Numerical Solutions of the Dirichlet Problem via a Density Theorem

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Received 19 January 1993; revised manuscript received 30 August 1993

Under mild conditions a certain subspace M, consisting of functions which are analytic in a simply connected domain Ω and continuous on the boundary Γ, is shown to have real parts which are dense, in the sup norm, in the set of all solutions to the Dirichlet problem for continuous boundary data. Similar results hold for L^p boundary data. Numerical solutions of sample Dirichlet problems are computed. © 1994 John Wiley & Sons, Inc.

I. INTRODUCTION

Boundary value problems which involve Laplace's equation in two dimensions can be solved numerically by a complex variable-based boundary element technique. This complex variable boundary element method (CVBEM) has been used to solve problems in groundwater flow [1], prediction of freezing front in soils [2], steady-state heat contaminant transport [3], St. Venant torsion [4], steady-state heat transfer [5], and other problems [6]. Also see the problems discussed in Chap. 1 of [7].

We establish constructive existence theorems for Dirichlet problems which apply to continuous boundary data or discontinuous L^p boundary data on a simply connected domain with a simple closed piecewise continuously differentiable boundary of finite length, allowing the consideration of domains with boundary corners which arise frequently in applications. The approximating function is shown to be the real part of a function analytic in the domain and continuous on its closure; the fact that this function is known throughout the domain, and not just at a series of mesh points, is of basic importance for many applications, e.g., in computing conformal maps, which we will discuss in a later paper, or in computing the torsional rigidity of a domain (among other applications), for which see below.

Sample computational results are given in Sect. III.

Numerical Methods for Partial Differential Equations, 10, 369–381 (1994)

CCC 0749-159X/94/030369-13

II. DENSITY THEOREMS

Let Ω be a simply connected domain in the complex plane with piecewise continuously differentiable boundary which is a simple closed curve of finite length, with parametrization

$$\varphi: [0,1] \longrightarrow \Gamma .$$

It is assumed that the map φ is continuous on $[0, 1]$ one-to-one on $[0, 1)$ with $\varphi(0) = \varphi(1)$, and is continuously differentiable, with nonzero derivative, except at a finite number of parameter points $\{c_1, \ldots, c_m\}$ corresponding to corners which are not cusps; at the corners, the right-hand and left-hand limits of the derivative exist, are not zero, and satisfy the condition $\varphi'(c_j^+) + \varphi'(c_j^-) \neq 0$ that c_j not be a cusp for each c_j.

A condition basic to the development is that for each point b on Γ, there be a continuous non-self-intersecting path Pb, joining b to infinity, which lies in the complement of $\Omega \cup \Gamma - \{b\}$; this property of the domain is slightly stronger than the domain being simply connected [8]. Use $Pb - b$, a curve joining 0 to infinity, as a branch cut to define a branch of the logarithm, $\log_{Pb}(w)$, analytic for w in the complex plane but not on the branch cut $Pb - b$, and thereby define the function

$$f_b(z) = (z - b) \log_{Pb}(z - b) \tag{1}$$

analytic on Ω and continuous on $\Omega \cup \Gamma$. Functions similar to (1), without consideration of the branch cuts, appear in the solution of various potential problems; see, for example, [9], p. 285.

Let M be the complex linear space spanned by the functions 1, z, and $f_b(z)$ for all b in Γ:

$$M = \text{sp}\{1, z, f_b(z) \colon b \text{ in } \Gamma\}. \tag{2}$$

Note that each function in M is analytic on Ω and continuous on $\Omega \cup \Gamma$.

Theorem 1. *Let g be a given continuous real-valued function defined on Γ and let $\varepsilon > 0$ be given. There is a function h(z) belonging to the subspace M described by (2) with*

$$\sup\{|\text{Re}[h(z)] - g(z)| \colon z \text{ on } \Gamma\} < \varepsilon . \tag{3}$$

Proof. Let M_r be the real vector space spanned by 1, Rez, Imz, and the real and imaginary parts of all the functions f_b,

$$M_r = \text{sp}\{1, \text{Re}z, \text{Im}z, \text{Re}f_b(z), \text{Im}f_b(z) \colon \text{for all } b \text{ in } \Gamma\}. \tag{4}$$

From the equations

$$\gamma_0 + \alpha_0 \text{Re}z + \beta_0 \text{Im}z + \sum_{j=1}^{m} \alpha_j \text{Re}f_{b_j}(z) + \beta_j \text{Im}f_{b_j}(z)$$

$$= \text{Re}\left\{(\gamma_0 + (\alpha_0 - i\beta_0)z + \sum_{j=1}^{m} (\alpha_j - i\beta_j) f_{b_j}(z)\right\} = \text{Re}[h(z)], \tag{5}$$

it follows that the assertion of the theorem is equivalent to the statement that M_r is dense in the space $C_r(\Gamma)$ of continuous real-valued functions defined on Γ.

