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THE RATIONAL METHOD FOR PEAK FLOW RATE ESTIMATION[1]

T. V. Hromadka II and R. J. Whitley[2]

ABSTRACT: The Rational Method continues to be the most widely used approach for estimating T-year return frequency peak flow rates for small catchments of about one square mile or less in area. The Balanced Design Storm unit hydrograph method is perhaps the second most widely used technique for estimating peak flow rates (and is the most widely used method for developing runoff hydrographs) but is generally considered to be more accurate than the Rational Method. In this paper, both of these T-year return frequency peak flow rate estimators are shown to be mathematically comparable. The close similarity between these two approximators may help explain why the Rational Method continues to be widely used even though other more computationally sophisticated techniques are readily available due to widespread computer software. (KEY TERMS: Rational Method; unit hydrograph; hydrologic models; surface water.)

INTRODUCTION

The unit hydrograph (UH) method is the most widely used approach for estimating a runoff hydrograph corresponding to a given rainfall event. Generally, a catchment UH is developed from a representative S-graph based upon a timing parameter called lag. The rainfall event is then modified by a loss function to estimate the catchment effective rainfall, which is then convoluted with the catchment UH (e.g., Hromadka *et al.*, 1987; Hromadka and Whitley, 1989; U.S. Army Corps of Engineers, 1982). The theoretical underpinnings of the UH method for both storm event simulation and design storm analysis is well documented (Hromadka and Whitley, 1989).

An important application of the UH method is the estimation of T-year return frequency peak flow rate values, Q_p, by use of the Balanced Design Storm concept (HEC Training Document TD-15; U.S. Army Corps of Engineers, 1982). An even more popular technique to estimate Q_p for small areas (where depth-area effects are negligible; usually, for areas less than about 1 square mile) is the Rational Method (e.g., Hromadka *et al.*, 1987). In this paper, the UH method will be used to derive the Rational Method equation, linking both of these methods for estimating peak flow rate values. The derived equations are then applied to Orange County, California, conditions for demonstration purposes.

PRELIMINARY DEVELOPMENT

A balanced design storm development for a T-year return frequency event simulation is provided in the U.S. Army Corps of Engineers HEC Training Document TD-15 (1982). In that development, rainfall T-year intensity-duration data are directly used to construct a balanced rainfall storm event such that the peak 5-minute rainfall depth is nested within the peak 10-minute rainfall depth and so forth until a 24-hour storm pattern is built. This storm pattern has the property that every peak duration of rainfall has a rainfall depth that has a T-year return frequency. Application of the UH method is also provided in TD-15, as are other topics such as loss rates, continuous depth-area adjustment, and the approximation of the convolution integral by a convolution matrix system analog.

For a computational unit time interval of Δt (e.g., 5-minutes), the design storm minus losses (i.e., the design storm effective rainfall) is developed as a sequence of unit effective rainfalls, $<f_i>$. Similarly, the

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[2]Respectively, Principal Engineer, Boyle Engineering Corporation, 1501 Quail St., Newport Beach, California 92658; and Professor, Department of Mathematics, University of California, Irvine, California 92717-9020.

UH is resolved into a sequence of UH runoffs, $<v_i>$. The corresponding sequence of unit runoffs ($<q_i>$ are developed by use of a convolution matrix system as demonstrated in Equation (1):

$$\begin{bmatrix} v_1 & 0 & 0 & 0 \cdots \\ v_2 & v_1 & 0 & 0 \\ v_3 & v_2 & v_1 & 0 \\ v_4 & v_3 & v_2 & v_1 \\ v_5 & v_4 & v_3 & v_2 \\ v_6 & v_5 & v_4 & v_3 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ \vdots \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ \vdots \end{pmatrix} \tag{1}$$

The sequence of unit runoffs, $<q_i>$, forms the runoff hydrograph corresponding to the balanced T-year return frequency design storm under study. The maximum value of the $\{q_i\}$, denoted by Q_p, is the peak flow rate estimate.

Because of the single peak shapes of both the T-year balanced design storm and the unit hydrograph, each has a largest unit interval value. Let e_1 be the maximum of the above set of unit effective rainfalls, $\{f_1, f_2, \ldots\}$. Let e_2 be the next largest effective rainfall. Similarly, e_i is the i^{th} largest value in the set $\{f_1, f_2, \ldots\}$. Similarly, let u_i be the i^{th} largest value in the set $\{v_1, v_2, \ldots\}$.

