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AN EXPANSION OF THE CVBEM MATRIX SYSTEM

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A matrix system expansion is developed for the complex variable boundary element method (or CVBEM). The expansion includes identification of matrix components that contain the approximation error due to basis function approximations. Bounds for error are developed by use of Taylor series expansions of the problem solution from each nodal point in the model discretization.

1 Introduction

The complex variable boundary element method, or CVBEM, has been the subject of several papers and books (e.g. Hromadka & Lai;[2] Harryman *et al.*,[1] Hromadka[3]). The basis of the CVBEM is the use of the Cauchy integral equation to develop approximations of two-dimensional boundary value problems involving the Laplace or Poisson equations.

An advantage of the CVBEM is the property that the resulting approximation function, $\hat{\omega}(z)$, is analytic in the simply connected domain, Ω, and continuous on the simple closed problem boundary, Γ. Thus, $\hat{\omega}(z)=\hat{\phi}(z)+i\hat{\psi}(z)$, where $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are the approximation function potential and stream functions, respectively, and each satisfy the Laplace equation in Ω. The general CVBEM technique is briefly described in the following discussion.

Let $\omega(z)=\phi(x,y)+i\psi(x,y)$ be a complex variable function which is analytic on $\Gamma\cup\Omega$, where Ω is a simply connected domain enclosed by the simple closed boundary Γ (Fig. 1). We define $\phi(x,y)$ to be the state variable and $\psi(x,y)$ the stream function, where ϕ and ψ are two-dimensional real valued functions. Since ω is analytic, ϕ and ψ are related by the Cauchy-Reimann equations

$$\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x} \tag{1}$$

and thus satisfy the two-dimensional Laplace equations in Ω, namely

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0 \tag{2}$$

The Cauchy integral theorem states that if we know the value of the complex function ω on the boundary Γ, and if ω is analytic on $\Gamma\cup\Omega$, then ω is given for any z in Ω by

$$\omega(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\omega(\zeta)\,d\zeta}{\zeta-z}, \quad z\in\Omega, \quad z\notin\Gamma \tag{3}$$

The CVBEM forms $\hat{\omega}$, an approximation of ω, using known values of either ϕ or ψ on the boundary Γ, and uses the Cauchy integral (3) to determine approximate values for ω on $\Omega\cup\Gamma$. The approximator, $\hat{\omega}$, is a two-dimensional analytic function in Ω that can be differentiated, integrated, or otherwise manipulated to obtain higher order operator relationships (Hromadka[3]).

Let the boundary Γ be a polygonal line composed of V straight line segments and vertices. We define nodal points with complex coordinates z_j, $j=1,...,m$ on Γ such that $m>V$. Nodal points are located at each vertex of Γ and are numbered in a counter-clockwise direction. Let Γ_j be the straight line segment joining z_j and z_{j+1} so that

$$\Gamma=\bigcup_{j=1}^{m}\Gamma_j$$

where $z_{m+1}=z_1$. Thus, m boundary elements, Γ_j, are defined on Γ, where Γ_m connects nodal coordinate z_m and z_1 (Fig. 2). The CVBEM defines a continuous global trial function, $G(z)$, by

$$G(z)=\sum_{j=1}^{m}N_j(z)(\bar{\phi}_j+i\bar{\psi}_j), \quad z\in\Gamma \tag{4}$$

where, for a piecewise linear polynomial global trial function, and $j=1,...,m$, $N_j(z)$ is given by

$$N_j(z)=\begin{cases}\dfrac{z-z_{j-1}}{z_j-z_{j-1}} & z\in\Gamma_{j-1}\\ 0 & z\notin\Gamma_j\cup\Gamma_{j-1}\\ \dfrac{z_{j+1}-z}{z_{j+1}-z_j} & z\in\Gamma_j\end{cases} \tag{5}$$

and where $\bar{\phi}_j$ and $\bar{\psi}_j$ are nodal values of the two conjugate components, evaluated at z_j. An analytic approximation is then determined by

$$\hat{\omega}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{G(\zeta)\,d\zeta}{\zeta-z}, \quad z\in\Omega, \quad z\notin\Gamma \tag{6}$$

Since usually only one of the two specified nodal values $(\bar{\phi}_j,\bar{\psi}_j)$ is known at each z_j, $j=1,...,m$, values for the unknown nodal values must be estimated as part of the CVBEM approach to developing an analytic approximation function. The CVBEM develops a matrix system for use in solving these unknown nodal values (see Hromadka & Lai[2]), solves the resultant matrix system, and uses these nodal value estimates along with the known nodal values in defining $\hat{\omega}(z)$.