The space $C_r(\Gamma)$ is mapped into $C_r[0, 1]$ by the linear isometry

$$\psi(f) = f \circ g , \tag{6}$$

the image consisting exactly of those functions g in $C_r[0,1]$ satisfying $g(0) = g(1)$, and thus

$$C_r[0,1] = \psi[C_r(\Gamma)] + \mathrm{sp}(g_0)$$

where g_0 is the function defined on $[0,1]$ by $g_0(t) = t$.

To establish the theorem, begin by assuming that ReM is not sup norm dense in $C_r(\Gamma)$, or equivalently that

$$\psi[M_r] + \mathrm{sp}(g_0) \tag{7}$$

is not dense in $C_r[0,1]$. In this case, by the Hahn-Banach theorem there is a nonzero continuous linear functional x^* on $C_r[0,1]$ which maps the subspace of Eq. (7) to zero. The standard representation theorem ([10], p. 150) shows that x^* corresponds to a right continuous function α of bounded variation on $[0,1]$, normalized to have $\alpha(0) = 0$, with

$$x^*(g) = \int_0^1 g(t)\,d\alpha(t) \tag{8}$$

for all g in $C_r[0,1]$.

Since the function identically 1 belongs to M_r,

$$\alpha(1) = \alpha(1) - \alpha(0) = \int_0^1 d\alpha = x^*(1) = 0. \tag{9}$$

Because the functions Ref_b and Imf_b belong to M_r for any b on Γ,

$$\int_0^1 f_b(\varphi(t))\,d\alpha(t) = 0. \tag{10}$$

Integrate by parts in (10) ([11], p. 195; [12], p. 283) to obtain

$$\int_0^1 \alpha(t)\,df_b(\varphi(t)) = 0. \tag{11}$$

Since $f_b(\phi(t))$ is absolutely continuous, (11) can be rewritten as the Lebesque-Stiltjes integral ([13], p. 419–420; [14], p. 264)

$$\int_0^1 \{1 + \log_{Pb}[\varphi(t) - b]\}\,\alpha(t)\varphi'(t)\,dt = 0. \tag{12}$$

From a more elementary point of view, the integral in (12) can be taken to be a Riemann-Stiltjes integral which is improper because of the singularity of $\log_{Pb}[\varphi(t) - b]$ at the point t_0 at which $\varphi(t_0) = b$, and so is understood to be the limit as ε_1 and ε_2 tend to zero of the integral from 0 to $t_0 - \varepsilon_1$ plus the integral from $t_0 + \varepsilon_2$ to 1 (with $t_0 = 1$ and $t_0 = 0$ requiring different notation).

Since $\alpha(1) = \alpha(0)$, the expression

$$g(z) = \alpha(\varphi^{-1}(z)) \tag{13}$$

defines a function on Γ. This function g is real valued and of bounded variation.

Use the definition of the line integral to write (12) as

$$\int_\Gamma [1 + \log_{Pb}(z - b)]\,g(z)\,dz = 0. \tag{14}$$

Define

$$Gr(t) = \int_0^t g(\varphi(\tau))\varphi'(\tau)\,d\tau$$
$$= \int_0^t \alpha(\tau)\varphi'(\tau)\,d\tau. \qquad (15)$$

Integrate by parts to show that

$$Gr(1) = -\int_0^1 \varphi(t)\,d\alpha(t) = -x^*(\psi(z)) = 0 \qquad (16)$$

and therefore, since $Gr(1) = Gr(0)$,

$$G(z) = Gr(\varphi^{-1}(z)) \qquad (17)$$

defines a continuous function on Γ, with derivative

$$G'(z) = g(z)\varphi'(\varphi^{-1}(z))(\varphi^{-1})'(z) = g(z) \qquad (18)$$

holding almost everywhere on Γ.

