The Existence of Approximate Solutions for Two-Dimensional Potential Flow Problems

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A proof is given of the existence of an approximate Complex Variable Boundary Element Method solution for a Birichlet problem. This constructive proof can be used as a basis for numerical calculations. © 1996 John Wiley & Sons, Inc.

I. INTRODUCTION

Most engineering problems involving two-dimensional potential flow in a domain, and related problems can be numerically solved by means of the Complex Variable Boundary Element Method (CVBEM) [1], including problems in groundwater flow [2], propagation of freezing fronts in algid soils [3], groundwater contaminant transport [4], steady-state heat transfer [5], and St. Venant torsion [6]. Some of these are Dirichlet problems, e.g., heat transport problems where the initial temperature is specified on the boundary of the domain, and others are mixed boundary value problems, e.g., heat transport problems where either the temperature or the heat flux are specified on various segments of the boundary of the domain.

The CVBEM uses analytic functions of the form

$$a_0 + a_0' z + \sum_{k=1}^{m} a_k (z - \beta_k) \log(z - \beta_k), \tag{1}$$

together with various ways of selecting the coefficients to approximate the harmonic function, which is the exact solution of the given problem by means of the real part of h, the imaginary part of h being the stream field function. A central theoretical issue is to establish that the solutions of these boundary value problems can indeed be approximated by the functions given in (1). This was done in [7] for the Dirichlet problem with continuous or L^p boundary data. The purpose of this article is to give a different proof of the results in [7], which, as opposed to the proof given in [7], is constructive in nature and which, therefore, can be used as the basis for numerical computations that are theoretically based. This constructive proof is based on the idea of moving the nodes

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slightly outside the domain, a technique that has been used in computations. This idea is made precise below.

II. RESULTS

Remark. The setting is as follows: Let Ω be a bounded simply connected domain in the complex plane with a piecewise continuously differentiable boundary Γ, which is a simple closed curve of finite length, parameterized by

$$\gamma\colon [0,1] \to \Gamma. \tag{2}$$

It is assumed that the map γ is continuous on [0, 1], one-to-one on [0, 1) with $\gamma(0) = \gamma(1)$, is continuously differentiable, with nonzero derivative, except at a finite number of parameter points $c_1, \ldots, c_r$ corresponding to corners that are not cusps; so that the right- and left-hand limits of the derivative exist at each corner, are not zero, and satisfy the condition that for each $j, \gamma'(c_j+) + \gamma'(c_j-)$ is not zero, so that c_j is not a cusp. With these hypotheses on the parameterization of Γ, there is a constant C_Γ with the property that the ratio of the shorter arc length between two points on the curve with their chord is bounded by this constant [8, pg. 31]: for z_1 and z_2 on the Γ, let the arc in the direction of z_1 to z_2 be the shorter arc, so that this arc length is obtained by integrating along Γ,

$$\operatorname{arc}(z_1, z_2) = \int_{z_1}^{z_2} |d\zeta|, \tag{3}$$

then we have

$$\frac{\operatorname{arc}(z_1, z_2)}{|z_1 - z_2|} \le C_\Gamma. \tag{4}$$

To correctly define the function (1), for each β_k on Γ we need to specify a continuous non-self-intersecting path P_{β_k}, joining β_k to infinity, which lies in the complement of $\Omega \cup \Gamma$. Then $P_{\beta_k} - \beta_k$ can be used as a branch cut to define a branch of the logarithm, $\log_{\beta_k}(z - \beta_k)$, which is analytic for z not on the branch cut $P_{\beta_k} - \beta$. Then (1) can, and should, be written as

$$a_0 + a_0' z + \sum_{k=1}^{m} a_k (z - \beta_k) \log_{\beta_k}(z - \beta_k). \tag{5}$$

These branch cuts and related matters are discussed in [9].

Theorem 1. *Let g be a function analytic in a domain containing $\Omega \cup \Gamma$. For any positive ϵ there is a function h, as given in (5), which is analytic in Ω and continuous on $\Omega \cup \Gamma$, with*

$$|g(z) - h(z)| < \epsilon, \tag{6}$$

for z in $\Omega \cup \Gamma$.