In general, for a UH of n Δt unit intervals, the peak flow rate from Equation (1) involves the sum of products of the u_i with the n largest e_i unit values. For example, a typical convolution matrix system (of Equation 1) written in terms of the above e_j and v_i unit values is:

$$\begin{bmatrix} u_7 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\ u_5 & u_7 & 0 & 0 & 0 & 0 & 0 \\ u_4 & u_5 & u_7 & 0 & 0 & 0 & 0 \\ u_2 & u_4 & u_5 & u_7 & 0 & 0 & 0 \\ u_1 & u_2 & u_4 & u_5 & u_7 & 0 & 0 \\ u_3 & u_1 & u_2 & u_4 & u_5 & u_7 & 0 \\ u_6 & u_3 & u_1 & u_2 & u_4 & u_5 & u_7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{pmatrix} e_5 \\ e_3 \\ e_1 \\ e_2 \\ e_4 \\ e_6 \\ e_7 \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Q_p \\ \vdots \end{pmatrix} \tag{2}$$

where Q_p is the peak flow rate given in this example by

$$Q_p = e_5u_6 + e_3u_3 + e_1u_1 + e_2u_2 + e_4u_4 + e_6u_5 + e_7u_7 + \ldots \tag{3}$$

Rearranging Equation (3) into decreasing magnitude terms with respect to the e_i values,

$$Q_p = e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_6 + e_6u_5 + e_7u_7 + \ldots \tag{4}$$

From Equation (4), the Q_p estimate is suggested by the upper bound Q_p (from the UH analog) given by the inner product

$$Q_p \leq \sum_{i=1}^{n} e_iu_i \tag{5}$$

where from the Equation (4) example, the upper bound value is obtained by simply rearranging the relatively small 5th and 6th terms. That is,

$$Q_p \approx \sum_{i=1}^{n} e_iu_i \tag{6}$$

The last result (Equation 6) will be used to derive a Rational Method peak flow rate equation, unifying the Q_p estimators of both the UH and Rational Methods. In order to proceed with the mathematical development, a direct approach for deriving the sequence of monitonically decreasing values of the e_i and u_i will be presented which addresses the case of continuous functions for both the effective rainfall, e(t), and the unit hydrograph, u(t), rather than using a discrete unit interval set of $\{e_i\}$ and $\{u_i\}$ as utilized in the previous discussion.

Rearranged Effective Rainfall Formulation

Design storm effective rainfall, or rainfall minus losses, is developed herein as a rearrangement (Hardy *et al.*, 1967) of the effective rainfall design storm pattern constructed from the T-year nested rainfall procedure outlined in TD-15 (U.S. Army Corps of Engineers, 1982). In order to develop effective rainfall quantities, a relationship for rainfall versus peak storm duration is needed.

Rainfall depth, of return frequency T, for storm peak duration t, may be described for short durations (usually less than about three hours) by

$$D(t) = at^b,\ a \text{ and } b > 0 \tag{7}$$

where D(t) = precipitation depth of return frequency T; and t = peak duration, in minutes. Figure 1 provides various T-year D(t) plots for Orange County, California.

Mean rainfall intensity, I(t), inches/hour for peak duration t, and return frequency T is obtained from Equation (7) by

$$I(t) = \frac{60}{t} D(t) = 60\ at^{b-1} \tag{8}$$

Equation (8) is the usual, rainfall intensity-duration relationship used in Rational Method studies. Instantaneous T-year rainfall intensity, i(t), is obtained from Equation (7) by

$$i(t) = 60 \frac{d}{dt} D(t) = 60\ abt^{b-1} = bI(t) \tag{9}$$

Figure 2 shows i(t) for T = 100 years based upon Figure 1.

Instantaneous effective rainfall, e(t), is obtained from Equation (9) for a phi-index (i.e., constant) loss function approach (e.g., Hromadka and Whitley, 1989) by

$$e(t) = i(t) - \phi,\ i(t) > \phi \tag{10}$$

where e(t) = instantaneous effective rainfall rate at peak storm time, t; and ϕ = phi-index, a constant.

Similarly, for instantaneous effective rainfall given as a constant fraction of rainfall,

$$e(t) = Ci(t) \tag{11}$$

where C is a positive constant greater than zero.