The function G satisfies

$$|G(z_1) - G(z_2)| \leq M_1 \int_{t_1}^{t_2} |\varphi'(t)|\,dt, \qquad (19)$$

where $M_1 = \sup\{|g(z)|: z \text{ on } \Gamma\}$, $\varphi(t_1) = z_1$, and $\varphi(t_2) = z_2$. The integral in (19) is the arclength of the curve between z_1 and z_2. The final inequality needed is also basic to the proof of the Sokhotski-Plemelj formulas used below; it is that ([15], p. 21) "the ratio of the small arc [length] of contour to the corresponding chord is bounded," which is true for curves with a finite number of corners which are not cusps ([15], p. 31; [16], Appendix 2). From this it follows that

$$|G(z_1) - G(z_2)| \leq M_2|z_1 - z_2|, \qquad (20)$$

i.e., G satisfies a Lipschitz condition on Γ.

Integrate by parts in Eq. (14), treating the integrals obtained as principal value integrals, modifying the proof of [15] (p. 18) to apply to curves with corners,

$$G(b)[\Theta(b)/\pi] = \frac{1}{\pi i}\int_\Gamma \frac{G(z)}{z - b}\,dz, \qquad (21)$$

where $\Theta(b)$ is the interior angle between the two tangents to the contour at the point b; thus $\Theta(b) = \pi$ except at corners.

Consider the function G_+ which is defined for ω in Ω by

$$G_+(\omega) = \frac{1}{2\pi i}\int_\Gamma \frac{G(z)}{z - \omega}\,dz \qquad (22)$$

and the function $G_-(\omega)$ defined for ω in the complement of $\Omega \cup \Gamma$ by the same integral as in (22). Given a point b on Γ, the Sokhotski-Plemelj formulas ([17], p. 88; [18], p. 94; [15], p. 32) for corners, show that

$$\begin{aligned} G_-(b) &= \left(1 - \frac{\Theta(b)}{2\pi}\right)G(b) + \frac{1}{2\pi i}\int_\Gamma \frac{G(z)}{z - b}\,dz, \\ G_-(b) &= -\frac{\Theta(b)}{2\pi}G(b) + \frac{1}{2\pi i}\int_\Gamma \frac{G(z)}{z - b}\,dz, \end{aligned} \qquad (23)$$

where $G_+(b)$ denotes the limit of $G_+(\omega)$ as ω tends to b nontangentially for ω in Ω, $G_-(b)$ the limit as ω tends to b nontangentially for ω in the complement of $\Omega \cup \Gamma$, and the integral is a singular integral; the existence of these limits following from the derivation of these formulas. Use Eq. (21) to see that

$$\begin{aligned} G_+(b) &= G(b), \\ G_-(b) &= 0. \end{aligned} \tag{24}$$

Thus the function $\hat{G}$ defined by

$$\hat{G}(z) = \begin{cases} G(z) & \text{for } z \text{ on } \Gamma \\ G_+(z) & \text{for } z \text{ in } \Omega \end{cases} \tag{25}$$

is continuous on $\Omega \cup \Gamma$ ([16], Appendix 2).

Use the fact that $G_-(\omega) = 0$, and expand the integrand in the integral of (22), for large ω, in a power series and conclude that

$$\int_\Gamma G(z)z^n \, dz = 0, \quad n = 0, 1, \ldots . \tag{26}$$

Integrate by parts in (26) to obtain

$$\int_\Gamma g(z)z^n \, dz = 0, \quad n = 1, 2, \ldots . \tag{27}$$

From (16) and (17), Eq. (27) also holds for $n = 0$. Define the function g_+ for ω in Ω by

$$g_+(\omega) = \frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z - \omega} \, dz \tag{28}$$

and the function $g_-(\omega)$ for ω in the complement of $\Omega \cup \Gamma$ by the same integral. Equation (27), for $n = 0, 1, \ldots$, shows that the analytic function $g_-(\omega)$ is zero for large ω and therefore is zero on its domain.

Differentiate the integral in (22) with respect to ω, and then integrate by parts, to obtain

$$G'_+(\omega) = g_+(\omega) \qquad \text{for } \omega \text{ in } \Omega . \tag{29}$$

The function g is differentiable on Γ, except for points in N_0, a set of points of measure zero, because α is of bounded variation ([14], p. 100). At the points where g is differentiable, it satisfies a local Lipschitz condition and the local version of the Sokhotski-Plemelj formulas ([15], p. 23; [18], p. 94, modified for corners) show that at a point z_0 where g is locally Lipschitz the limit of $g_+(\omega)$, as ω tends nontangentially to z_0, is $g(z_0)$, since $g_-(\omega)$ is zero.