Proof. Choose $0 = t_1 < t_2 < \cdots < t_n = 1, t_{n+1} = t_1$, and set

$$\beta_k = \gamma(t_k) \tag{7}$$

for $k = 1, 2, \ldots, n+1$, with $\beta_{n+1} = \beta_1$. Let

$$\delta = \max_{1 \le k \le n} |\beta_{k+1} - \beta_k|. \tag{8}$$

The nodes (7) will be used to define the function h of Eq. (5), which will be shown to satisfy (6) for the proper choice of coefficients $a_0, a_0', a_1, \ldots, a_n$ and small enough δ.

Let Γ_k denote the arc of Γ joining β_k to β_{k+1}. Define $\hat{g}$ on Γ by defining it for z on Γ_k by

$$\hat{g}(z) = \frac{g_{k+1}(z-\beta_k) + g_k(\beta_{k+1} - z)}{\beta_{k+1} - \beta_k}, \tag{9}$$

where

$$g_k = g(\beta_k). \tag{10}$$

Use $\hat{g}$ to define a function h analytic for z in Ω by

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\hat{g}(\zeta)}{\zeta - z}d\zeta. \tag{11}$$

To evaluate h, the integral over Γ must be written as a sum of integrals over the arcs Γ_k on which $\hat{g}$ is defined by (9):

$$2\pi i h(z) = \sum_{k=1}^{n}\int_{\Gamma_k} \frac{\hat{g}(\zeta)}{\zeta - z}d\zeta. \tag{12}$$

The k-th term in this sum can be rewritten as

$$\left(\frac{g_{k+1}(z-\beta_k) + g_k(\beta_{k+1} - z)}{\beta_{k+1} - \beta_k}\right)\int_{\Gamma_k}\frac{d\zeta}{\zeta - z} + \left(\frac{g_{k+1} - g_k}{\beta_{k+1} - \beta_k}\right)\int_{\Gamma_k} d\zeta. \tag{13}$$

The second term in (13) integrates to $g_{k+1} - g_k$, and so, when summed in Eq. (12), sums to zero.

A convenient way to avoid some technical difficulties that arise in choosing branch cuts for the integrals in (12) is to differentiate h twice with respect to z. The first derivative of the first term in (13) is

$$\frac{g_{k+1} - g_k}{\beta_{k+1} - \beta_k}\int_{\Gamma_k}\frac{1}{\zeta - z}d\zeta + \frac{g_{k+1}}{\beta_{k+1} - z} - \frac{g_k}{\beta_k - z}. \tag{14}$$

The second and third terms are obtained by differentiating under the integral and then integrating $1/(\zeta - z)^2$; they sum to zero in the equation for the derivative of h. Differentiating again and combining terms, gives

$$2\pi i h''(z) = \sum_{k=1}^{n}\left[\frac{g_{k+1} - g_k}{\beta_{k+1} - \beta_k} - \frac{g_k - g_{k-1}}{\beta_k - \beta_{k-1}}\right]\frac{1}{z - \beta_k}, \tag{15}$$

where $\beta_0 = \beta_n$ and $g_0 = g_n$.

Integrate (15) twice to see that h has the form given in (5) with

$$2\pi i a_k = \frac{g_{k+1} - g_k}{\beta_{k+1} - \beta_k} - \frac{g_k - g_{k-1}}{\beta_k - \beta_{k-1}} \tag{16}$$

for $k = 1, \dots, n$; note that the coefficient a_k is the same for the functions $g(z) + a + bz$ and $g(z)$. The function h is analytic except on the branch cuts $P_{\beta_1}, \dots, P_{\beta_n}$, and so is continuous on $\Omega \cup \Gamma$, even at each β_k, since the limit of $(z - \beta_k)\log_{\beta_k}(z - \beta_k)$, as $z \to \beta_k$ through values of z in $\Omega \cup \Gamma$, is zero.

For z on Γ, the value of $h(z)$ is equal to the limit of $h(\omega)$, as ω in Ω tends to z nontangentially, to which the Sokhotski–Plemelj formula [8, pg. 32] applies:

$$h(z) = \left(1 - \frac{\Theta(z)}{2\pi}\right)\hat{g}(z) + \frac{1}{2\pi i}\int_\Gamma \frac{\hat{g}(\zeta)}{\zeta - z}d\zeta, \tag{17}$$

where the integral is the Cauchy Principal Value of the singular integral, and $\Theta(z)$ is the interior angle between the two tangents to the curve Γ at z; thus, $\Theta(z) = \pi$ for z not a corner point. Since g is analytic on $\Omega \cup \Gamma$,

$$g(z)\frac{\Theta(z)}{\pi} = \frac{1}{\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta. \tag{18}$$

Using (17) and (18) write

$$h(z) - g(z) = \left(1 - \frac{\Theta(z)}{2\pi}\right)(\hat{g}(z) - g(z)) + \frac{1}{2\pi i}\int_\Gamma \frac{\hat{g}(\zeta) - g(\zeta)}{\zeta - z}d\zeta. \tag{19}$$

To show that $|h(z) - g(z)|$ is small and, in fact, bounded by a constant times δ, several estimates are needed.

