Engineering Analysis with Boundary Elements 11 (1993) 233–237

CVBEM error reduction using the approximate boundary method

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(Received 3 September 1992; revised version received 5 February 1993; accepted 5 February 1993)

The complex variable boundary element method or CVBEM has been shown to be an effective computational tool for approximately two-dimensional problems involving the Laplace or Poisson equations. In this paper, the approximate boundary method (ABM) for error reduction in the CVBEM boundary condition approximation is further examined. A norm for evaluating accuracy and approximation improvement is used to assess the utility of the ABM.

Key words: CVBEM, boundary elements, approximate boundary, numerical methods

INTRODUCTION

The objective of this paper is to present the results of a study into the error reduction afforded by utilization of the approximate boundary method to refining approximations of the complex variable boundary element method or CVBEM. The CVBEM is a mathematical modeling technique that approximates solutions to boundary value problems, such as two-dimensional ideal fluid flow and steady-state heat transfer, which are governed by the two-dimensional Laplace equation. Given known and usually mixed conditions at specified points, or nodes, on the boundary of a singly or multiply connected domain, the CVBEM utilizes the Cauchy integral formula to produce an approximation function which is analytic on the interior of the problem domain, continuous on the problem boundary, and thus satisfies the two-dimensional Laplace equation. The approximation function produced by the CVBEM approaches the analytic solution as the number of specified nodal values increases and the boundary element lengths decrease.

THE COMPLEX VARIABLE BOUNDARY ELEMENT METHOD

Let $\omega(z) = \phi(x,y) + i\psi(x,y)$ be a complex variable function which is analytic on $\Gamma \cup \Omega$, where Ω is a simply connected domain enclosed by the simple closed boundary Γ. We define $\phi(x,y)$ to be the state variable and $\psi(x,y)$ the stream function, where ϕ and ψ are two-dimensional real valued functions. Since ω is analytic, ϕ and ψ are related by the Cauchy–Reimann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{1}$$

and thus satisfy the two-dimensional Laplace equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \text{ and } \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{2}$$

The Cauchy integral theorem states that if we know the value of the complex function ω on the boundary Γ, and if ω is analytic on $\Omega \cap \Gamma$, then ω on Ω is given by

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \in \Omega, \qquad z \notin \Gamma \tag{3}$$

The CVBEM forms $\hat{\omega}$, an approximation of ω, using known values of either ϕ or ψ on the boundary Γ, and uses the Cauchy integral (eqn (3)) to determine approximate values for ω on $\Omega \cup \Gamma$.

Let the boundary Γ be a polygonal line composed of V straight-line segments and vertices. Define nodal points with complex coordinates z_j, $j = 1, \ldots, m+1$ on Γ such that $m > V$. Nodal points are located at each vertex of Γ and are numbered in a counter-clockwise direction. Let Γ_j be the straight-line segment joining z_j and z_{j+1}, so that $\Gamma = \cup_{j=1}^{m+1}\Gamma_j$. Thus, $m+1$ boundary elements, Γ_j, are defined on Γ, where Γ_{m+1} connects

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nodal coordinates z_{m+1} and z_1. The CVBEM defines a continuous global trial function, $G(z)$, by

$$G(z) = \sum_{j=1}^{mH} N_j(z) = \sum_{j=1}^{m+1} (\bar{\phi}_j + i\bar{\psi}_j) \tag{4}$$

where, for a piecewise linear polynomial global trial function, and $j = 1, \ldots, m+1$, $N_j(z)$ is given by

$$N_j(z) = \begin{cases} \dfrac{z - z_{j-1}}{z_j - z_{j-1}} & z \in \Gamma_{j-1} \\ 0 & z \notin \Gamma_j \cup \Gamma_{j+1} \\ \dfrac{z_{j+1} - z}{z_{j+1} - z_j} & z \in \Gamma_j \end{cases} \tag{5}$$

and where $\bar{\phi}_j$ and $\bar{\psi}_j$ are nodal values of the two conjugate components, evaluated at z_j. An analytic approximation is then determined by

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G(a)\mathrm{d}a}{a - z}, \qquad z \in \Omega, \qquad z \notin \Gamma \tag{6}$$