It is noted that the effective rainfall formulations of Equations (10) and (11) are both monitonically decreasing rearrangements of the effective rainfall design storm pattern of TD-15 (U.S. Army Corps of Engineers, 1982), or other nested design storm pattern approaches that provide consistent return frequency rainfalls for all durations. It is also noted that other effective rainfall formulations can be developed analogous to the above.

Rearranged S-Graph Formulation

For Southern California, the "Valley-Developed" (Figure 3) S-Graph is representative of fully urbanized regions with extensive storm drain systems. It is equivalent to the U.S. Army Corps of Engineers' "LACDA Urban" S-Graph (Hromadka *et al.*, 1987; Hromadka and Whitley, 1989), and closely approximates the standard SCS unit hydrograph. This S-Graph can be rearranged analogous to the depth-duration relationship used for rainfall. A log-log best fit plot provides an approximation of the resulting mass curve by

$$M(\ell) = c\ell^d \tag{12}$$

where $M(\ell)$ = peak mass (percent) of the S-Graph for peak time duration of ℓ; and ℓ = catchment lag, in percent. Figure 4 shows a log-log fit of Figure 3 in a monitonically decreasing rearrangement form, $M(\ell)$, resulting in $M(\ell) = 2.42\ \ell^{0.68}$. By ignoring the last 10 percent of mass in Figure 4, Figure 5 provides a log-log fit of $M(\ell) = 1.42\ell^{0.84}$ which is used in the following development. Figure 6 provides a cubic polynomial fit to $M(\ell)$ by $-0.77 + \ell - 0.003\ \ell^2 + 0.00004\ \ell^3$. In both fits, agreement is close for the initial 80 percent of UH mass.

Catchment lag is related to the usual catchment time of concentration, Tc, by

$$\ell = \left(\frac{t}{Tc}\right)\left(\frac{100\%}{0.8}\right) \tag{13}$$

where lag = 0.8Tc by calibration to rainfall-runoff data; and Tc is the time of concentration (i.e., the sum of normal depth travel times) in minutes. Other linear relationships may be used in Equation (13) of the form lag = γTc, γ a constant.

From Equation (12), M(t) is given by

$$M(t) = c\left(\frac{100}{0.8}\right)^d \left(\frac{t}{Tc}\right)^d = c(125)^d \left(\frac{t}{Tc}\right)^d \tag{14}$$

and the rate of change in M(t) is

$$\frac{dM(t)}{dt} = \frac{125^d\, cdt^{d-1}}{Tc^d} \tag{15}$$

Regression Equations: $D(t) = at^b$
(D= Depth in inches, t= duration in minutes)

Return Frequency (years)	a	b
2	0.095	0.426
5	0.131	0.438
10	0.170	0.427
25	0.200	0.434
50	0.225	0.434
100	0.259	0.427