Let $\mathcal{U}$ be the unit ball and ρ: $\mathcal{U} \to \Omega$ be the conformal map given by the Riemann mapping theorem, extended as a continuous map of the closure of $\mathcal{U}$ to the closure of Ω by the Caratheodory-Osgood theorem ([18], p. 346). Consider

$$\tilde{g}_+(\omega) = g_+ \circ \rho(\omega), \qquad \omega \text{ in } \mathcal{U} . \tag{30}$$

If $z = e^{i\theta}$ in $\partial\mathcal{U}$ does not belong to $\rho^{-1}(N_0)$, and $w = re^{i\theta}$, $0 < r < 1$, then as r tends to 1, $\rho(\omega)$ tends to $\rho(z)$ nontangentially for almost all z ([19], p. 45) and therefore $\tilde{g}_+(\omega)$ tends to $g(\rho(z))$. Since $\rho^{-1}(N_0)$ is a set of measure zero ([19], p. 45), $\tilde{g}_+$ is analytic in $\mathcal{U}$ and has a radial limit a.e. the function $g(\rho(z))$, which is a function of bounded variation. Since $\tilde{g}_+$ is in H^∞, by [19], (p. 42) $\tilde{g}_+$ has a continuous extension to the closure of $\mathcal{U}$ which is actually absolutely continuous on the boundary. Since the boundary values of $\tilde{g}_+$

are real a.e., and $\tilde{g}_-$ extends continuously, its boundary values are real. If $\tilde{g}_+$ were not constant it would be an open map onto a bounded domain with real boundary, which is not possible, and so $\tilde{g}_+$ is constant and therefore g_+ is constant in Ω. Then g on Γ, being the nontangential limit of g_+ a.e. is constant a.e., and being right continuous is therefore constant. Equation (13) shows that then α is constant and therefore the linear functional x^* is zero, contrary to assumption. Q.E.D.

The result of Theorem 1 was previously known to hold only under very restrictive conditions ([6], Chap. 3) on both the boundary and the given boundary value function.

Note that if Reh approximates g to within ε in sup norm on Γ, then it approximates the solution to the Dirichlet problem to within ε in sup norm on all of $\Omega \cup \Gamma$ by the maximum principle because the approximating function h of Theorem 1 is analytic on all of Ω and continuous on $\Omega \cup \Gamma$. [In fact, if $|\mathrm{Re}h_n(z) - g(z)| < 1/n$ on Γ, then $\{h_n\}$ converges uniformly to h continuous on $\Omega \cup \Gamma$, analytic in Ω, with Re$h = g$ on Γ.]

Corollary 2. Given a real-valued function g in $L^p(\Gamma)$, $1 \le p < \infty$, and a positive number ε, there is a function h, analytic on Ω and continuous on $\Omega \cup \Gamma$, of the form given in Eq. (5), with

$$\left\{ \int_\Gamma |g(z) - \mathrm{Re}[h(z)]|^p \, |dz| \right\}^{1/p} < \varepsilon. \tag{31}$$

Proof. Given g as stated, consider $\psi(g)$, which is in $L_r^p[0,1]$ since $|\varphi'(t)|$ is bounded below by a positive number. The real-valued continuous functions on $[0,1]$ are dense in $L_r^p[0,1]$, so there is an f_1 in $C_r[0,1]$ with

$$\int_0^1 |g(\varphi(t)) - f_1(t)|^p \, dt$$

as small as desired. There is a function f_2 in $C_r[0,1]$, arbitrarily close to f_1 in $L_r^p[0,1]$ norm, with $f_2(0) = f_2(1)$; this f_2 is then also close to $\psi(g)$ in $L_r^p[0,1]$. The function $g_2 = f_2 \circ \varphi^{-1}$ is continuous on Γ, and, because $|\varphi'(t)|$ is bounded above on [0,1], g_2 is close to g in $L^p(\Gamma, |dz|)$. By Theorem 1, there is a function $h(z)$, of the type desired, which is close to g_2 in sup norm on Γ and therefore close to g in $L^p(\Gamma, |dz|)$. Q.E.D.

The above proof of the existence of a solution to the Dirichlet problem does not supply an *a priori* error bound in the sense that from the proof it is not possible before calculation begins to say how large m must be in Eq. (5) in order that a given boundary function g be approximated to within a preassigned positive ε. It is also not possible to prescribe the best choice of the points $b_1, \ldots, b_m$; it is easy to see that theoretically it will suffice to choose points from any given dense set of points on Γ, but as a practical matter some placements of nodes lead to much higher accuracy. However, in all the numerical work which has been done, useful accuracy has been obtained for small m and simple choices $b_1, \ldots, b_m$.