2 Matrix system development and analysis

In order to develop the CVBEM approximation function, $\hat{\omega}(z)$, defined on $\Omega\cup\Gamma$, the unknown nodal values of $\hat{\omega}(z_j)$ need to be determined. For m nodes, a system of m complex valued linear equations are developed by taking the Cauchy principal value of eqn (6) evaluated at each node j; that is,

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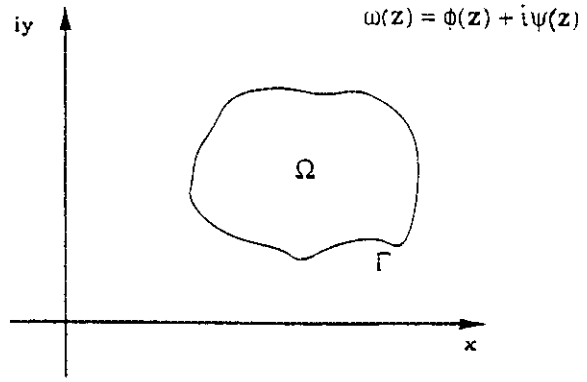

Figure 1: Problem domain and boundary.

$$\hat{\omega}_j = \lim_{z \to z_j^-} \frac{1}{2\pi i} \int_\Gamma \frac{G(\zeta)\, d\zeta}{\zeta - z_j}; \quad z_j^- \in \Omega \text{ and } z_j^- \notin \Gamma \tag{7}$$

In solving for the $\hat{\omega}_j$, j=1,2,...,m, complex numbers are determined such that

$$\hat{\omega}_j = \sum_{k=1}^{m} C_{jk} \bar{\omega}_k \tag{8}$$

where C_{jk} are complex numbers determined by integrating eqn (7) for each trial function $N_k(\zeta)$ used in $G(\zeta)$ and also where C_{jk} is with respect to the term $(\zeta - z_j)$ in the denominator of eqn(7). It is noted that the C_{jk} values are complex constants that depend on the trial function used, $N_k(\zeta)$, the nodal point locations, z_j, and the geometry of Γ; that is, the C_{jk} values do not depend on the nodal values of $\omega(z_j)$. In eqn (8), the $\bar{\omega}_k$ is the nodal point value used in eqn (7) at node k.

Expanding eqn (8),

$$\begin{aligned} \hat{\omega}_j = \hat{\phi}_j + i\hat{\psi}_j &= \sum_{k=1}^{m} (\alpha_{jk} + i\beta_{jk})(\bar{\phi}_k + i\bar{\psi}_k) \\ &= \sum_{k=1}^{m} (\alpha_{jk}\bar{\phi}_k - \beta_{jk}\bar{\psi}_k) \\ &+ i \sum_{k=1}^{m} (\alpha_{jk}\bar{\psi}_k + \beta_{jk}\bar{\phi}_k) \end{aligned} \tag{9}$$

where $C_{jk} = \alpha_{jk} + i\beta_{jk}$, and $\bar{\omega}_k = \bar{\phi}_k + i\bar{\psi}_k$;

$$i = \sqrt{-1}; \text{ and } \alpha_{jk} \text{ and } \beta_{jk}$$

are real constants.