If we let $q_{j,j+1}$ be the internal angle defined by nodal point coordinates z_j, z_{j+1}, and z_0, then for z_0 in Ω

$$\hat{\omega}(z_0) = \sum_{j=1}^{m+1} [\omega_{j+1}(z_0 - z_j) - \omega_j(z_0 - z_{j+1})] \frac{H_j}{z_{j+1} - z_j} \tag{7}$$

where ω_j and ω_{j+1} are nodal values at coordinates z_j and z_{j+1}, and

$$H_j = \ln \left| \frac{z_{j+1} - z_0}{z_j - z_0} \right| + iq_{j,j+1} \tag{8}$$

Since usually only one of the two specified nodal values $(\bar{\phi}_j, \bar{\psi}_j)$ is known at each z_j, $j = 1, \ldots, m+1$, values for the unknown nodal values must be estimated. The CVBEM develops a matrix system for use in solving for these unknown nodal values (see Ref. 1), solves the resultant matrix system, and uses these nodal values estimates along with the known nodal values in defining $\hat{\omega}(z)$ in eqn (7).

THE APPROXIMATE BOUNDARY METHOD

Once the CVBEM approximation function is produced, it is useful to examine and reduce the approximation error on the problem domain. Since modeling error is reduced as nodal points are added along the boundary,[1] a scheme to optimize the addition of nodes with respect to error reduction is needed. One method of doing this is the approximate boundary method (ABM). The maximum modulus theorem[2] guarantees that the maximum error in the approximation function occurs on the boundary of the problem domain. Thus, the problem of reducing the approximation error can be attacked as a problem of reducing modeling error on the boundary Γ. The ABM involves the construction of an appropriate boundary, $\hat{\Gamma}$, upon which the approximation function achieves the problem's boundary conditions, which then indicates where additional nodal points should be placed so that the geometric deviations between the approximation boundary, $\hat{\Gamma}$, from the true boundary, Γ, are small. Generally, the problem boundary Γ is a polygonal line which is well fitted by discretization into straight-line boundary elements. Consequently, the construction of $\hat{\Gamma}$ is analogous to plotting level curves of $\hat{\omega}(z)$ corresponding to the problem's boundary conditions along each boundary element. The additional nodal points are then used to produce a more refined CVBEM approximation.

Given the approximation function $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, we wish to reduce $|\hat{\omega}(z) - w(z)|$, $z \in \Omega$, the approximation error in modulus form. Reducing the approximation error on Γ reduces the error on Ω. The ABM attempts to develop a $\hat{\Gamma}$ that is geometrically 'close' to Γ. In other words, the goal is to minimize the maximum normal distance between $\hat{\Gamma}$ and Γ, with respect to Γ. To determine the approximation boundary at a point z_0 on Γ, locate the point $\tilde{z}_0$ along the normal to Γ at z_0 for which $\hat{\phi}(\tilde{z}_0) = \bar{\phi}(\tilde{z}_0)$ if ϕ is the known boundary condition, or $\hat{\psi}(\tilde{z}_0) = \bar{\psi}(\tilde{z}_0)$ if γ is the known boundary condition. Once $\hat{\Gamma}$ is determined, additional nodes are placed along Γ wherever $|\tilde{z}_0 - z_0|$ is large.

The ABM affords the user several distinct advantages. It is intuitively comforting to visually compare the approximate boundary to the true boundary and use geometric closeness as a measure of error. The user simply considers displacement of the problem boundary rather than using more abstract error approximation techniques. Nodal points can be easily added to the boundary, based solely on visual determinations. In this way, the approximate boundary provides a direct visual representation of the sensitivity of the approximation function in accommodating boundary conditions, variations in the boundary geometry, and the addition of nodal points.

The ABM is not, however, without difficulties. Using a direct application of the CVBEM, the approximation function has a value of zero on the exterior of the problem domain. Thus, an analytic continuation of the

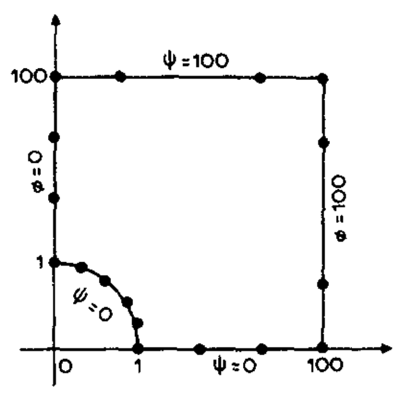

Fig. 1 The problem boundary for $z + 1/z$.