For z on Γ_k, as long as

$$0 < \delta < \frac{L}{2C_\Gamma}, \tag{20}$$

where L is the length of Γ, the arc from β_k to β_{k+1} is shorter than the arc in the other direction, and (3) can be applied:

$$|g_{k+1} - g(z)| = \left|\int_z^{\beta_{k+1}} g'(\zeta)d\zeta\right| \le B_1 \mathrm{arc}(z, \beta_{k+1}) \le B_1 C_\Gamma |\beta_{k+1} - \beta_k|, \tag{21}$$

where the integral in (21) is taken along the arc Γ_k and

$$B_1 = \max\{|g'(\zeta)|\colon \zeta \in \Gamma\}. \tag{22}$$

Similarly,

$$|g(z) - g_k| \le B_1 C_\Gamma |\beta_{k+1} - \beta_k|. \tag{23}$$

Thus, for z on Γ_k, since

$$\hat{g}(z) - g(z) = \frac{(g_k - g(z))(\beta_{k+1} - z) + (g_{k+1} - g(z))(z - \beta_k)}{\beta_{k+1} - \beta_k}, \tag{24}$$

$$|\hat{g}(z) - g(z)| \le B_1 C_\Gamma(|\beta_{k+1} - z| + |z - \beta_k|) \le B_1 C_\Gamma \mathrm{arc}(\beta_k, \beta_{k+1}) \le B_1 C_\Gamma^2 \delta. \tag{25}$$

For ζ and z on Γ_k, integrate along Γ_k from z to ζ

$$\begin{aligned} |g'(\zeta) - g'(z)| &= \left|\int_z^\zeta g''(\tau)d\tau\right| \le B_2 \mathrm{arc}(z, \zeta) \\ &\le B_2 \mathrm{arc}(\beta_k, \beta_{k+1}) \le B_2 C_\Gamma |\beta_{k+1} - \beta_k|, \end{aligned} \tag{26}$$

where

$$B_2 = \max\{|g''(\tau)|\colon \tau \in \Gamma\}. \tag{27}$$

Then

$$\begin{aligned} |g_{k+1} - g_k - (\beta_{k+1} - \beta_k)g'(z)| &= \left|\int_{\beta_k}^{\beta_{k+1}} (g'(\zeta) - g'(z))d\zeta\right| \\ &\le B_2 C_\Gamma |\beta_{k+1} - \beta_k| \mathrm{arc}(\beta_k, \beta_{k+1}) \le B_2 C_\Gamma^2 |\beta_{k+1} - \beta_k|^2. \end{aligned} \tag{28}$$

Then

$$\left|\frac{g_{k+1}-g_k}{\beta_{k+1}-\beta_k} - g'(z)\right| \le B_2 C_\Gamma^2 |\beta_{k+1}-\beta_k| \le B_2 C_\Gamma^2 \delta. \tag{29}$$

Let

$$w(z) = \hat{g}(z) - g(z). \tag{30}$$

For z on Γ_k, not equal to either β_{k+1} or β_k,

$$w'(z) = \frac{g_{k+1}-g_k}{\beta_{k+1}-\beta_k} - g'(z). \tag{31}$$

For any z_1 and z_2 on Γ with, say, the shorter arc on Γ being from z_1 to z_2, since $w'(z)$ is piecewise continuous on Γ, it can be integrated to obtain

$$|w(z_2) - w(z_1)| \le \left|\int_{z_1}^{z_2} w'(\zeta)d\zeta\right| \le B_2 C_\Gamma^2 \delta \operatorname{arc}(z_1, z_2) \le B_2 C_\Gamma^3 \delta |z_2 - z_1|. \tag{32}$$

Thus, $w(z)$ satisfies a Lipschitz condition of order one with the Lipschitz constant $B_2 C_\Gamma^3 \delta$.