For the analysis of approximation error, we reconsider the above equations given new trial functions $N_j^*(\zeta)$ with the special attribute that the $N_j^*(\zeta)$ are 'perfect' in the sense that

$$G^*(\zeta) = \sum_{j=1}^{m} N_j^*(\zeta)\omega_j = \omega(\zeta) \tag{10}$$

for all $\zeta \in \Gamma$, and in eqn (10) necessarily $\bar{\omega}_j = \omega_j$ for all nodes j. For this ideal case,

$$\begin{aligned} \omega_j &= \uparrow \omega(z_j^-) = \uparrow \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta)\, d\zeta}{\zeta - z_j^-} \\ &= \sum_{k=1}^{m} (\alpha_{jk}^* + i\beta_{jk}^*)(\phi_k + i\psi_k) \end{aligned} \tag{11}$$

where $\uparrow$ is notation for the Cauchy principal value; and α_{jk}^* and β_{jk}^* are new real constants that depend on $\omega(\zeta)$ as defined on Γ. Note that the nodal values used in eqn (11) are exact, whereas the nodal values used in eqn (9) are approximate.

In matrix form, for m nodes on Γ,

$$\{\phi_k\}_{m\times 1} = [\alpha_{jk}^*]_{m\times m}\{\phi_k\}_{m\times 1} + [-\beta_{jk}^*]_{m\times m}\{\psi_k\}_{m\times 1} \tag{12a}$$

$$\{\psi_k\}_{m\times 1} = [\alpha_{jk}^*]_{m\times m}\{\psi_k\}_{m\times 1} + [\beta_{jk}^*]_{m\times m}\{\phi_k\}_{m\times 1} \tag{12b}$$

where $\{\phi_k\}$ and $\{\psi_k\}$ are column vectors of the exact nodal values ω_k; $[\alpha_{jk}^*]$ and $[\beta_{jk}^*]$ are square matrices composed of the α_{jk}^* and β_{jk}^* terms from eqn (11), respectively.

In eqn (12), either eqn (12a) or (12b) can be used to determine unknown nodal values (recall, only one unknown nodal value occurs in the typical boundary value problem under consideration).

In comparing eqns (9) and (11), both systems of equations utilize the known nodal point boundary condition values. The unknown component, however, is completely dependent on whether $N_j(\zeta)$ or $N_j^*(\zeta)$ is used. Obviously, $N_j(\zeta)$ is used in practice (and hence the α_{jk} and β_{jk} are constants independent of the problem boundary conditions). The error in estimating the unkown nodal point component, for each node then is a result of the difference between $N_j(\zeta)$ and $N_j^*(\zeta)$ which is fully represented by the differences between α_{jk}^* and α_{jk} or β_{jk}^* and β_{jk}.

For a given boundary value problem, let

$$E_{jk}^{\alpha} = \alpha_{jk}^* - \alpha_{jk} \tag{13a}$$

$$E_{jk}^{\beta} = \beta_{jk}^* - \beta_{jk} \tag{13b}$$

Then from eqns (12a) and (12b), a new matrix expansion is

$$\begin{aligned} \{\phi_k\} &= [\alpha_{jk}]\{\phi_k\} + [-\beta_{jk}]\{\psi_k\} \\ &+ [E_{jk}^{\alpha}]\{\phi_k\} + [-E_{jk}^{\psi}]\{\psi_k\} \end{aligned} \tag{14a}$$

$$\begin{aligned} \{\psi_k\} &= [\alpha_{jk}]\{\psi_k\} + [\beta_{jk}]\{\phi_k\} \\ &+ [E_{jk}^{\alpha}]\{\psi_k\} + [-E_{jk}^{\psi}]\{\phi_k\} \end{aligned} \tag{14b}$$

where $m \times m$ matrices $[E_{jk}^{\alpha}]$ and $[E_{jk}^{\beta}]$ depend on the given boundary value problem solution on Γ, and where the exact nodal values of $\phi_k + i\psi_k$ result due to use of the ideal $N_j^*(\zeta)$ basis functions.

In eqns (14a) and (14b) it is seen that all error of approximation is due to the contribution of matrices $[E_{jk}^{\alpha}]$ and $[E_{jk}^{\beta}]$. To examine these later matrices, we will assume that the solution to the boundary value problem $\omega(\zeta)$, is analytic on a larger simply connected region $\bar{\Omega}$ such that $\Omega \cup \Gamma \subset \bar{\Omega}$. Furthermore, define a circle R_j at each nodal point j such that the center of R_j is z_j and the radius r_j is the larger of the distances $|z_j - z_{j-1}|$ and $|z_{j+1} - z_j|$; that is, $r_j = \max\{|z_j - z_{j-1}|, |z_{j+1} - z_j|\}$. Then $\bar{\Omega}$ is also assumed to contain each R_j (and hence the disc interior of each R_j).

