# DERIVATION OF THE RATIONAL METHOD FOR PEAK FLOW ESTIMATES

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Abstract

Estimating peak flow rates for design storm runoff is a central problem in flood control design and planning. For small catchments (e.g., less than one square mile), two methods are commonly used to calculate the peak flow rate. The popular rational method has been used successfully for years for small catchments. The balanced design storm unit hydrograph method is more computationally intensive, but has become more practical due to the widespread availability of powerful computers. the balanced design storm unit hydrograph method is the preferred method because it is thought to yield more accurate results. A recent paper[1] established that the two methods yield comparable estimates of the peak flow rate. This paper will show that the two methods are, under reasonable restrictions, mathematically equivalent.

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1 Hromadka and Whitley, *The Rational Method for Peak Flow Rate Estimation.*

**Background**

Suppose a region of land exists whose topography is such that any rain that falls on any part of the region flows through a certain point when it runs off the region. Such a region is called a *catchment* and the point through which the rain flows is called the *point of concentration.* (The area of the catchment appears in several equations and is denoted by A.) The flow through the point of concentration is called the *runoff.* The peak runoff flow rate is of great interest when designing flood control systems.

The runoff is equal to the *effective rainfall* (denoted by e(t), where t is time), which is the actual rainfall less any losses. These losses are due to the fact that some of the rainfall is absorbed into the soil and some adheres to the vegetation in the region, as well as several other factors. There are two common methods of deriving the effective rainfall from the actual rainfall. This loss is denoted by j. The second method assumes the effective rainfall is equal to some constant (denoted by k, k<1) times the actual rainfall.

If precipitation is continuous and constant, eventually the runoff will reach a maximum and remain at that level until the precipitation ceases. This maximum is generally reached when all areas of the catchment are contributing to the runoff. The earliest time at which this occurs is called the *time of concentration* and is denoted by $T_C$.

If one linear unit of rain falls during one unit time period and no rain falls after that one unit time period, one could draw a graph showing the runoff, expressed as a proportion of the total runoff, on the vertical axis and time, expressed in unit time periods, on the horizontal axis. Such a graph is called an *S-graph*, since it is shaped like an S. The time at which half the runoff has drained off the catchment is called the *lag time* and is denoted by $\ell$. If we scale the horizontal axis to be units of the lag time, the resulting graph is called a *normalized S-graph*.

If we take the derivative of the normalized S-graph with respect to time, the result is called a *unit hydrograph* (UH). (The unit hydrograph is usually quantized over some small period of time, such as five minutes.) Thus the unit hydrograph represents the flow through the point of concentration against time for a single unit of rainfall. The flow is zero when the rain begins falling, rises to a maximum at some point, and then falls back to zero as all the runoff flows out of the catchment. The time at which the maximum occurs is called the *time to peak* and is denoted by $T_p$.

Suppose we were able to collect data on rainfall depth, in particular, the maximum amount of precipitation that fell during a specified interval over the course of a year. For example, if the interval was five minutes, we might know a maximum of 0.12 inches of rain fell in a five minute interval during 1988, a maximum of 0.09 inches fell in a five minute interval during 1989, and so forth. Further suppose we could collect this data for the entire history of the planet. We could then determine the median of the collected data points. Of course, since half the data points are below the median and half are above it, then the maximum amount of precipitation would be below the median half the time and above it the other half of the time (at least statistically). This level is called the *mean precipitation* *depth*. The specified interval (in this case, five minutes) is called the *storm duration*. The number of years that must pass before the level will be above the mean precipitation is called the *return frequency*. In this case the return frequency is two years, since we chose the median. The storm duration is denoted by $\tau$ and the return frequency is denoted by T. The mean precipitation depth is denoted by $D(\tau)$.

We could also determine a rainfall depth such that four fifths of the data points were below that depth and one fifth of the data points were above it. That level would be the mean precipitation depth for a storm duration of five minutes and a return frequency of five years. The mean precipitation depths can be calculated for any storm duration and any storm return frequency. These numbers are typically determined for storm return frequencies of two, five, ten, twenty-five, fifty and one hundred years.

The unit hydrograph is used to model the runoff from a storm. To do this, a storm must be input to the model. Such a storm is usually specified in terms of the amount of precipitation over each of a series of time intervals. For example, we might specify that 0.05 inches falls during the first five minute interval, 0.07 inches the second five minute interval, and so forth. A storm so specified is called a *design storm*.