Use the standard device of writing the singular integral of (19) in the form

$$\int_\Gamma \frac{w(\zeta)}{\zeta - z} d\zeta = \int_\Gamma \frac{w(\zeta) - w(z)}{\zeta - z} d\zeta + w(z) \int_\Gamma \frac{d\zeta}{\zeta - z}. \tag{33}$$

The first integral exists, since w satisfies a Lipschitz condition of order one, and has modulus bounded by

$$B_2 C_\Gamma^3 \delta \int_\Gamma |d\zeta| \le B_2 C_L^3 L \delta, \tag{34}$$

L denoting the length of Γ, while the second term in (33) is $w(z) i \Theta(z)$. Combine these estimates and apply them to (19) to show:

$$h(z) - g(z) = w(z) + \frac{1}{2\pi i} \int_\Gamma \frac{w(\zeta) - w(z)}{\zeta - z} d\zeta, \tag{35}$$

from which

$$|h(z) - g(z)| \le \delta \left(B_1 C_\Gamma^2 + \frac{B_2 C_\Gamma^3 L}{2\pi} \right). \tag{36}$$

For small enough δ, this shows that (6) holds on Γ and, therefore, holds in $\Omega \cup \Gamma$ by the maximum modulus theorem. Q.E.D.

Lemma 1. *Let Ω be a bounded simply connected domain with boundary Γ, which is piecewise twice differentiable. Given $\epsilon > 0$, there is a domain Ω' with piecewise continuously differentiable boundary Γ'*

$$\Omega' \supset \Omega \cup \Gamma \tag{37}$$

and a bicontinuous map Ψ of Γ' onto Γ with

$$|\Psi(z') - z'| < \epsilon \qquad \text{for all } z' \text{ on } \Gamma'. \tag{38}$$

Proof. One simple and useful case is when Ω is convex, or more generally when Ω is starlike with respect to a point z_0 in Ω, [10] so that the line $\{\alpha z_0 + (1-\alpha)z\colon\ 0 \le \alpha \le 1\}$ is contained in Ω for any z in Ω. Define Ψ by defining its inverse

$$\Psi^{-1}(z) = -\rho z_0 + (1+\rho)z \qquad \text{for each } z \text{ on } \Gamma. \tag{39}$$

If Ω is contained in the ball $\{z\colon\ |z| \le R\}$, then for $0 < \rho < \epsilon/R$ the conclusions of the lemma will hold, noting that the line joining z_0 to a point z on the boundary of Ω can only intersect the boundary at that point and, therefore, (37) is satisfied.

Next suppose that Γ is two times differentiable with no corners. Define the map Ψ by

$$\Psi(\gamma(t)) = \gamma(t) - i\rho\gamma'(t), \tag{40}$$

where the term $-i\gamma'(t)$, rather than the outward pointing normal $-i(\gamma'(t)/|\gamma'(t)|)$, is used for simplicity. Equation (40) is indeed continuously differentiable and (38) holds for $0 < \rho < \epsilon/\max_{0\le t\le 1}|\gamma'(t)|$. It remains to show that Eq. (40) defines a simple curve and that (37) holds.

To simplify the notation, let

$$M_1 = \max_{0\le t\le 1} |\gamma'(t)| \tag{41}$$

$$M_2 = \max_{0\le t\le 1} |\gamma''(t)| \tag{42}$$

and

$$m_1 = \min_{0\le t\le 1} |\gamma'(t)|. \tag{43}$$

Suppose that

$$\Psi(t) = \Psi(s), \tag{44}$$

then

$$\gamma(t) - \gamma(s) = i\rho(\gamma'(t) - \gamma'(s)). \tag{45}$$

first, note that

$$|\gamma'(t) - \gamma'(s)| = \left|\int_s^t \gamma''(t)dt\right| \le M_2|t-s|. \tag{46}$$

Second, note that

$$|t-s| = \left|\int_{\gamma(s)}^{\gamma(t)} (\gamma^{-1})'(z)dz\right| \le \operatorname{arc}(\gamma(t),\gamma(s)) \max_{z\in\Gamma} |(\gamma^{-1})'(z)| \le \frac{C_\Gamma|\gamma(t)-\gamma(s)|}{m_1}. \tag{47}$$