As part of the proof of Theorem 1, it is shown that an analytic function on Ω which has boundary values a.e. equal to a real-valued right-continuous function of bounded variation is constant. This is shown by a heavy-handed use of the Riemann mapping theorem and the Caratheodory-Osgood continuous extension (from which a proof of the existence of a solution to the Dirichlet problem follows directly via the Poisson integral formula), as well as the use of more subtle properties of the Riemann map. An easier proof of this fact would allow Theorem 1 to provide an independent proof of the existence of a solution to the Dirichlet problem for the domains covered by the theorem.

Note that the functions of the form (5) are not necessary equivalent to the linear superposition of integrals of Cauchy type which are ordinarily used in the CVBEM. Any family of functions which are analytic in D and continuous on $D \cup \Gamma$, whose real parts approximate any given continuous function to any given degree of accuracy, can be used in place of (5); the latter approximating property is, however, usually difficult to show.

III. COMPUTATIONAL EXAMPLES

A computer program was written to approximate the solution of Dirichlet problems by means of functions given by Eq. (5). Given a parametrization φ of the simple closed curve Γ, an input number m of points, $t_1, \ldots, t_m$ are chosen equally spaced in $[0, 1]$, giving rise to the points $b_1 = \varphi(t_1), \ldots, b_m = \varphi(t_m)$, called *nodes*, on Γ. Given a real-valued function g defined on Γ, the coefficients γ_0, and α_j and β_j, $j = 0, 1, \ldots, m$ of Eq. (5) are found by finding the best solution, in the sense of least squares, to the overdetermined system of linear equations

$$\mathrm{Re}[h(z_j)] = g(z_j), \quad j = 1, 2, \ldots, 3m + 5,$$

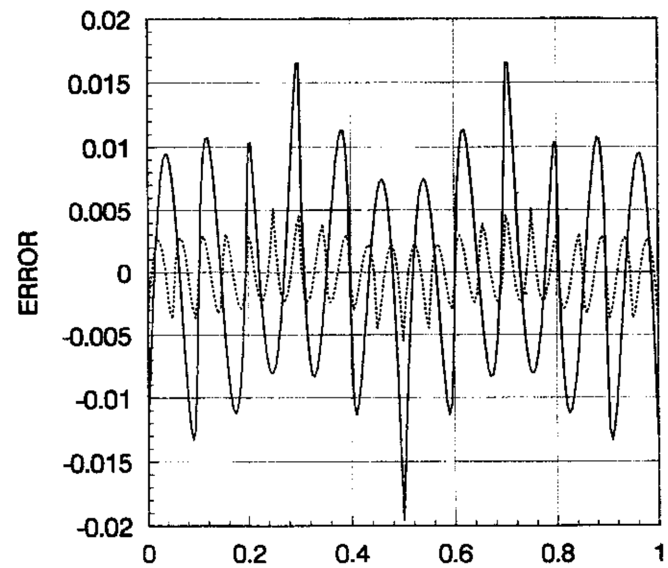

FIG. 1. The errors given by (34) for the ellipse (32) with $a = 2$, $b = 1$, with boundary condition (33) for 10 and 20 nodes.

the evaluation points z_j being obtained by applying φ to $3m + 5$ equally spaced points in $[0, 1]$. Once these coefficients are known, $h(z)$ can be easily evaluated for each z in the domain or on its boundary Γ.

Example A

The problem domain is an ellipse, parametrized by

$$\varphi(t) = a \cos(2\pi t) + ib \sin(2\pi t), \tag{32}$$

t in $[0, 1]$; for this problem $a = 2$ and $b = 1$. The boundary conditions are given by

$$g(z) = \frac{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}{2}. \tag{33}$$

The solution to this problem with boundary condition (33) can be directly used to calculate the torsional rigidity of the domain ([7], p. 206).