Design storms start with a small amount of rainfall, increase to a peak, and then diminish to zero. A storm designed such that a specified proportion of the total rainfall occurs before the peak is called a *balanced design storm*. The proportion is denoted by $\theta$.

A design storm is created by first choosing a return frequency, say twenty years, a time interval, say five minutes, and a $\theta$, say 0.5. The design storm peak is chosen to be the mean precipitation depth for a storm duration of five minutes and a return frequency of twenty years. The next highest storm intensity is chosen to be the mean precipitation depth for a storm duration of ten minutes minus the mean precipitation depth for a storm duration of five minutes. The third highest storm intensity is chosen to be the mean precipitation depth for a storm duration of fifteen minutes minus the mean precipitation depth for a storm duration of ten minutes. This process is continued until a suitable number of peaks have been chosen.

We then start with the peak. Since our $\theta$ is 0.5, the highest peak will be in the center of the design storm. The second highest peak will occur just before the highest peak. The third highest peak will occur just after the highest peak. The fourth highest peak will occur just before the second, the fifth will occur just after the third, the sixth will occur just before the fourth, the seventh will occur just after the fifth, and so forth. After all the peaks have been placed in their proper places, the balanced design storm is complete.

A graph of the runoff through the point of concentration is created as follows. The unit hydrograph is multiplied by the amount of rainfall during the first time interval (recall that the unit hydrograph gives the runoff per unit rainfall). This gives the amount of runoff due to the rainfall during the first time interval. The unit hydrograph is then multiplied by the amount of rainfall during the second time interval and shifted one time interval to the right. This gives the amount of runoff due to the rainfall during the second time interval. (This graph is zero during the first time interval, since obviously no runoff due to the rainfall during the second interval occurs during the first interval.) This process of scaling and shifting continues for the entire design storm. The resulting hydrographs (they are no longer unit hydrographs) are added together to get the hydrograph showing the total runoff due to the design storm. The process of scaling and shifting is called *convolution*.

## Unit Hydrograph

Unit hydrographs (UHs) for a catchment may be derived from normalized S-graphs. The S-graph, which is developed from regional rainfall-runoff data, is typically expressed by $S(\ell)$, where $\ell$ is a precentage of the catchment lag. Catchment lag is related to the usual time of concentration, $T_c$, by

$$lag = \gamma T_c \tag{1}$$

In several flood control districts in California, $\gamma = 0.80$. We can make the S graph a function of $T_c$ by making the substitution $S(\ell) = S((100t)/(\gamma\ T_c))$. The UH is obtained from the derivative of S(t) with respect to time, t, thus the UH also becomes a function of $T_c$. For $T_c = 1$ and catchment area A = 1, a normalized UH results. For $T_c \neq 1$ or $A \neq 1$, the catchment UH, $u(t,T_c,A)$ is related to the normalized UH, U(t), by

$$u(t,T_c,A) = \frac{A}{T_c} U\left(\frac{t}{T_c}\right) \tag{2}$$

where

$$\int_0^\infty u(t,T_c,A)dt = A\int_0^\infty U\left(\frac{t}{T_c}\right)\frac{dt}{T_c} = AU_0 . \tag{3}$$

Note that $U_0$ is a constant. Hereafter, the catchment UH, $u(t,T_c,A)$, will simply be written as u(t) where no confusion occurs.

## Rainfall Depth-Duration Description

Precipitation depth-duration relationships, for a given return frequency T, are generally given by the power law analog,

$$D(\tau) = a\tau^b , \tag{4}$$

where a > 0 is a function of return frequency, and is held constant for a selected design storm return frequency; b is typically a constant for large regions (e.g., entire counties); $D(\tau)$ is the rainfall depth; and $\tau$ is the selected duration of time.