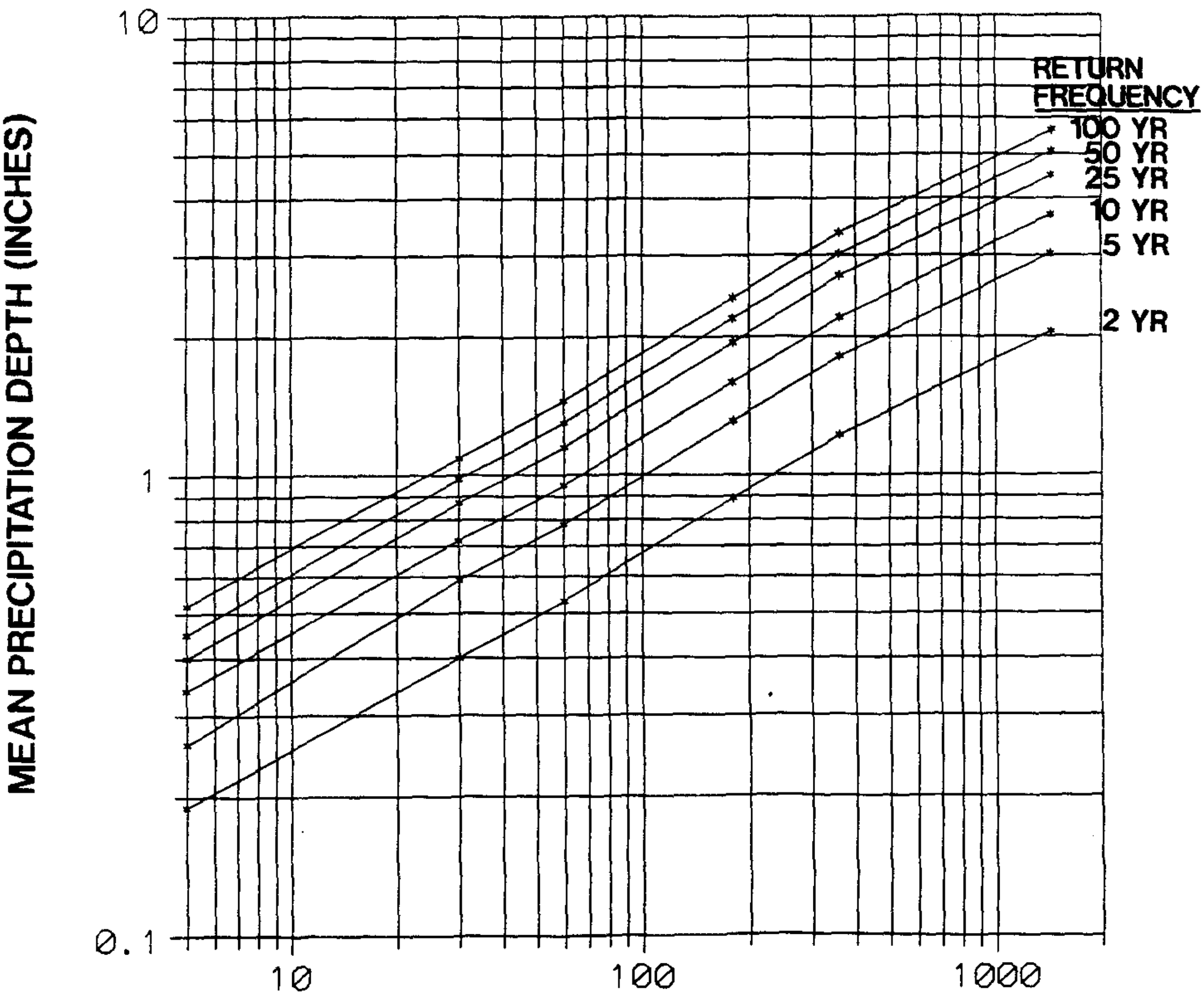

Figure 1. Precipitation Depths Versus Peak Storm Duration in Orange County, California.

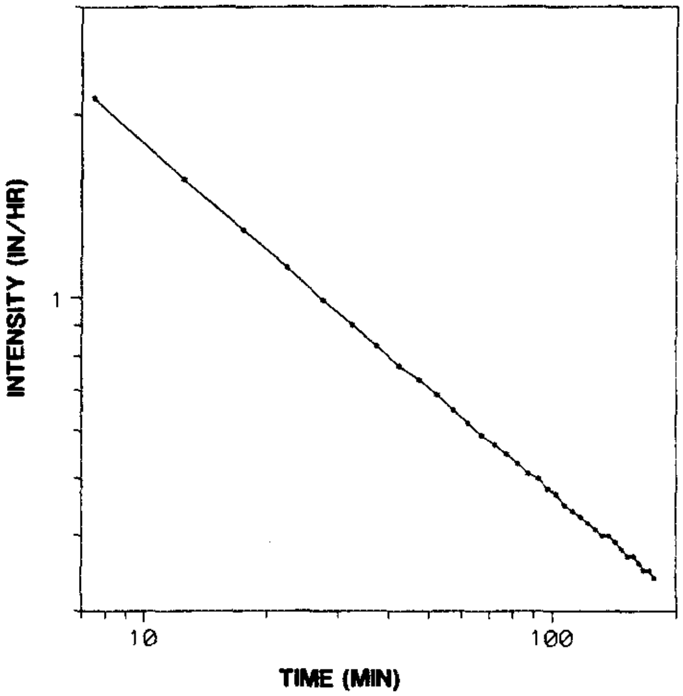

Figure 2. Instantaneous Rainfall Intensity, i(t), for T = 100 Years.