This example compares the accuracy of the approximation for 10, 20, 30, and 40 nodes. The $L^2(\Gamma, |dz|)$ norm of the difference between the approximate solution and the boundary condition, i.e., the value of Eq. (31) for $p = 2$ is, for 10, 20, 30, and 40 nodes, respectively,

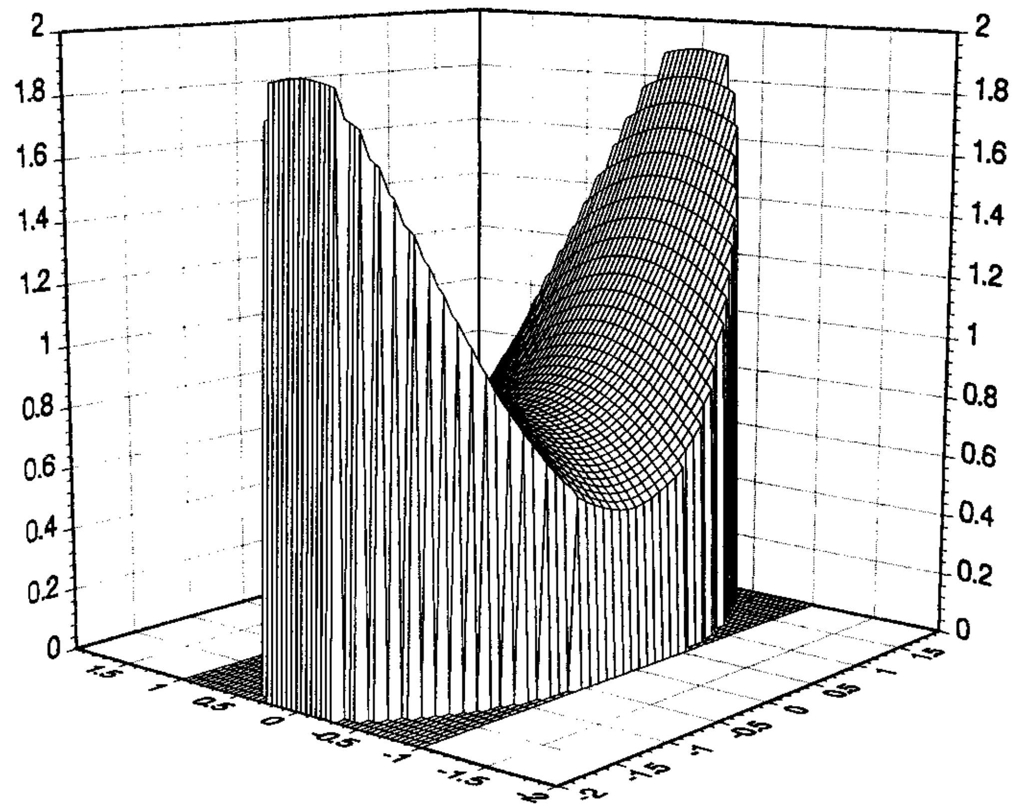

FIG. 2. The solution for an ellipse (32) with $a = 2$, $b = 1$ for boundary condition (33).

2.43×10^{-2}, 6.84×10^{-3}, 4.20×10^{-3}, and 1.82×10^{-3}. In Fig. 1, a graph is given comparing the error

$$\text{error}(t) = \text{Re}[h(\gamma(t)) - g(\phi(t))] \tag{34}$$

for 10 and 20 nodes.

The solution $h(z)$ is easily computed on $\Omega \cup \Gamma$, and so the approximate solution $\text{Re}[h(z)]$ can be drawn as a solution surface over the domain. This is done in Fig. 2 for example A with 20 nodes. The error is so small that the difference between the exact solution and the approximate solution is not discernible in the graph; the jaggedness in the plot of the surface comes from plotting it at a mesh of 50×50 points, not from any error in the approximation.

Example B

This example shows how the accuracy of the approximation depends on the shape of the figure. The boundary condition is the same function (33) used in example A, and the number of nodes is 20. The domains are ellipses for the three sets of parameters $(a = 2,\ b = 1)$, $(a = 3,\ b = 1)$, and $(a = 4,\ b = 1)$, For $p = 2$, (31) is, respectively, 6.84×10^{-3}, 1.90×10^{-2}, and 3.71×10^{-2}. In Fig. 3, a graph compares the error of (34) for $(a = 2,\ b = 1)$ with the larger error for $(a = 4,\ b = 1)$. Because the error r of (31) is given by the square root of an integral along Γ, it is better to compare the values of r of divided by the square root of the arclength of Γ; for the three curves considered here these values are 2.2×10^{-3}, 5.2×10^{-3}, and 9.0×10^{-3}.

