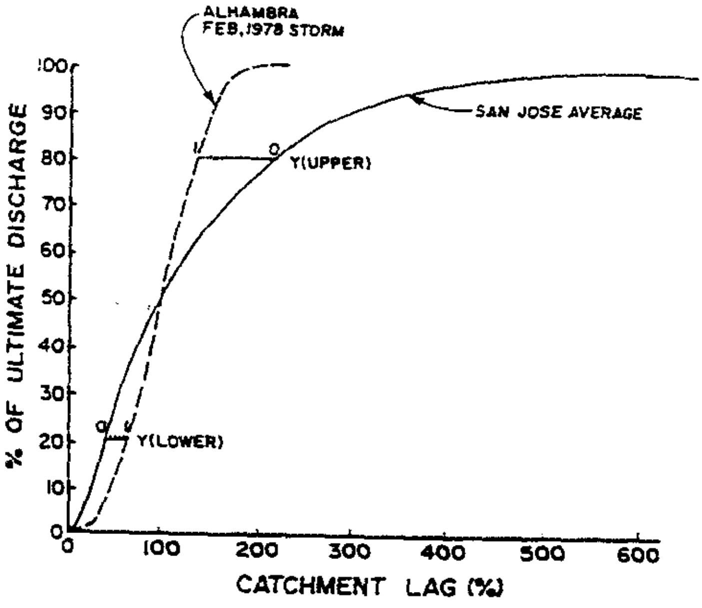

Figure 9. Normalized Summation Graph Shape Factor, Y, Definition.

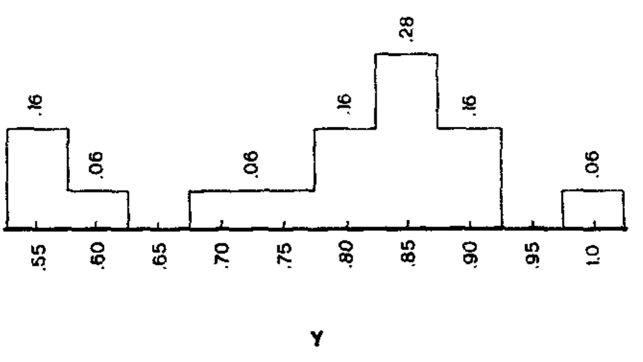

Figure 10. Distribution of Y, Regionalized from Data from Several Catchments.

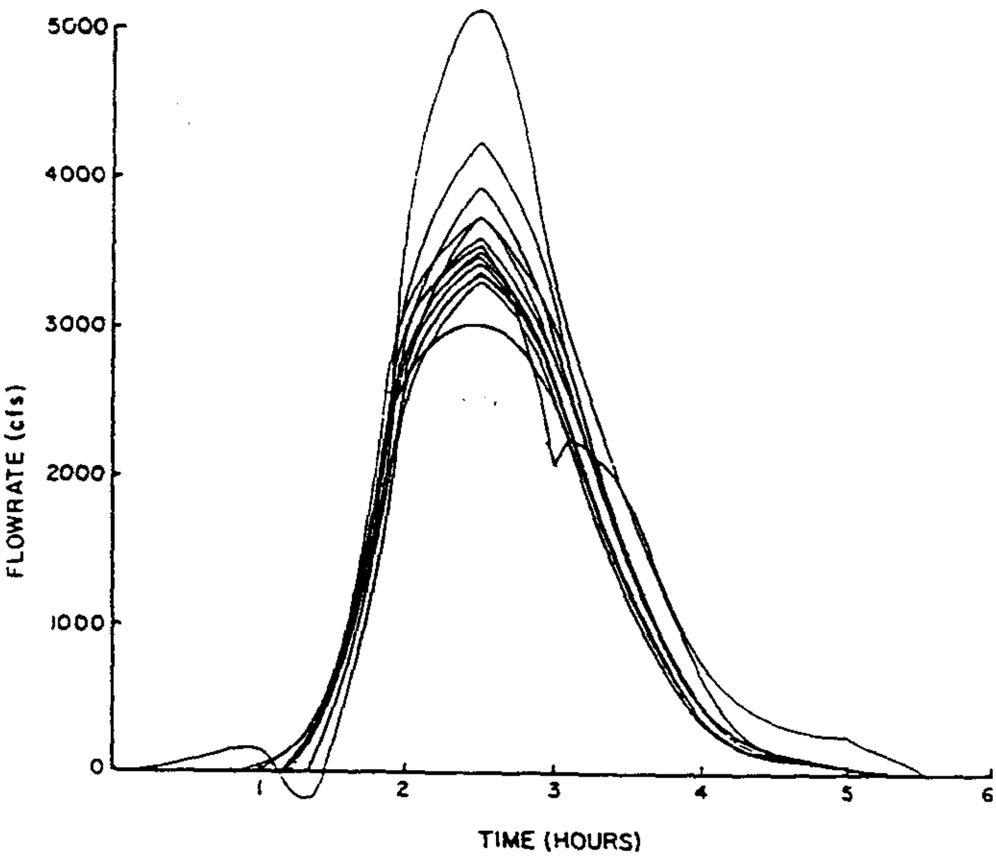

Figure 11. Probable Runoff Hydrographs, using Model M1, for a Hypothetical Storm Event.

Discussion and Conclusions

In the use of the above rainfall-runoff model, $M^D(t)$ is given by

$$M^D(t) = E[Q_M{}^D(t)] = \lambda \int_{s=0}^{t} P_g{}^D(t-s)\, E[\eta(s)]\, ds \tag{76}$$

The model uncertainty is then evaluated from Eq. (73) by

$$E_M{}^D(t) = \lambda \int_{s=0}^{t} (P_g{}^D(t-s))\, [\eta(s)] - E[\eta(s)])\, ds \tag{77}$$

where from Eq. (60),

$$[h_M(s)] = [\eta(s)] - E[\eta(s)] \tag{78}$$

Hence it is seen that the distribution of $[\eta\ (s)]$ includes the effects of both the rainfall-runoff model itself and the associated uncertainty, where from Eqs. (68) and (70),

$$E[\eta\ (s)] = \sum_{j=1}^{m} \sum_{\langle \Delta \rangle_j} a^i{}_{\langle \Delta \rangle_j} \sum_{k} \lambda_{jk}\, (1+E\ [X_{jk}])\, \phi_j(s-E)[\theta_{jk}] - \alpha^i{}_{\langle \Delta \rangle_j})$$

$$\text{for } F^D(t)\ \epsilon\ [\xi_D] \tag{79}$$

The various stochastic distributions utilized are estimated from regional rainfall-runoff data and the chosen model structure. Because runoff data are available for the precise catchment point under study (i.e., we have a stream gauge), the various distributions represented by Figs. 7 through 11 can be rescaled to correspond to the selected study point (because from the stream gauge data being studied, we can estimate the expected value for lag and ultimate discharge). However, in order to utilize these distributions at ungauged points in the catchment, or at other catchments where there are no runoff data, a method of transferring these distributions is needed. That is, a method is needed for estimating the expected values for lag and ultimate discharge (or other description variables used) for the point under study. Given these estimates, the various distributions can be rescaled, and a distribution $[\eta(s)]$ can be estimated from the rainfall-runoff data pool.

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