THE THEORETICAL UNDERPINNING OF THE RATIONAL METHOD

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Key Words: Rational Method, Unit Hydrographs, Models, Hydrology, Surface Water

Abstract

The Rational Method equation for estimating peak flow rates for stormwater runoff is derived from the balanced design storm unit hydrograph approach presented in the U.S. Army Corps of Engineers HEC Training Document 15. The new form of the Rational Method equation is $Q_p = (\alpha I - \phi)A$, instead of the well known $Q_p = (I - \phi)A$; or $Q_p = \alpha CIA$, instead of the well known $Q_p = CIA$, depending upon the respective loss function used in the unit hydrograph effective rainfall model. The above fixed constant α is found to depend upon the type of unit hydrograph used (i.e., S-Graph) and the log-log slope of the rainfall depth-duration curve, and is easily determined by equating to a known unit hydrograph design storm model peak flow rate result. This new development provides significant foundation for use of the well-known Rational Method equation in small catchments where depth-area effects are negligible.

INTRODUCTION

1. Unit Hydrographs

Unit hydrographs (UH) for a catchment may be developed from normalized S-graphs. The S-graph, which is developed from regional rainfall-runoff data, is typically expressed by $S(\ell)$ where ℓ is a proportion (percent) of catchment lag. Catchment lag is related to the usual time of concentration, T_C, by

$$\text{lag} = \gamma T_C \tag{1}$$

In several flood control districts in California, $\gamma = 0.80$. Then

$S(\ell) = S\left(\frac{t100}{\gamma T_C}\right)$, where now UH is a function of T_C, and is obtained from the derivative of S(t) with respect to time t.

For $T_C = 1$ and catchment area A = 1, a normalized UH results, U(t). For $T_C \neq 1$ or $A \neq 1$, the catchment UH, $u(t, T_C, A)$, is related to U(t) by

$$\frac{A}{T_C} U\left(\frac{tc}{T_C}\right) \tag{2}$$

where

$$\int_0^\infty u(t, T_C, A)\, dt = A \int_0^\infty U\left(\frac{t}{T_C}\right) \frac{dt}{T_C} = AU_o \tag{3}$$

where U_o is a constant. Hereafter, the catchment UH, $U(t, T_C, A)$, will simply be written as u(t) where no confusion occurs.

2. Rainfall Depth-Duration Description

Precipitation depth-duration relationships, for a given return frequency, is generally given by the power law analog,

$$D(t) = at^b \tag{4}$$

where a>o is a function of return frequency, and is held constant for a selected design storm return frequency; "b" is typically a constant for large regions (e.g., entire counties); D(t) is the rainfall depth; and t is the selected duration of time.

Mean rainfall intensity, I(t), is

$$I(t) = \frac{1}{t} D(t) = at^{b-1} \tag{5}$$

and instantaneous rainfall intensity, i(t), is

$$i(t) = \frac{d}{dt} D(t) = abt^{b-1} = bI(t). \tag{6}$$

With respect to HEC TD-15 (1984), a balanced design storm pattern (of nested uniform return frequency rainfall depths) can be described by the time coordinates $t^{\pm}$ shown in Figure 1. For a proportioning of rainfall quantities by allocation of a θ proportion prior to time $t^{\pm} = 0$ (see Fig. 1), instantaneous rainfall intensities are given by

$$i^{-}(t^{-}) = i^{-}(\theta t) = i(t) \tag{7}$$

or

$$i^{-}(t^{-}) = i\left(\frac{t^{-}}{\theta}\right) = \left(\frac{1}{\theta}\right)^{b-1} i(t^{-}) \tag{8}$$

Similarly,

$$i^{+}(t^{+}) = \left(\frac{1}{1\theta}\right)^{b-1} i(t^{+}) \tag{9}$$

3. <u>Peak Flow Rate Estimates from the Balanced Design Storm Unit Hydrograph Procedure</u>