Equations (45), (46), and (47) show that (44) cannot hold for $s \ne t$, and, therefore, Γ' is a simple closed curve, if

$$0 < \rho < \frac{m_1}{C_\Gamma M_2}. \tag{48}$$

Since the point $\Psi(\gamma(t))$ lies on the line though $\gamma(t)$ in the direction of the outward pointing normal, the curve Γ' will remain outside of Ω, and, therefore, (37) will hold, if it is shown that Γ' and Γ do not intersect. Suppose that these two curves do intersect so that

$$\Psi(s) = \gamma(t). \tag{49}$$

Then

$$\gamma(t) - \gamma(s) = -i\rho\gamma'(s). \tag{50}$$

Write

$$\begin{aligned}\gamma(t) - \gamma(s) &= \int_s^t \gamma'(u)du \\ &= \int_s^t \left[\gamma'(s) + \int_s^u \gamma''(\tau)d\tau\right] du \\ &= \gamma'(s)(t-s) + \int_s^t \int_s^u \gamma''(\tau)d\tau du,\end{aligned} \tag{51}$$

so that

$$\rho m_1 \le |\gamma'(s)(i\rho + t - s)| \le M_2(t-s)^2. \tag{52}$$

From (47) and (50),

$$|t - s| \le \frac{C_\Gamma \rho M_1}{m_1} \tag{53}$$

and (52) and (53) cannot hold if

$$0 < \rho < \frac{m_1^3}{M_2 M_1^2 C_\Gamma^2}, \tag{54}$$

which completes the proof that Γ' is a simple closed curve containing $\Omega \cup \Gamma$.

In considering the remaining case, where Γ has a finite number of corners, a geometrical discussion will make the result clear. This clarity is, however, purchased at the price of not calculating explicit bounds on ρ, in contrast to the paragraphs above. In brief, consider the curve Γ', defined by Eq. (40) except at the values of t corresponding to the corner points. Because of the smoothness conditions on Γ, near a corner $\gamma(t_0)$ the curves Γ and Γ' are closely approximated by two straight lines intersecting in that corner. If the interior angle at that corner is acute, then there is a gap between the values of $\Psi(\gamma(t_o+))$ and $\Psi(\gamma(t_0-))$; join those two values by a straight line. If the interior angle at that corner is obtuse, then $\Psi(\gamma(t))$ for $t < t_0$ and $\Psi(\gamma(t))$ for $t > t_0$ intersect for t near t_0; remove the two small arcs of Γ' that extends beyond this intersection. When these adjustments are made to Γ', the conditions of the lemma hold for sufficiently small ρ. Q.E.D.

For any domain Ω arising in an actual engineering problem, it will be clear that the curve Γ' of lemma 1 exists: just think of drawing it by hand; and so drawn, it can be made even smoother than Γ, not less smooth as in the lemma. An alternate theoretical construction of Γ' can be accomplished by the use of a conformal map ϕ of the complement of $\Omega \cup \Gamma$ onto the unit circle, the existence of which is guaranteed by the Riemann mapping theorem, and letting Γ' be the analytic curve $\phi^{-1}\{z\colon |z| = r\}$ for r less than but close to one. A practical difficulty with this approach is that, for the map Ψ of the lemma to exist, the curve Γ must be reparameterized by the continuous extension of the unknown map ϕ to Γ; and a further technical difficulty lies in showing that this extension has the smoothness required by Theorem 1 [10, chap. 3].

Unlike the intuitive and theoretical approaches discussed in the paragraph above, the proof of lemma 1 shows how to construct Γ' in a way that can be made part of an actual computation. The smoothness requirement in the lemma that the parameterization of Γ be piecewise two-times differentiable is stronger than is necessary, but leads to a simple proof and contains most problems of practical interest.

Theorem 2. (The Dirichlet Problem). *Let Ω be a bounded domain with piecewise twice continuously differentiable boundary Γ, and let a continuous real-valued function u be given on Γ. For any $\epsilon > 0$, there is a CVBEM function h as given in (5), which is analytic in Ω and has $Re(h)$ continuous on $\Omega \cup \Gamma$ with*

$$|Re(h)(z) - u(z)| < \epsilon \qquad \text{for all } z \text{ in } \Omega \cup \Gamma. \tag{55}$$

Proof. The standard existence theorem for the Dirichlet problem states that there is a function f analytic in Ω with $Re(f)$ continuous on $\Omega \cup \Gamma$ and $Ref(z) = u(z)$ for z on Γ. Since $Re(f)$ is continuous on the compact set $\Omega \cup \Gamma$, it is uniformly continuous there and so there is a $\delta > 0$ with

$$|Ref(z) - Ref(w)| < \frac{\epsilon}{2} \qquad \text{for } z \text{ and } w \text{ in } \Omega \cup \Gamma \text{ and } |z - w| < \delta. \tag{56}$$