Mean rainfall intensity, $I(\tau)$, is

$$I(\tau) = \frac{1}{\tau} D(\tau) = a\tau^{b-1} \tag{5}$$

and instantaneous rainfall intensity, $i(\tau)$, is

$$i(\tau) = \frac{d}{d\tau} D(\tau) = ab\tau^{b-1} = bI(\tau) . \tag{6}$$

With respect to HEC TD-15 (1984), a balanced design storm pattern (of nested uniform return frequency rainfall depths) can be described by the time coordinates $\tau^{\pm}$ shown in Figure 1. For a proportioning of rainfall quantities by allocation of a $\theta$ proportion prior to time $\tau^{\pm} = 0$ (see Figure 1), instantaneous rainfall intensities are given by

$$i^-\left(\tau^-\right) = i^-(\theta\tau) = i(\tau) \tag{7}$$

or

$$i^-(\tau^-) = i\left(\frac{\tau^-}{\theta}\right) = \left(\frac{1}{\theta}\right)^{b-1} i(\tau^-). \tag{8}$$

Similarly,

$$i^+(\tau^+) = \left(\frac{1}{1-\theta}\right)^{b-1} i(\tau^+). \tag{9}$$

For example, the HEC TD-15 balanced design storm is given by θ = 0.50; in several California flood control districts, θ = 2/3 describes the balanced design storms.

## Peak Flow Rate Estimates from the Balanced Design Storm Unit Hydrograph Procedure

Let $v(t) = u(\eta T_c - t)$, i.e, *v(t)* is a time-reversed plot of the UH, *u(t)*. From Figure 1, and aligning the UH peak to occur at time $t^+ = 0$,

$$v^+(t^+) = u(T_p - t^+) \tag{10a}$$

$$v^-(t^-) = u(T_p + t^-) \tag{10b}$$

Then the balanced design storm UH procedure estimates the peak flow rate, $Q_p$, by

$$\begin{aligned} Q_p &= \int_{t^+=0}^{T_p} e^+(t^+)v^+(t^+)dt^+ + \int_{t^-=0}^{\eta T_c - T_p} e^-(t^-)v^-(t^-)dt^- \\ &= \int_0^{T_p} i^+(t^+)v^+(t^+)dt^+ + \int_0^{\eta T_c - T_p} i^-(t^-)v^-(t^-)dt^- - \phi\left[\int_0^{T_p} v^+(t^+)dt^+ + \int_0^{\eta T_c - T_p} v^-(t^-)dt^-\right] \end{aligned} \tag{11a,b}$$

where $\eta T_c$ is the total duration of the UH and $T_p$ is the time to peak of the UH.

In (11b) a "phi index" (or constant) loss function has been used to compute rainfall excess; Equation (12) shows the relationship between the intensity i(t) and the excess e(t). A necessary constraint imposed is that $i(\eta T_c) \geq \phi$.

$$e(t) = i(t) - \phi \tag{12}$$

The last term of Equation (11b) is solved by

$$\phi\left[\int_0^{T_p} v^+(t^+)dt^+ + \int_0^{\eta T_c - T_p} v^-(t^-)dt^-\right] = \phi\left[\int_0^{\infty} v(t)dt\right] = \phi A U_0 \tag{13}$$

The next step in the mathematical development is to replace the time-based coordinate system with a dimensionless system based on $T_c$. This is done by introducing a variable s defined by

$$s = \frac{t}{T_c} \tag{14}$$

Then $t = T_c s$ and $dt = T_c ds$.

The balanced design storm instantaneous rainfall intensities, $i^{\pm}(t^{\pm})$, can now be rewritten in terms of $s^{\pm}$ (analagous to $t^{\pm}$) by

$$i^{+}\left(t^{+}\right)=\left(\frac{1}{1-\theta}\right)^{b-1} ab\left(s^{+}T_c\right)^{b-1}=\left(\frac{T_c}{1-\theta}\right)^{b-1} i\left(s^{+}\right) \tag{15a}$$

$$i^{-}\left(t^{-}\right)=\left(\frac{T_c}{\theta}\right)^{b-1} i\left(s^{-}\right) \tag{15b}$$

Similarly, the $v^{\pm}\left(t^{\pm}\right)$ functions can be rewritten in terms of coordinates $s^{\pm}$ by

$$v^{+}\left(t^{+}\right)=u\left(T_p-t^{+}\right)=\frac{A}{T_c}U\left(\frac{T_p-t^{+}}{T_c}\right)=\frac{A}{T_c}U\left(t_p-s^{+}\right) \tag{16a}$$

$$v^{-}\left(t^{-}\right)=u\left(T_p+t^{-}\right)=\frac{A}{T_c}U\left(\frac{T_p+t^{-}}{T_c}\right)=\frac{A}{T_c}U\left(t_p+s^{-}\right) \tag{16b}$$

where $t_p = {T_p}/{T_c}$ is a constant for a given S-graph type.