Let $v(t) = v(\eta T_c - t)$; that is, v(t) is a time-reversed plot of the UH, u(t). From Fig. 1, and aligning the UH peak to occur at time $t^{\pm} = 0$,

$$v^{+}(t^{+}) = u(T_p - t^{+}) \tag{10a}$$

$$v^{-}(t^{-}) = u(T_p + t^{-}) \tag{10b}$$

Then the balanced design storm UH procedure estimates the peak flow rate, Q_p, by

$$Q_p = \int_{t^+=0}^{T_p} e^+(t^+)\, v^+(t^+)\, dt^+ + \int_{t^-=0}^{\eta T_c - T_p} e^-(t^-)\, v^-(t^-)\, dt^- \tag{11a}$$

$$= \int_0^{T_p} i^+(t^+)\, v^+(t^+)\, dt^+ + \int_0^{\eta T_c - T_p} i^-(t^-)\, v^-(t^-)\, dt^-$$

$$- \phi \left[\int_0^{T_p} v^+(t^+)\, dt^+ + \int_0^{\eta T_c - T_p} v^-(t^-)\, dt^- \right] \tag{11b}$$

where ηT_c is the total duration of the UH, and T_p is the time to peak of the UH, and T_p is the time to peak of the UH. In (11b) a "phi index" (or constant) loss function is used to compute rainfall excess; also, a necessary constraint imposed is that $i(\eta T_c) \geq \phi$.

The last term of Eq. (11b) is solved by

$$\phi = \left[\int_0^{T_p} v^+(t^+)\, dt^+ + \int_0^{\eta T_c - T_p} v^-(t^-)\, dt^- \right] = \phi \left[\int_0^{\infty} u(t)\, dt \right] = \phi A U_o \tag{12}$$

The first two integrals of (11b) are rewritten by including Eqs. (8) and (9),

$$\int_0^{T_p} i^+(t^+)\, v^+(t^+)\, dt^+ = \left(\frac{1}{1\theta}\right)^{b-1} \int_0^{T_p} i(t^+)\, v^+(t^+)\, dt^+ \tag{13a}$$

$$\int_{t^-=0}^{\eta T_c - T_p} i^-(t^-)\, v^-(t^-)\, dt^- = \left(\frac{1}{1\theta}\right)^{b-1} \int_{t^-=0}^{\eta T_c - T_p} i(t^-)\, v^-(t^-)\, dt^- \tag{13b}$$

The next step in the mathematical development is to introduce a T_c-based coordinate system defined by

$$s = \frac{t}{T_c} \tag{14}$$

Then $t = sT_c$, $dt = T_c\, ds$.

The balanced design storm instantaneous rainfall intensities, $i^{\pm}(t^{\pm})$, can now be rewritten in terms of $s^{\pm}$ (analogous to $t^{\pm}$) by

$$i^{+}(t^{+}) = \left(\frac{1}{1-\theta}\right)^{b-1} ab(s^{+}T_c)^{b-1} = \left(\frac{T_c}{1-\theta}\right)^{b-1} i(s^{+}) \tag{15a}$$

and

$$i^{-}(t^{-}) = \left(\frac{T_c}{\theta}\right)^{b-1} i(s^{-}) \tag{15b}$$

Similarly, the $v^{\pm}(t^{\pm})$ functions can be rewritten in terms of coordinates $s^{\pm}$ by

$$v^{+}(t^{+}) = u(T_p{}^{-}t^{-}) = \frac{A}{T_c} U\left(\frac{T_p - t^{+}}{T_c}\right) = \frac{A}{T_c} U\,(t_p{}^{-}s^{+}) \tag{16a}$$

$$v^{-}(t^{-}) = u(T_p{}^{-}t^{-}) = \frac{A}{T_c} U\left(\frac{T_p + t^{-}}{T_c}\right) = \frac{A}{T_c} U\,(t_p{+}s^{-}) \tag{16b}$$

where $t_p = T_p/T_c$ is a constant for a given S-graph type.