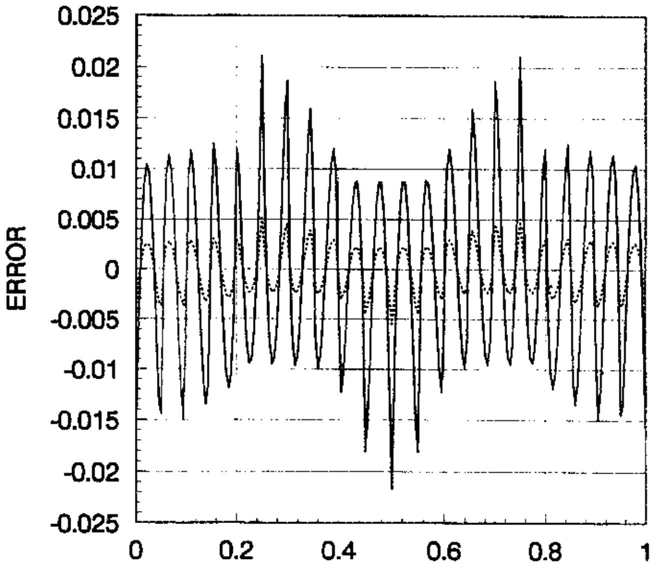

FIG. 3. Error (34) for the ellipses (32) with $a = 2$, $b = 1$ and $a = 4$, $b = 1$, for boundary condition (33) with 20 nodes.

Example C

The Dirichlet problem is solved for the rhombus with vertices at $(L,0)$, $(0,1)$, $(-L,0)$, and $(0,-1)$ for $L = 1,2,3$, and 20 nodes the error curves for $L = 1$ and $L = 3$ being given in Fig. 4.

$$\varphi(t) = \begin{cases} L(1-4t) + i4t, & 0 \le t \le 0.25 \\ L(1-4t) + i(2-4t), & 0.25 \le t \le 0.50 \\ L(4t-3) + i(2-4t), & 0.50 \le t \le 0.75 \\ L(4t-3) + i(4t-4), & 0.75 \le t \le 1. \end{cases} \tag{35}$$

The r values are, respectively, 3.86×10^{-3}, 1.02×10^{-2}, and 2.22×10^{-2}. Compare these values with the r values for the progressively more elongated ellipses of example A to see how the presence of corners affects the accuracy. As in example B, a better measure of the error is r divided by the square root of the perimeter of the figure, i.e., 1.62×10^{-3}, 3.41×10^{-3}, and 6.24×10^{-3}.

The next three examples are of discontinuous boundary values. The boundary function chosen is 1 on the top half of the figure and 0 on the bottom half.

Example D

The domain is the ellipse, with $a = 2$, $b = 1$, of example A with the boundary condition

$$g(t) = \begin{cases} 1, & 0 \le t \le 0.5 \\ 0, & 0.5 \le t \le 1. \end{cases} \tag{36}$$

The error curve of Fig. 5 is not much improved by adding nodes, the r values for 10, 20, 30, and 40 nodes being 2.82×10^{-1}, 2.01×10^{-1}, 1.64×10^{-1}, and 1.42×10^{-1}. This is also true for the remaining examples.

Example E

The domain is the top half of an ellipse:

$$\phi(t) = \begin{cases} a\cos(2\pi t) + ib\sin(2\pi t), & 0 \le t \le 0.5 \\ a(4t-3) + 0i, & 0.5 \le t \le 1 \end{cases} \tag{37}$$

with $a = 2$ and $b = 1$, and boundary condition (36).

Example F

The curve is the rhombus (35) with $L = 1$, i.e., a rotated square, and the boundary condition (36). The error curve of Figs. 5 and 6 is almost identical with that of Fig. 7.

The last three examples illustrate that the dominant cause of inaccuracy is the discontinuity in the boundary condition, the presence of corners in the domain having a much smaller effect.

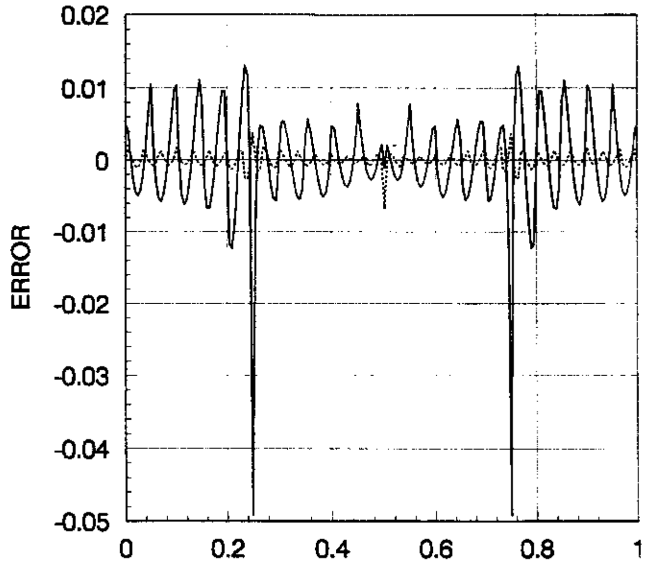

FIG. 4. Error (34) for the rhombus (35) with $L = 1$ and $L = 3$ for boundary condition (33) with 20 nodes.