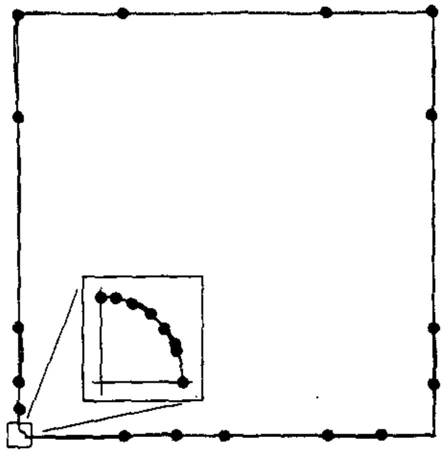

Fig. 2. The 25-node approximate boundary.

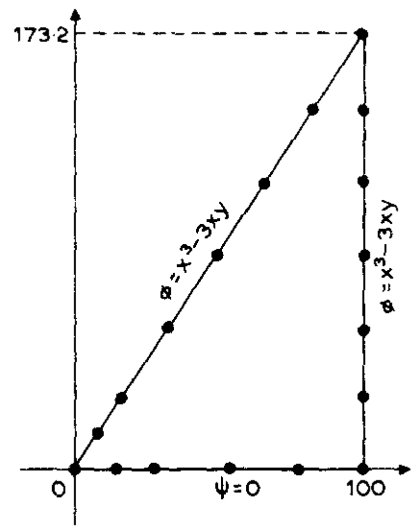

Fig. 4. The problem boundary for z^3.

approximation function to the exterior of the domain is needed in order to determine the approximate boundary. Another limitation of the ABM is the relative tedium and often difficulty in locating the approximate boundary between nodal points. These issues are discussed in the following application problems.

PROBLEM SETUP

Several sample problems were considered in this study to demonstrate the approximate boundary method. While the ABM uses geometric deviation of the approximate boundary, $\hat{\Gamma}$, from the problem boundary, Γ, as the determining factor in the placement of additional nodal points, the error of interest is the CVBEM approximation error $|\hat{\omega}(z) - \omega(z)|$, $z \in \Omega$. A useful measure of the approximation error is the maximum normed deviation of $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$ from $\omega(z) = \phi(z) + i\psi(z)$ over all z in Ω. However, since this maximum error must occur, in the limit, on the problem boundary, we use the error measure, E_n, given by

$$E_n = \|\hat{\omega}(z) - \omega(z)\| = \max_{z \in \Gamma} \sqrt{([\hat{\phi}(z) - \phi(z))^2 + (\hat{\psi}(z) - \psi(z))^2]} \quad (9)$$

where n is the number of nodes located on Γ. The objective of this study is to use the ABM with initial CVBEM approximations of the sample problems to obtain a 25-node and a 40-node approximation for each sample problem and to calculate E_n in each case.

The first step in using the ABM to refine the initial CVBEM approximation is to locate the approximate boundary for $\omega(z)$. Since $\omega(z) = 0$ for z on the exterior of $\Gamma \cup \Omega$, an analytic continuation of $\omega(z)$ is needed. The analytic continuation method used requires that branch cuts originating from each nodal point and extending outward, away from $\Gamma \cup \Omega$, be introduced so that the finite-series expansion of the approximation function can be for-

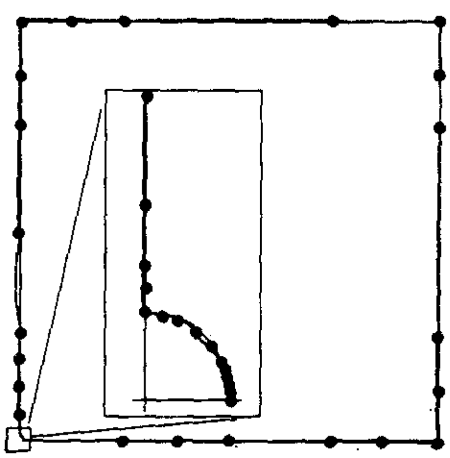

Fig. 3. The 40-node approximate boundary.

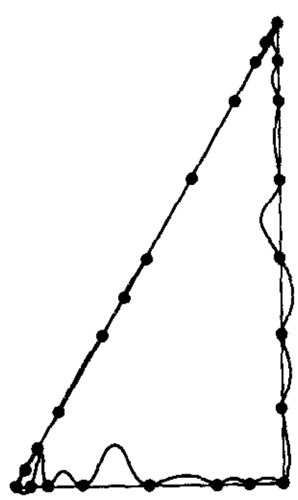

Fig. 5. The 25-node approximate boundary.