Combining Eqs. 11 through 16 gives

$$Q_p = \left(\frac{1}{1-\theta}\right)^{b-1} A\int_0^{\frac{T_p}{T_c}} a(s^{+})^{b-1} \left(\frac{1}{T_c}\right) U(t_p{}^{-}s^{+})\, T_c\, ds^{+} + \left(\frac{T_c}{\theta}\right)^{b-1}$$

$$\int_0^{\eta\frac{T_p}{T_c}} a(s^{-})^{b-1} \frac{1}{T_c} U(T_p{+}s^{-})\, T_c\, ds^{-} - \phi A U_o \tag{17}$$

where it is recalled that it is assumed $i(\eta T_c) \geq \phi$.

Equation (17) is rearranged to give

$$Q_p = A \quad a(T_c)^{b-1} \left(\frac{1}{1-\theta}\right)^{b-1} \int_0^{\frac{T_p}{T_c}} b(s^{+})^{b-1}\, U(t_p{}^{-}s^{+})\, ds^{+}$$

$$+ \left(\frac{1}{\theta}\right)^{b-1} \int_0^{\eta - T_p} b(s^{-})^{b-1}\, U(t_p{+}s^{-}) ds^{-} - \phi\, U_o \tag{18a}$$

$$= A\,[a(t_c)^{b-1}\, \alpha - \phi U_o] \tag{18b}$$

where α is constant. For a given S-graph, and a given precipitation region where exponent "b" is a constant, then t_p and η are constants, and Eq. (18) can be simplified by including (5) as

$$Q_p = [\alpha I(T_c) - \phi U_o] A \tag{19}$$

where α is a constant for the given S-graph and precipitation region.

For English units, $U_o = 1.008$, which is simplified to be simply $U_o = 1$. Then,

$$Q_p = [\alpha I(T_c) - \phi] A \tag{20}$$

In comparison, a Rational Method peak flow rate estimator, for an equivalent mathematical structure for estimating rainfall excess by a phi-index (constant loss function), is

$$Q_R = [I(T_c) - \phi] A \tag{21}$$

Application

In (20), the single "calibration" constant, α, can be determined by equating (20) to (11a) for a single peak flow rate estimate (again, observing $i(\eta T_c) \geq \phi$). Several California Hydrology Manuals (see references) use two S-graphs, one for "Urbanized" and another for "Undeveloped" regions. By equating (20) to (11a), $\alpha = 0.99$ for the "Urbanized S-graph and $\alpha = 0.86$ for the "Undeveloped" S-graph. In these a determinations, the rainfall exponent (b) of Eqs. (4) to (6) was $b = 0.55$. Additionally, the constraint of $\eta T_c \geq \phi$ resulted in T_c limitations of 45-minutes to 180-minutes for 10-year to 100-year storm events (and typical loss rates of 0.4 inch/hour), respectively.

Constant Fraction Loss Rate

Another popular loss function is to use a constant proportion loss rate function, to estimate rainfall excess, given by

$$e(t) = ki(t) \tag{22}$$

Using (22) in the above development results in the balanced design storm UH procedure peak flow rate estimator, Q_p, given by

$$Q_p = k\alpha I(T_c)\ A \qquad (23)$$

where in (23), α is the same constant (and same values) used in (20), and the constraint of $i(\eta T_c) \geq \phi$ is eliminated. The corresponding well-known Rational Method peak flow rate estimator, Q_R, is

$$Q_R = kI(T_c)\ A \qquad (24)$$

From the above example, Eq. (23) results in

$$\left.\begin{array}{l} Qp = kI(Tc), \text{ for Urbanized areas} \\ Qp = 0.86\ kI(Tc), \text{ for Undeveloped areas} \end{array}\right\} \qquad (25)$$

where again in (25), the rainfall exponent is b=0.55.