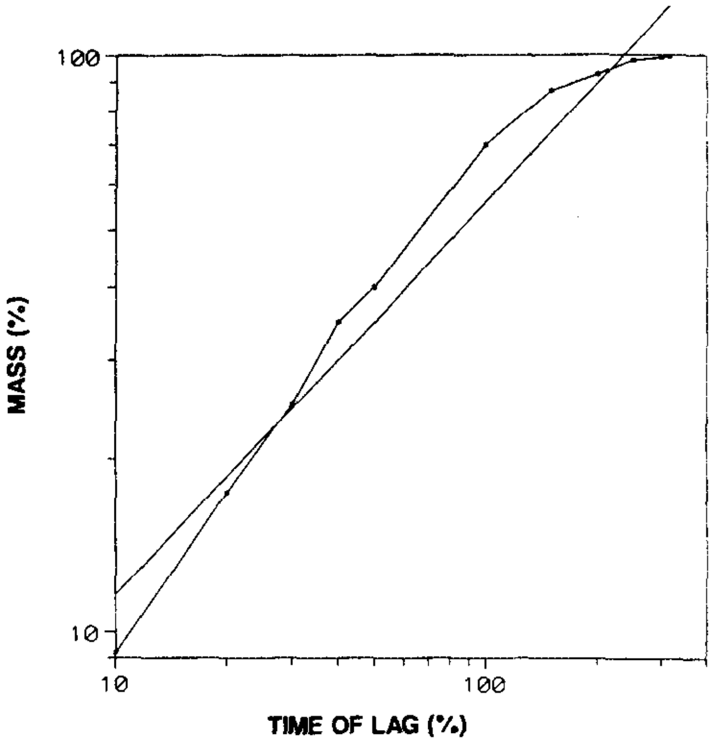

Figure 4. Log-Log Plot of Rearranged S-Graph of Figure 3.

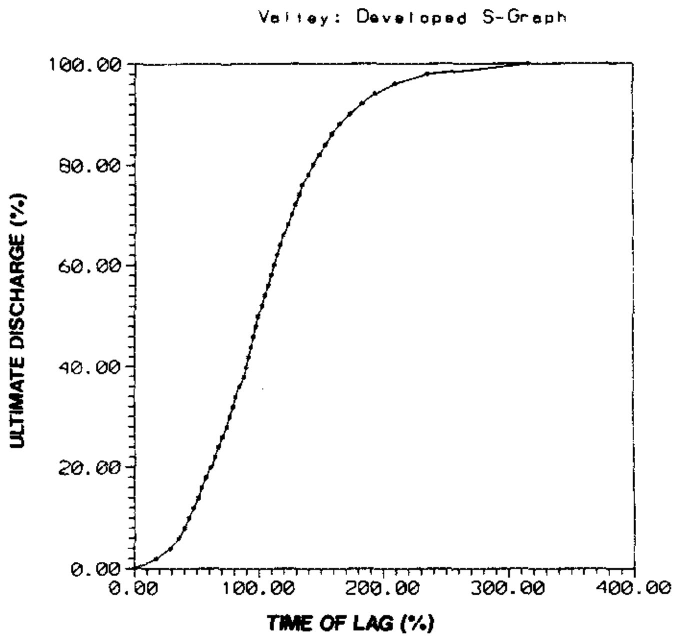

Figure 3. "Valley-Developed" S-Graph or "LACDA Urban" S-Graph for UH Development.

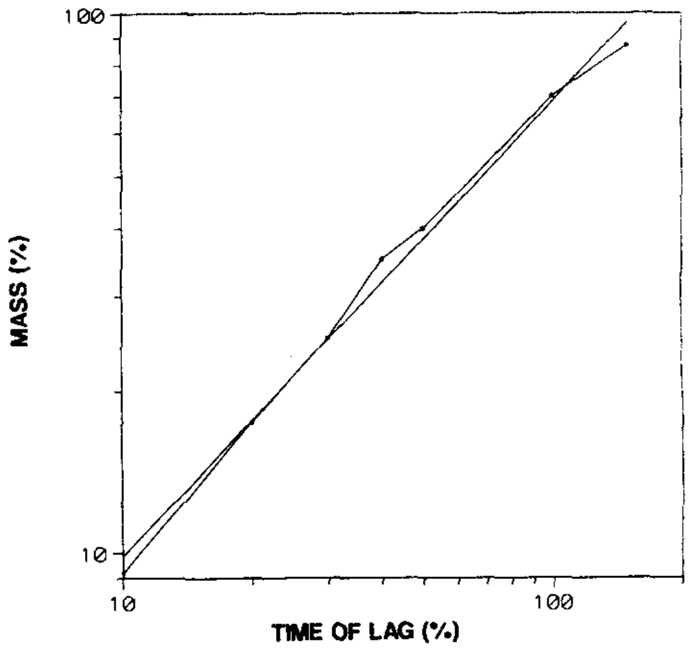

Figure 5. Log-Log Plot of Peak 80 Percent of Rearranged S-Graph of Figure 4.