Choose $\rho_n \to 0$; then the maps $\Psi_n(\gamma(t)) = \gamma(t) - i\rho_n\gamma'(t)$ of Lemma 1 satisfy

$$|\Psi_n(z) - z| \to 0 \qquad \text{uniformly for } z \text{ on } \Gamma, \tag{57}$$

and, letting Ω'_n denote the interior of the curve Γ'_n,

$$\Omega'_n \supset \Omega \cup \Gamma \qquad \text{for all } n. \tag{58}$$

Choose a point w_0 in Ω. From the Riemann mapping theorem there is a unique one-to-one analytic map of $D = \{z\colon |z| < 1\}$ onto Ω'_n, with $\phi_n(0) = w_0$ and $\phi'_n(0) > 0$, which, by the Carathéodory extension theorem, is also a continuous one-to-one map of the closure $\bar{D} = \{|z| \le 1\}$ of D onto the closure of Ω'_n. And, by the same theorems, there is the corresponding one-to-one continuous map ϕ of $\bar{D}$ *onto* $\Omega \cup \Gamma$, *which takes 0 to* w_0, has $\phi'(0) > 0$, and is a one-to-one analytic map of D onto Ω. Since the parameterization $\Psi'_n = \gamma(t) - i\rho_n\gamma'(t)$ of Γ'_n converges uniformly for $0 \le t \le 1$ to the parameterization $\gamma(t)$ of Γ, Rado's Theorem [10, pg. 26, or 11, pg. 62] applies to show that

$$\phi_n \to \phi \qquad \text{uniformly on } \{z\colon |z| \le 1\}. \tag{59}$$

Consider the composition $f \circ \phi \circ \phi_n^{-1}(z)$, which is analytic on Ω'_n, a domain containing $\Omega \cup \Gamma$. From Theorem 1 there is a CVBEM function h, of the form given by Eq. (5), which is analytic in Ω, continuous on $\Omega \cup \Gamma$, and satisfies

$$|h(z) - f \circ \phi \circ \phi_n^{-1}(z)| < \frac{\epsilon}{2} \qquad \text{for } z \text{ in } \Omega \cup \Gamma. \tag{60}$$

For $z = \phi_n(w)$ in $\Omega'_n \cup \Gamma'_n$,

$$|\phi \circ \phi_n^{-1}(z) - z| = |\phi(w) - \phi_n(w)| < \delta \quad \text{for large enough } n, \tag{61}$$

the δ being the δ of Eq. (56), and (61) holding uniformly for z in $\Omega'_n \cup \Gamma'_n$ and *a fortiori* uniformly for z in $\Omega \cup \Gamma$. Take n large enough; then, for all z on Γ,

$$\begin{aligned} |Re(h(z)) - u(z)| &\le |Reh(z) - Ref \circ \phi \circ \phi_n^{-1}(z)| \\ &+ |Ref \circ \phi \circ \phi_n^{-1}(z) - Ref(z)| + |Ref(z) - u(z)| < \epsilon \end{aligned} \tag{62}$$

by combining the results above. Q.E.D.

Theorem 3. (The L^p Dirichlet Problem). *Let Ω be a bounded domain with piecewise twice continuously differentiable boundary Γ, and let a real-valued function u be given on Γ, which* *belongs to* $L^p(\Gamma)$, *for* $1 \leq p < \infty$. *For any* $\epsilon > 0$, *there is a CVBEM function* h *as given in (5), which is analytic in* Ω *and has* $Re(h)$ *continuous on* $\Omega \cup \Gamma$ *with*

$$\int_{\Gamma} |Re(h)(z) - u(z)|^p |dz| < \epsilon. \tag{63}$$

Proof. See the proof of Corollary 2 in [7]. Q.E.D.

It is possible to derive CVBEM approximation results for mixed boundary value problems of the type most common in applications, where the harmonic function u is prescribed on part of the boundary of the domain, and its normal derivative is prescribed on the remainder of the boundary, using the representations given in [11]. The results are similar to those obtained for the Dirichlet problem, but several difficulties arise for the mixed problems, which are distinctive, e.g., the CVBEM functions are bounded, but the solution to a mixed problem need not be bounded, and will be addressed in a further article.

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