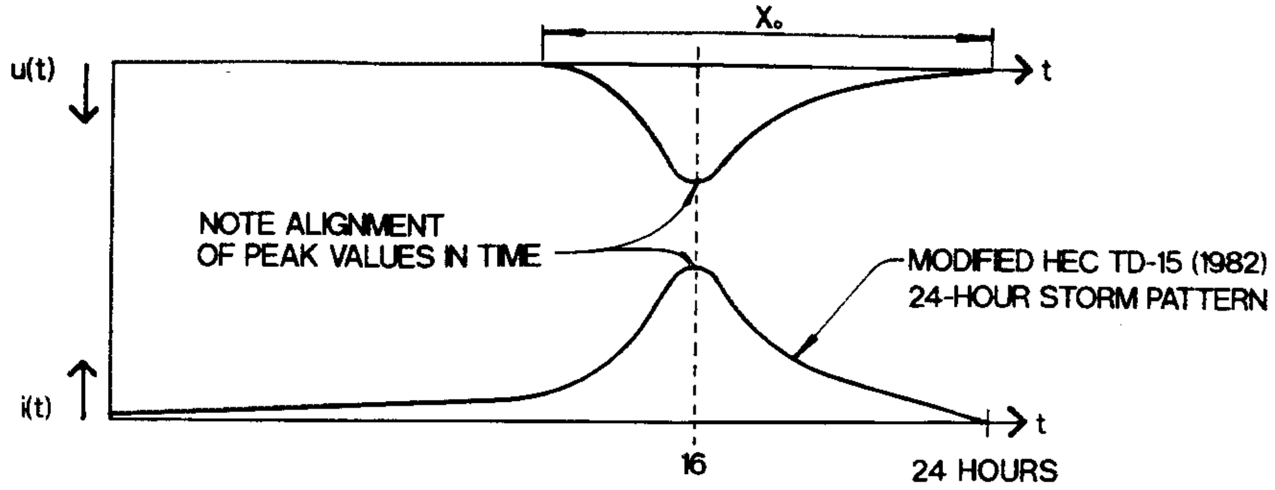

FIGURE 1

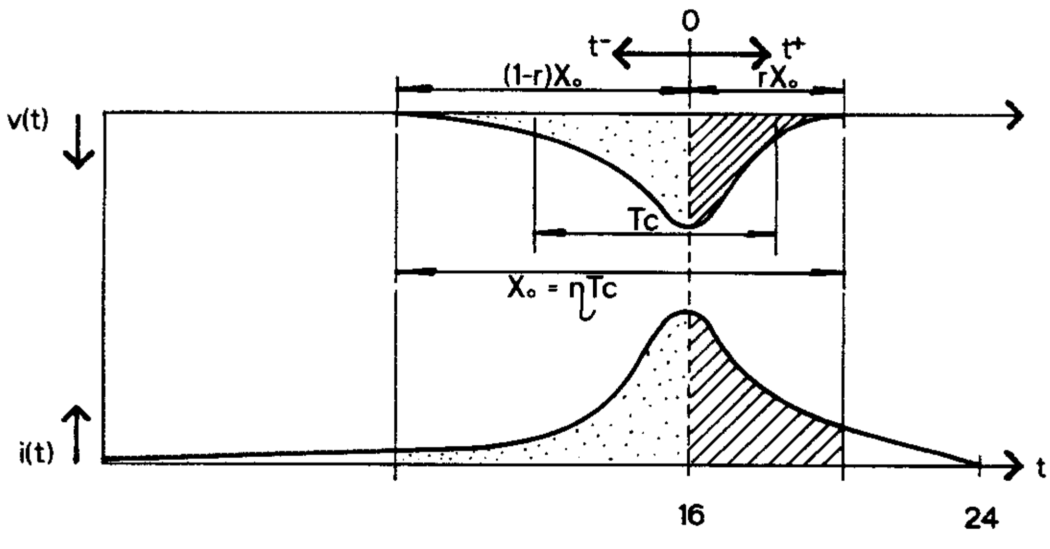

FIGURE 2