Because $\omega(z)$ is analytic on $\bar{\Omega}$, then $\omega(z)$ can be expanded as a Taylor series at each node, $T_j(z)$, where the radius of convergence of

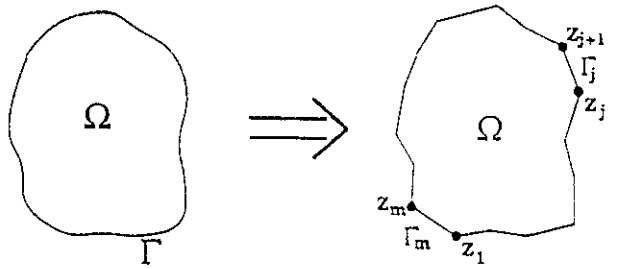

Figure 2: Discretization.

$T_j(z)$ is greater than r_j of circle R_j (by assumption of $\omega(z)$ being analytic over Ω). Then for node j, the Taylor series expansion of $\omega(\zeta)$ for $\zeta \in \Gamma$, centered at z_j, is

$$T_j(\zeta) = \omega_j + \left.\frac{d\omega}{dz}\right|_{z_j} (\zeta - z_j) + \eta(\zeta) \tag{15}$$

where $\eta(\zeta)$ is the Taylor series remainder term given by

$$\eta(\zeta) = \frac{(\zeta - z_j)^2}{2\pi i} \int_{R_j} \frac{\omega(u)\, du}{(u-\zeta)(u-z_j)^2} \tag{16}$$

where the contour integration of eqn (16) is on the circle R_j (centered at z_j), u is the local coordinate on R_j; and $\zeta \in \Gamma_j$ or Γ_{j-1}.

A bound for the Taylor series remainder, $\|\eta(\zeta)\|$, can be developed. Let M be the maximum value of $\omega(z)$ over $\bar{\Omega}$ (such a maximum exists by the Maximum Modulus Theorem), then from eqn (16),

$$\begin{aligned} \|\eta(\zeta)\| &\le \frac{|\zeta - z_j|^2}{2\pi i} \int_{R_j} \frac{|\omega(u)|\,|du|}{|u-\zeta|\,|u-z_j|^2} \\ &\le \frac{r^2 M 2\pi r_j}{2\pi(r_j - r) r_j^2} \end{aligned} \tag{17}$$

where r_j is the radius of circle R_j, and $r = |\zeta - z_j|$.

Simplifying,

$$\|\eta(\zeta)\| \le \frac{M r^2}{(r_j - r) r_j} \tag{18}$$

where necessarily $0 < r \le r_j$.

On Γ_j, for $\zeta \in \Gamma_j$,

$$\begin{aligned} E(\zeta) &= T_j(\zeta) - G(\zeta) \\ &= \left(\omega_j + \left.\frac{d\omega}{dz}\right|_{z_j} (\zeta - z_j) + \eta(\zeta)\right) \\ &\quad - \left(\omega_j + \left(\frac{\omega_{j+1} - \omega_j}{z_{j+1} - z_j}\right)(\zeta - z_j)\right) \\ &= \left[\left.\frac{d\omega}{dz}\right|_{z_j} - \left(\frac{\omega_{j+1} - \omega_j}{z_{j+1} - z_j}\right)\right](\zeta - z_j) + \eta(\zeta) \end{aligned} \tag{19}$$

From eqns (18) and (19), as the maximum distance between nodes decreases, then $r_j \to 0$ for all j and therefore $\|\eta(\zeta)\| \to 0$, giving a convergence bound for the CVBEM matrix system.

3 Conclusions

A matrix system expansion is developed for the complex variable boundary element method (or CVBEM). The expansion includes identification of matrix components that contain the approximation error due to basis function approximations. Bounds for error are developed by use of Taylor series expansions of the problem solution from each nodal point used in the nodal model. The resulting error bound is used to demonstrate convergence of the approximation to the exact solution as the distance between nodes on the problem boundary decreases.

References

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