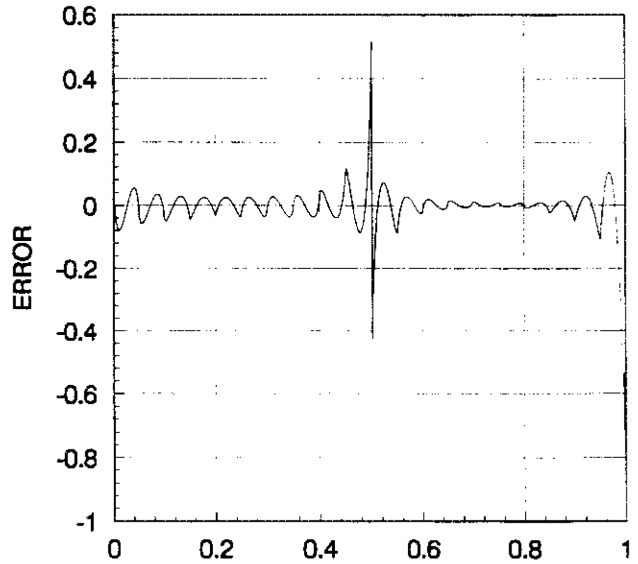

FIG. 5. The boundary is the ellipse (32) with $a = 2$, $b = 1$, 20 nodes and boundary condition (36).

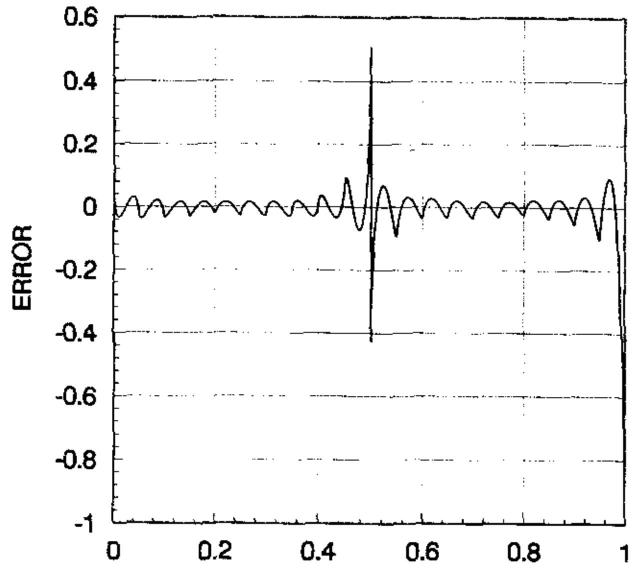

FIG. 6. Error for the figure of (36), $a = 2$, $b = 1$, for boundary condition (36) and 20 nodes.

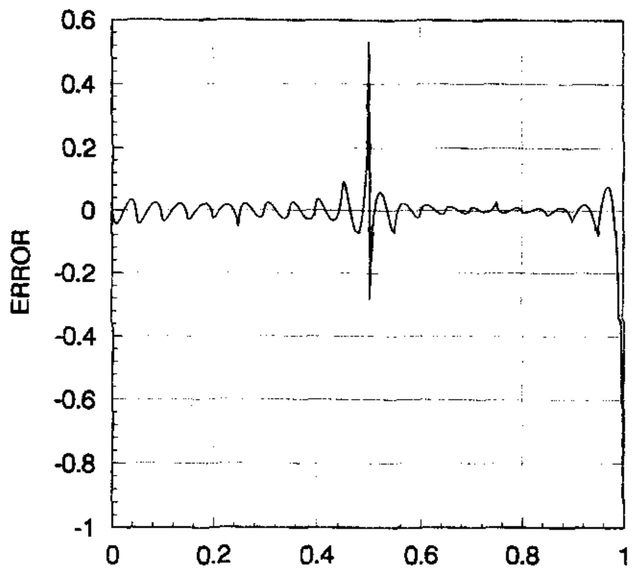

FIG. 7. Error for the rhombus (35) with $L = 1$, boundary condition (37) and 20 nodes.

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