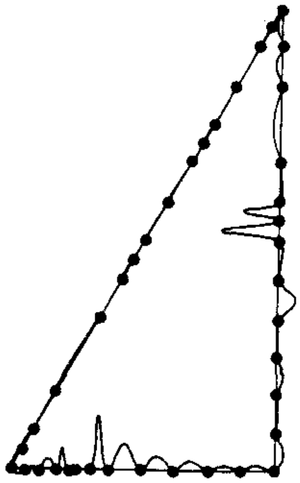

Fig. 6. The 40-node approximate boundary.

mulated.[1] In this way, the analytic continuation of $\omega(z)$ is available, except across each branch cut. Using the analytic continuation of $\omega(z)$, the approximate boundary is guaranteed to pass through the designated nodal points z_j, $j = 1, \dots m+1$ because each known nodal value (via the boundary conditions) is matched by $\hat{\omega}(z)$ evaluated at each node. Hence, locating the approximate boundary is accomplished by searching along the normal to Γ at points interior of each boundary element. In practice, the approximate boundary is determined at a single point between each node. Once $\hat{\Gamma}$ is found, additional nodal points are placed at locations where the normal distance with respect to Γ between $\hat{\Gamma}$ and Γ is large. Using these additional nodal points, a more refined CVBEM approximation function is formulated. This process is continued until a 25-node approximation function, $\omega_{25}(z)$, is produced and its error, E_{25}, calculated. Afterwards, the same process is used to add nodes until a 40-node approximation function, $\omega_{40}(z)$, is produced and its error, E_{40}, calculated.

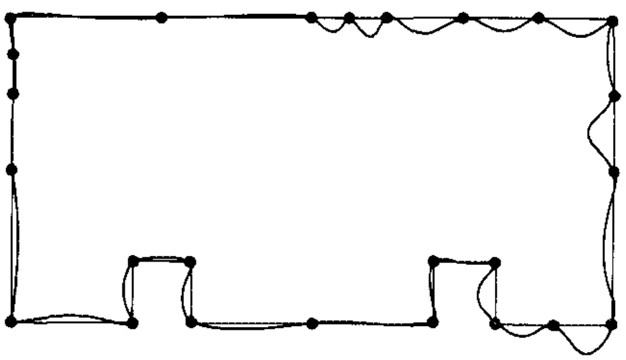

Fig. 8. The 25-node approximate boundary.

APPLICATIONS

The considered example problems have known solutions. However, the considered problems are formulated as mixed boundary value problems which require the estimation of unknown conjugate function values by means of the Cauchy integral, and the application of the ABM demonstrates the technique's utility. Because the application problems have known solutions, the CVBEM results can be examined for accuracy.

Example 1: $\omega(z) = z + 1/z$ (ideal fluid flow over a cylinder)

Figure 1 shows the problem boundary, the 17 nodes initially defined on Γ, and the known values of ϕ and ψ. Both ϕ and ψ are known at the points (100, 0), (100, 100), (0, 100), and (0, 1). This problem provides a good test case for demonstrating the ABM at locations along the problem boundary where streamlines and potential lines interface as part of the boundary conditions. The 25-node and 40-node approximate boundaries are shown in Figs 2 and 3, respectively. The maximum errors for each approximation are $E_{25} = 0{\cdot}3502$ and $E_{40} = 0{\cdot}2534$, indicating a 27·6% reduction in approximation error.

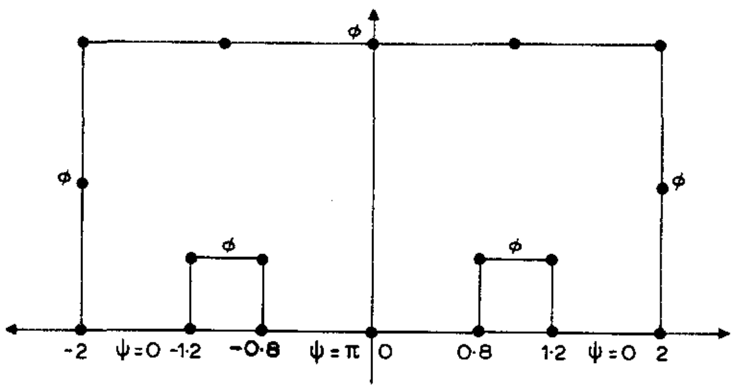

Fig. 7. The problem boundary for $\ln[(z+1)/(z-1)]$.