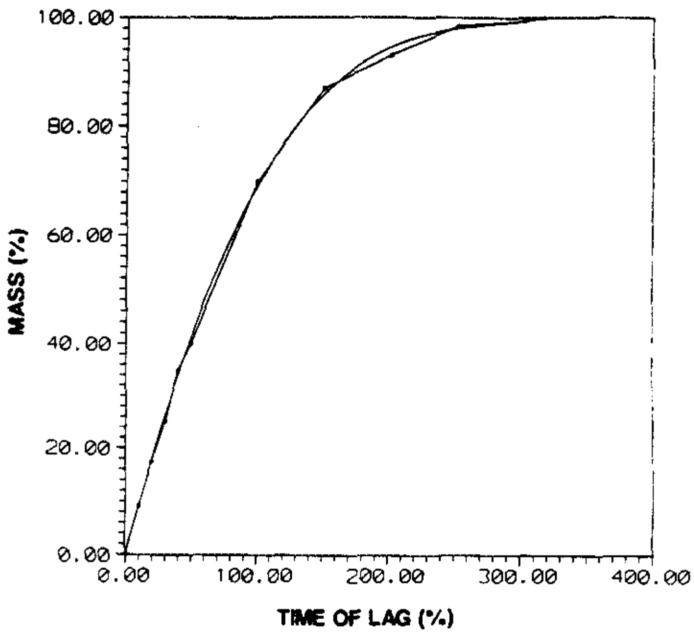

Figure 6. Cubic Polynomial Fit of Rearranged S-Graph of Figure 3.

Peak Flow Rate Formulation

The peak flow rate, Q_p, is related to the product of the above rearranged e(t) and M(t) functions by

$$Q_p(x) \le \frac{1.0083}{100} A \int_0^x e(t) \frac{dM}{dt} dt \tag{16}$$

where A = catchment area; 1.0083 = conversion factor for inch/hour (from e(t) to cfs) (Imperial units used due to familiar form of Rational Equation); $Q_p(x)$ = peak flow rate, for peak storm duration x, in minutes; and 100 = conversion factor for percent to decimal (from M(t)). Combining Equations (9), (10), (15), and (16),

$$Q_p(x) \le \frac{1.0083}{100} A \int_0^x \left(60abt^{b-1} - \phi\right) \frac{125^d cd}{Tc^d} t^{d-1} dt \tag{17}$$

Simplifying Equation (17),

$$Q_p(x) \le \left(\frac{1.0083}{100}\right) \frac{A\,125^d cdx^d}{Tc^d} \left[\frac{60abx^{b-1}}{b+d-1} - \frac{\phi}{d}\right] \tag{18}$$

Example

In Orange County, California, and T = 100 years, a = 0.259, b = 0.427, c = 1.42, d = 0.84; thus from Equation (8), $I(Tc) = 15.54(Tc)^{-0.573}$.

In order to develop Q_p, we must find the value of x_o such that M(x) = 100 percent in Equation (14),

$$100 = c(125)^d \left(\frac{x_o}{Tc}\right)^d \tag{19}$$

giving

$$x_o = \left(\frac{Tc}{125}\right)\left(\frac{100}{c}\right)^{\frac{1}{d}} \tag{20}$$

For Orange County,

$$x_o = 1.267Tc\,. \tag{21}$$

Substituting Equation (21) into Equation (18) gives

$$Q_p \le \left(\frac{1.0083}{100}\right) \frac{A\,125^d cd\left(\frac{Tc}{125}\right)^d \left(\frac{100}{c}\right)}{Tc^d} \left[\frac{60ab\left(\frac{Tc}{125}\right)^{b-1}\left(\frac{100}{c}\right)^{\frac{b-1}{d}}}{b+d-1} - \frac{\phi}{d}\right] \tag{22}$$

or

$$Q_p \le (1.0083)A \left[\frac{bd\left(\frac{100}{c}\right)^{\frac{b-1}{d}}}{125^{(b-1)}(b+d-1)} I(t) - \phi\right] \tag{23}$$

A limitation to the above results (for the phi-index loss function) is that $i(t) > \phi$. For x_o (see Equation 19 to Equation 21) = ηTc, $\eta > 1$, this implies $bI(x_o) > \phi$. This is satisfied when

$$Tc < \frac{1}{\eta}\left[\frac{\phi}{60ab}\right]^{\frac{1}{b-1}} \tag{24}$$

For Orange County and T = 100 years, a = 0.259, b = 0.427, η = 1.267, and Equation (24) is given by the requirement

$$Tc < 21.45\,\phi^{-1.745}\ . \tag{25}$$

For ϕ = 0.3 inches/hour (average conditions in Orange County), TC < 175 minutes, which is generally satisfied for small catchment areas.