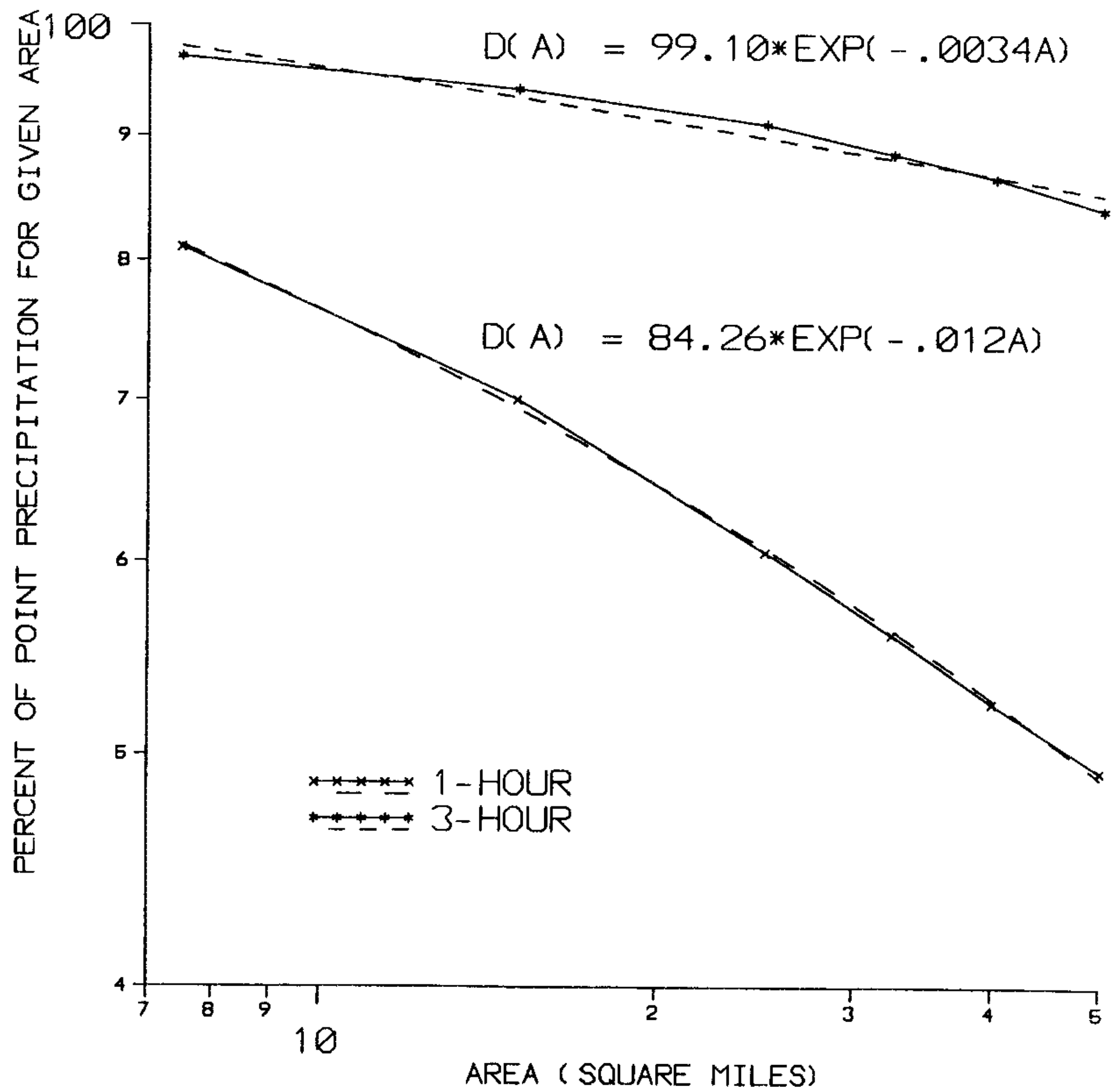

FIGURE 3. U.S. Army Corps of Engineers Sierra Madre Depth-Area Curves