Combining Equations 11 through 16 gives

$$Q_p=\left(\frac{T_c}{1-\theta}\right)^{b-1} A\int_0^{\frac{T_p}{T_c}} ab\left(s^{+}\right)^{b-1}\frac{1}{T_c}U\left(t_p-s^{+}\right)T_c ds^{+} + \left(\frac{T_c}{\theta}\right)^{b-1} A\int_0^{\eta-\frac{T_p}{T_c}} ab\left(s^{-}\right)^{b-1}\frac{1}{T_c}U\left(t_p+s^{-}\right)T_c ds^{-} - \phi A U_0 \tag{17}$$

where it is still assumed $i(\eta T_c) \geq \phi$.

Equation (17) is rearranged to give

$$Q_p = A\left[a\left(T_c\right)^{b-1}\left(\left(\frac{1}{1-\theta}\right)^{b-1}\int_0^{\frac{T_p}{T_c}} b\left(s^{+}\right)^{b-1}U\left(t_p-s^{+}\right)ds^{+}+\left(\frac{1}{\theta}\right)^{b-1}\int_0^{\eta-\frac{T_p}{T_c}} b\left(s^{-}\right)^{b-1}U\left(t_p+s^{-}\right)ds^{-}\right)-\phi U_0\right]$$

This can be simplified by defining a constant, $\alpha$, as

$$\alpha=\left(\frac{1}{1-\theta}\right)^{b-1}\int_0^{\frac{T_p}{T_c}} b\left(s^{+}\right)^{b-1}U\left(t_p-s^{+}\right)ds^{+}+\left(\frac{1}{\theta}\right)^{b-1}\int_0^{\eta-\frac{T_p}{T_c}} b\left(s^{-}\right)^{b-1}U\left(t_p+s^{-}\right)ds^{-} \tag{19}$$

We can then rewrite Equation (18) in much simpler form as

$$Q_p = A\left[a\left(T_c\right)^{b-1}\alpha-\phi U_0\right] \tag{20}$$

For a given S-graph, and a given precipitation region where the exponent b is a constant, then $t_p$ and $\eta$ are constants, and Equation (20) can be simplified by including Equation (5) as

$$Q_p=\left[\alpha I\left(T_c\right)-\phi U_0\right]A \tag{21}$$

where $\alpha$ is a constant for the given S-graph and precipitation region.

For English units, $U_0$ = 1.008, which is approximated as $U_0$ = 1. Then

$$Q_p = [\alpha I(T_c) - \phi]A \tag{22}$$

In comparison, the Rational Method peak flow rate estimator, for an equivalent mathematical structure using a phi-index loss function for estimating rainfall excess, is

$$Q_R = [I(T_c) - \phi]A \tag{23}$$

## Application

In Equation (22), the single calibration constant, $\alpha$, can be determined by equating (22) to (11a) for a single peak flow rate estimate (again, observing $i(\eta T_c) \geq 0$). Several California Hydrology Manuals (see references) use two S-graphs, one for "urbanized" regions and another for "undeveloped" regions. By equating (22) to (11a), $\alpha = 0.99$ for the urbanized S-graph and $\alpha = 0.86$ for the undeveloped S-graph. In these $\alpha$ determinations, the rainfall exponent b of Equations (4) to (6) was b = 0.55. Additionally, the constraint of $i(\eta T_c) \geq \phi$ resulted in $T_c$ limitations of 45 minutes to 180 minutes for 10 year to 100 year storm events (and typical loss rates of 0.4 inches per hour), respectively.

## Constant Fraction Loss Rate

Another popular loss function is to use a constant proportion loss rate function to estimate rainfall excess, given by

$$e(t) = ki(t) \tag{24}$$

Using Equation (24) in the above development results in the balanced design storm UH procedure peak flow rate estimator, $Q_p$, given by

$$Q_p = k\alpha I(T_c)A \tag{25}$$

where in Equation (25), $\alpha$ is the same constant (and same values) as used in Equation (22), and the constraint of $i(\eta T_c) \geq \phi$ is eliminated. The corresponding well-known Rational Method peak flow rate estimator, $Q_R$, is

$$Q_R = kI(T_c)A \tag{26}$$

From the above example, Equation (25) results in (??? no A)

$$Q_p = kI(T_c) \text{ for urbanized areas}$$

$$Q_p = 0.86kI(T_c) \text{ for undeveloped areas} \tag{27}$$

where again in Equation (27), the rainfall exponent is b = 0.55.