Comparison of Rational Method and UH Peak Flow Rate Formulations

For the phi-index loss function, the usual Rational Method peak flow rate estimate, Q_R, is given by

$$Q_R = 1.0083A\,(I-\phi) \tag{26}$$

where I = I(Tc).

In comparison, from Equation (23)

$$Q_p \le 1.0083A\left[\left(\frac{bd\left(\frac{100}{c}\right)^{\frac{b-1}{d}}}{125^{b-1}(b+d-1)}\right)I-\phi\right] = 1.0083A\,[\alpha I-\phi] \tag{27}$$

where α follows from the equation.

Similarly, for a constant proportion loss function, the Rational Method peak flow rate, Q_R, is

$$Q_R = 1.0083\ ACI \tag{28}$$

whereas from the above UH development,

$$Q_p \le 1.0083AC\left[\frac{bd\left(\frac{100}{c}\right)^{\frac{b-1}{d}}}{125^{b-1}(b+d-1)}\right]I = 1.0083AC\alpha I \tag{29}$$

where again, I = I(Tc), and α is the same value as in Equation (27). It is noted from Equations (27) and (29) that α is dependent upon the values of parameters c and d (which describe the rearranged S-Graph characteristics) and the rainfall parameter b. The b value describes the slope of the log-log rainfall intensity-duration plot, and is generally a constant for large regions (e.g., b = 0.427 for the entire Orange County, California).

In Orange County, California, for the "Valley-Developed" S-Graph of Figure 3,

$$Q_p\ (\text{Equation } 27) \le 1.0083A\,(1.173I-\phi) \tag{30}$$

$$Q_p\ (\text{Equation } (29) \le 1.0083AC\,(1.173)\,I \tag{31}$$

which implies α = 1.173.

A comparison of the UH peak flow rate estimators to the corresponding Rational Method estimators show a very similar structure, which may partially explain why the Rational Method continues to be widely used in design practice due to its acceptable results.

It is noted that Equations (27) and (29) are maximal estimators (see Equations 2 through 6). Consequently, the coefficient of 1.173 seen in Equations (30) and (31) is an upper bound. Additionally, the log-log approximation used in Equation (12) is also a conservative approximation. By a further analysis of Equation (16), an evaluation of Q_p is possible. Define the function $\chi^*(x,\phi,Tc)$ by

$$\chi^*(x,\phi,Tc) = \int_o^x (i(t)-\phi)u(t,Tc)\,dt \tag{32}$$

where

$$u(t,Tc) = \left.\frac{dM(s)}{ds}\right|_{Tc,s=t}$$

Then Equation (32) is a restatement of Equation (16), where the asterisk notation refers to the upper bound.

For any time t, the u(t, Tc) function can be rescaled for another time of concentration value, Tc′, by

$$u(t,Tc') = u(t,Tc)\left(\frac{Tc}{Tc'}\right)^{d-1} \tag{33}$$

Consequently, combining Equations (32) and (33),

$$\chi^*(x,\phi,Tc') = \left(\frac{Tc}{Tc'}\right)^{d-1}\int_o^x (i(t)-\phi)u(t,Tc)\,dt \tag{34}$$

From Equation (34), variation in the catchment Tc (or lag) results in a constant proportional change in the integration.

In Equation (32), u(t, Tc) is a result of rearranging (see Equations 2 to 6) the original UH v(t,Tc). Consequently,

$$v(t,Tc) = u(g(t), Tc) \tag{35}$$

where g(t) is a remapping of the point order of u(t, Tc). Define ε(t,Tc) by

$$\varepsilon(t,Tc) = \frac{u(g(t),Tc)}{u(t,Tc)}, \text{ for } u(t,Tc) > 0 \tag{36}$$

Then from Equations (15) and (32),

$$\varepsilon(t,Tc) = \frac{cd125^d(g(t))^{d-1}}{Tc^d} \Big/ \frac{cd125^d t^{d-1}}{Tc^d} = \left(\frac{g(t)}{t}\right)^{d-1} \tag{37}$$

and ε(t, Tc) = ε(t), a function of t only. Thus, ε(t) is independent of ϕ and Tc, and ε(t) is infinitely discontinuous, although bounded and measurable (Hardy *et al.*, 1967).