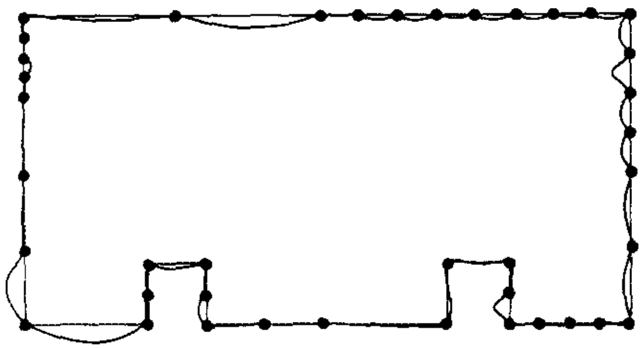

Fig. 9. The 40-node approximate boundary.

Example 2: $\omega(z) = z^3$ (ideal fluid flow around a corner)

Figure 4 shows the problem boundary, the 18 nodes initially defined on Γ, and the known values of ϕ and ψ. Both ϕ and ψ are known at the points (100, 0) and (0, 0). This example considers the ABM as applied to a large spatial domain problem where the boundary conditions change considerably, and nonlinearly, along the boundary. The 25-node and 40-node approximate boundaries are shown in Figs 5 and 6, respectively. The maximum errors for each approximation are $E_{25} = 162,326{\cdot}90$ and $E_{40} = 74,054{\cdot}67$, indicating a 54·5% reduction in approximation error. Note that the ABM plots (Figs 5 and 6) for the 25- and 40-node application, show a decrease in departure area, and in most cases, a decrease in departure magnitude, between Γ and $\hat{\Gamma}$. (Also see later Figs 8 and 9). Further research is needed to describe what such departures between Γ and $\hat{\Gamma}$ indicate, and whether changes in departure magnitude and in frequency imply approximation magnitude bounds.

Example 3: $\omega(z) = \ln[(z+1)/(z-1)]$ (steady-state heat transfer between a source and sink of equal strength)

Figure 7 shows the problem boundary, the 18 nodes initially defined on Γ, and the known values of ϕ and ψ. Both ϕ and ψ are known at the points (−2, 0), (−1·2, 0), (−0·8, 0), (0·8, 0), (1·2, 0), and (2, 0). The coupled source and sink problem involves a singularity at coordinates (−1, 0) and (1, 0). Consequently, this problem demonstrates the ABM in the vicinity of singularities of the logarithmic type. The 25-node and 40-node approximate boundaries are shown in Figs 8 and 9, respectively. The maximum errors for each approximation are $E_{25} = 3{\cdot}7076$ and $E_{40} = 3{\cdot}3743$, indicating a 9·0% reduction in approximation error.

Conclusions

The approximate boundary method is studied with respect to the usual relative error norm. Example applications demonstrate comparative uses of the approximate boundary technique; namely, an easy-to-use error indicator, and a tool for aiding additional nodal point placement. Because of the speed of available graphics programs, error displays analogous to the ABM may find increased use in computational mechanics applications.

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1.BASIC FORMULATIONS/MATHEMATICAL ASPECTS

93/6/II.1.065

Van Genderen, A.J. and Van der Meijs, N.P.

A frontal computation scheme for the Schur algorithm to efficiently solve large boundary-element problems, *CompEuro 1992*, The Hague, The Netherlands, 1992, 568-573

The Schur algorithm provides an approximate inverse for partially specified staircase band matrices. A frontal computation scheme is developed for the Schur algorithm that achieves an $O(b^2)$ space bound, where b is the width of the staircase band. This property allows certain classes of BE problems, such as those that occur with VLSI capacitance modelling, to be solved in linear time and using constant memory. An example is presented and experimental results are given to illustrate the algorithm.

93/6/II.1.066

Dumont, N.A. and de Souza, R.M.