| Storm duration (minutes) | Maximum depth (inches) | Delta from previous duration |
|---|---|---|
| 5 | 20 | 20 |
| 10 | 38 | 18 |
| 15 | 53 | 15 |
| 20 | 65 | 12 |
| 25 | 74 | 9 |

We place the peak duration delta in the center of our design storm.

| | | | | |
|---|---|---|---|---|
| | | 20 | | |

Then we put the second highest duration delta on the left of what we have so far.

| | | | | |
|---|---|---|---|---|
| | 18 | 20 | | |

The third highest delta goes on the right of what we have built so far.

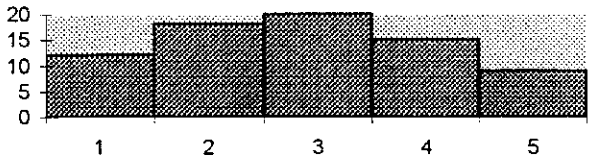
| | | | | |
|---|---|---|---|---|
| | 18 | 20 | 15 | |

The fourth delta goes on the left.

| | | | | |
|---|---|---|---|---|
| 12 | 18 | 20 | 15 | |

The final delta goes on the right.

| | | | | |
|---|---|---|---|---|
| 12 | 18 | 20 | 15 | 9 |



| Unit hydrograph | Interval 1 rainfall | Interval 2 rainfall | Interval 3 rainfall | Interval 4 rainfall | Total runoff |
|---|---|---|---|---|---|
| | | | | | |
| | 2.00 | 9.00 | 12.00 | 6.00 | |
| | | | | | |
| 5 | 10 | | | | 10 |
| 12 | 24 | 45 | | | 69 |
| 18 | 36 | 108 | 60 | | 204 |
| 20 | 40 | 162 | 144 | 30 | 376 |
| 15 | 30 | 180 | 216 | 72 | 498 |
| 9 | 18 | 135 | 240 | 108 | 501 |
| 1 | 2 | 81 | 180 | 120 | 383 |
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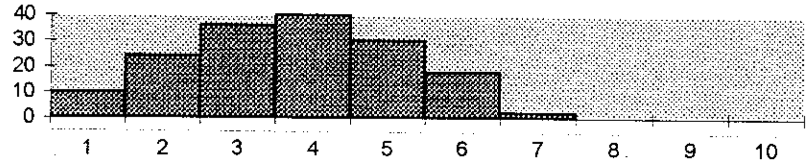
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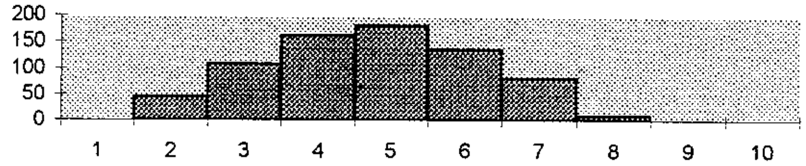
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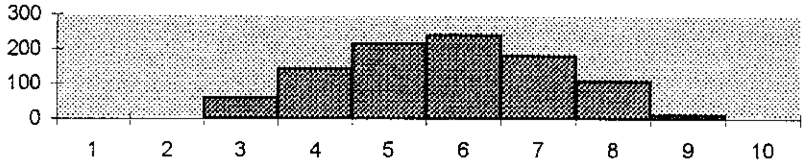
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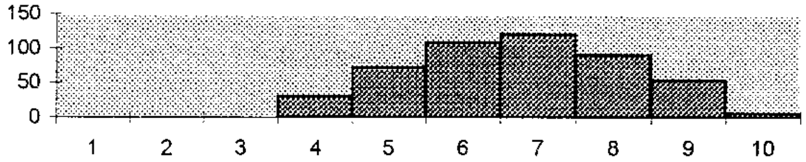
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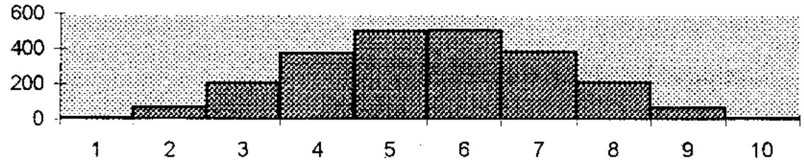
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