Combining equations, a function χ(x, ϕ, Tc) is given by

$$\chi(x,\phi,Tc) = \int_0^x (i(t) - \phi)u(t,Tc)\varepsilon(t)\,dt \tag{38}$$

where u(t, Tc) = 0 for x > x_o, and M(x_o, Tc) = 100 percent.

Thus

$$Q_p = \chi(x_o,\phi) = \int_0^{x_o} i(t)u(t,Tc)\varepsilon(t)dt - \phi\int_0^{x_o} u(t,Tc)\varepsilon(t)dt \tag{39}$$

where in Equation (39) x_o and Tc are known simultaneously.

The second integral of Equation (39) is solved by

$$\phi\int_0^{x_o} u(t,Tc)\varepsilon(t)dt = \phi(M(x_o) - M(0)) = \phi A \tag{40}$$

The first integral of Equation (39) is solved by using the mean value theorem (Hardy *et al.*, 1967)

$$\int_0^{x_o} i(t)u(t,Tc)\varepsilon(t)dt = \varepsilon_o\int_0^{x_o} i(t)u(t,Tc)\,dt \tag{41}$$

where the right-hand integral of Equation (41) is solved previously (see Equation 18), and ε_o depends on x_o, where x_o - ηTc.

From Equations (40) and (41), the resulting Qp formula is

$$Q_p = 1.0083\,\varepsilon_o\,(I(Tc) - \phi)A \tag{42}$$

To evaluate ε_o, and its variation with respect to x_o, numerous Q_p estimates were developed by using HEC-1 UH procedures, for various Tc, ϕ, and the S-graph of Figure 3. Again, the limitations imposed by Equation (24) were observed. From the simulations, it was concluded that

$$0.97 < \varepsilon(\eta Tc) < 1.02, \text{ with mean of } 0.98 \text{ and standard deviation of } 0.01 \tag{43}$$

that is,

$$Q_p = (I(Tc) - \phi)A; \text{(Valley-Developed)}. \tag{44}$$

Similarly, using the constant proportion loss function for Equation (28),

$$Q_p = CIA; \text{(Valley-Developed)}. \tag{45}$$

It is noted that Equations (44) and (45) are for the "Valley-Developed" S-Graph. Another Orange County, California, S-Graph is the "Valley-Undeveloped." For the latter S-Graph,

$$Q_p = (0.86I(Tc) - \phi)A; \text{(Valley-Undeveloped)} \tag{46}$$

or

$$Q_p = 0.86\,CIA; \text{(Valley-Undeveloped)} \tag{47}$$

where in this case, $0.83 < \varepsilon_o < 0.89$ with mean 0.86 and standard deviation of 0.01. The above two Q_p formulae (Equations 44 and 46; or Equations 45 and 47) represent two different S-graph (unit hydrograph) response functions. Other S-Graph types (e.g., "Mountain," "Desert") would have different parameter values.

CONCLUSIONS

The Rational Method peak flow rate estimator is shown to closely match the Balanced Design Storm unit hydrograph estimator for peak flow rate given loss functions of the phi-index type or constant fraction loss type (among others). The Rational Method equation can be significantly improved by including an additional multiplicative constant that corresponds to the S-Graph or unit hydrograph type (e.g., SCS, Mountain, Desert, Valley, etc.). Both modeling techniques are shown to result in very similar mathematical structures in the estimate of the T-year return frequency peak flow rate. This similarity in structure provides additional foundation and insight nto the application of the widely used Rational Method.

Because the Balanced Design Storm unit hydrograph method has gained widespread use among floodplain management and flood control agencies, the direct linkage to the simple Rational Method equation should provide more insight as to that equation's appropriateness, as well as a similar level of confidence in its usage. The Rational Method can now be applied for a wide variety of unit hydrograph responses by simply developing a single multiplicative constant (that depends only on the type of unit hydrograph). A table containing S-Graph type versus the peak flow rate adjustment factor value can be readily made and included in watershed management hydrology manuals or similar documents.

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