The hybrid boundary element method for the analysis of solids, *Boundary Elements XV*, Vol.1, ed. C.A. Brebbia & J.J. Renčis, Computational Mechanics Publications, 1993, 551-564

The hybrid boundary element method, as developed in the Civil Engineering Department at PUC/RJ, may already be considered a well established formulation for problems of elasticity and potential. Among other publications, several articles have been presented at BEM International Conferences, since 1987, dealing with the basic theory, body forces, special applications and transient problems.

The present paper describes the implementation of a three-dimensional analysis program. In a first step, the basic equations are introduced and the most relevant numerical aspects are discussed. Then follows a general outline of the program, as regards stress analysis and post processing of results. Some examples are displayed for illustration of the capabilities of the program, mainly concerning the ease of data handling, a characteristic of the hybrid variational formulation implemented.

93/6/II.1.067

Hromadka, T.V. II and Whitley, R.J.

Expansion of the CVBEM into a series using intelligent fractals (IFs), *Boundary Elements XV*, Vol.1, ed. C.A. Brebbia & J.J. Rencis, Computational Mechanics Publications, 1993, 571-578

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the complex variable boundary element method (or CVBEM) into a series. The entire approximation effort can be written as a sum of Cauchy integrals of incremental changes in basis functions.

93/6/II.1.068

Huber, O., Lang, A. and Kuhn, G.

Evaluation of the stress tensor in 3D elastostatics by direct solving of hypersingular integrals, *Comp. Mech.*, 1993, **12**(1/2), 39-50

A new method of direct numerical evaluation of hypersingular BIs is applied to the differentiated form of the Somigliana-identity (hypersingular identity) in 3D elastostatics. Through this method it is possible to evaluate the stress tensor on the boundary of a complex 3D structure in a very accurate manner by employing the direct BEM. The geometry of the elements and their arrangements over the boundary of the structure are not subjected to any restrictions. Numerical examples are included.

93/6/II.1.069

Kamiya, N. and Andoh, E.

Standard eigenvalue analysis by boundary-element method, *Commun. Num. Meth. Eng.*, 1993, 9(6), 489-495

A method for analysing eigenvalues of the Helmholtz equation using a standard existing subroutine for eigenvalue determination is presented. It is based on the BIE formulation known as the multiple reciprocity method. The formulation, having polynomial matrices in terms of the eigenvalue as the coefficient matrices, is transformed into the standard type eigenvalue problem. The resulting formulation makes it possible to determine the required eigenvalues only by boundary discretization without any initial rough estimation.

93/6/II.1.070

Kamiya, N. and Andoh, E.

Helmholtz eigenvalue analysis by boundary element method, *J. Sound Vib.*, 1993, **160**(2), 279-287

A new and robust scheme for the eigenvalue analysis of the Helmholtz differential equation by the BEM is presented. Unlike the existing methods in which a highly complicated transcendental equation including the unknown wavenumbers appears, the developed method can reduce the computational task greatly with the help of the Multiple Reciprocity BE formualtion in terms of the fundamental solution for the Laplace equation and related simple calculations for polynomials. The Newton method is employed for determination of the desired eigenvalues. The solutions of 2D problems with various homogeneous boundary conditions are included.

93/6/II.1.071

Kamiya, N. and Koide, M.

Adaptive boundary element for multiple subregions, *Comp. Mech.*, 1993, 12(1/2), 69-80

The sample point error analysis and related adaptive BE refinement, proposed by one of the present authors, is extended to the problem with subregion partition which is often required for maintaining higher accuracy and for treatment of composite dissimilar materials. The study is devoted to regularization of the requirement that the interface between neighbouring subregions should be discretized by the unified criterion for the both, while, in general, the error influences on the point on the interface from one region differs from the other. Two examples concerning the 2D Laplace equation are included.

93/6/II.1.072

Lei, X.-Y., Wang, X. and Huang, M.-K.

An effective boundary element approach for higher order singularities, *Int. J. Solids Struct.*, 1993, **30**(15), 2109-2115

A new BE approach which introduces a 'source line' is presented. It is very effective for higher order singularity. The scheme is discussed in some detail in connection to plane elasticity. Numerical results for meshes with unequal BEs are included. Higher precision than the general BEM is obtained for both deflection and force.

93/6/II.1.073

Lu, P. and Mahrenholtz, O.

A modified variational boundary element formulation for potential problems, *Boundary Elements XV*, Vol.1, ed. C.A. Brebbia & J.J. Rencis, Computational Mechanics Publications, 1993, 565-570

This paper presents a modified variational boundary element