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Expansion of the CVBEM into a series using fractals

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The study of fractals, such as leading to the Koch snowflake, among other graphical displays, has advanced considerably in the last several years. In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the complex variable boundary element method (or CVBEM) into a series.

Key words: Boundary element method, CVBEM, complex variables, fractals, series expansions.

INTRODUCTION

The complex variable boundary element method (CVBEM) has been the subject of several papers and books.[1,2] The basis of the CVBEM is the use of the Cauchy integral equation to develop approximations of two-dimensional boundary value problems involving the Laplace and Poisson equations.

An advantage of the CVBEM is the property that the resulting approximation function, $\hat{\omega}(z)$, is analytic in the simply connected domain, Ω, and continuous on the problem boundary, Γ. Thus, $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, where $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are the potential and stream functions, respectively, and satisfy the Laplace equation in Ω. The general CVBEM technique is briefly described in the following discussion.

Let $\omega(z) = \phi(x, y) + i\psi(x, y)$ be a complex variable function which is analytic on $\Gamma \cup \Omega$, where Ω is a simply connected domain enclosed by the simple closed boundary Γ (Fig. 1). We define $\phi(x, y)$ to be the state variable and $\psi(x, y)$ the stream function, where ϕ and ψ are two-dimensional real valued functions. Since ω is analytic, ϕ and ψ are related by the Cauchy–Reimann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{1}$$

and thus staisfy the two-dimensional Laplace equations in Ω, namely

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ and } \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{2}$$

The Cauchy integral theorem states that if we know the value of the complex function ω on the boundary Γ, and if ω is analytic on $\Gamma \cup \Omega$, then ω in Ω is given by

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\,d\zeta}{\zeta - z}, z \in \Omega,\ z \notin \Gamma \tag{3}$$

The CVBEM forms $\hat{\omega}$, an approximation of ω, using known values of either ϕ or ψ on the boundary Γ, and uses the Cauchy integral (eqn (3)) to determine approximate values for ω on $\Omega \cup \Gamma$. The approximator, $\hat{\omega}$, is a two-dimensional analytic function in Ω that can be differentiated, integrated, or otherwise manipulated to obtain higher order operator relationships.[1]

Let the boundary Γ be a polygonal line composed of V straight line segments and vertices. Define nodal points with complex coordinates z_j, $j = 1, \ldots, m$ on Γ such that $m > V$. Nodal points are located at each vertex of Γ and are numbered in a counter-clockwise

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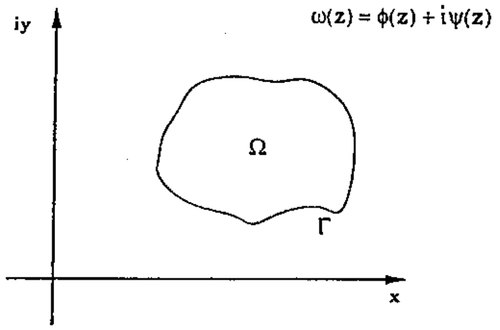

Fig. 1. Problem domain and boundary.

direction. Let Γ_j be the straight line segment joining z_j and z_{j+1} so that

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j.$$

Thus, m boundary elements, Γ_j, are defined on Γ, where Γ_m connects nodal coordinate z_m and z_1 (Fig. 2). The CVBEM defines a continuous global trial function, $G(z)$, by

$$G(z) \sum_{j=1}^{m} N_j(z)(\bar{\phi}_j + i\bar{\psi}_j) \tag{4}$$

where, for a piecewise linear polynomial global trial function, and $j = 1, \ldots m$, $N_j(z)$ is given by

$$N_j(z) = \begin{cases} \dfrac{z - z_{j-1}}{z_j - z_{j-1}} & z \in \Gamma_{j-1} \\ 0 & z \notin \Gamma_j \cup \Gamma_{j-1} \\ \dfrac{z_{j+1} - z}{z_{j+1} - z_j} & z \in \Gamma_j \end{cases} \tag{5}$$

and where $\bar{\phi}_j$ and $\bar{\psi}_j$ are nodal values of the two conjugate components, evaluated at z_j. An analytic approximation is then determined by

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\,d\zeta}{\zeta - z}, z \in \Omega,\ z \notin \Gamma \tag{6}$$

If we let $q_{j,j+1}$ be the difference in the polar coordinate angles defined by nodal point coordinates z_{j+1} and z_0, and z_j and z_0, for z_0 in Ω, then

$$\hat{\omega}(z_0) = \sum_{j=1}^{m} [\omega_{j+1}(z_0 - z_j) - \omega_j(z_0 - z_{j+1})]\frac{H_j}{z_{j+1} - z_j} \tag{7}$$

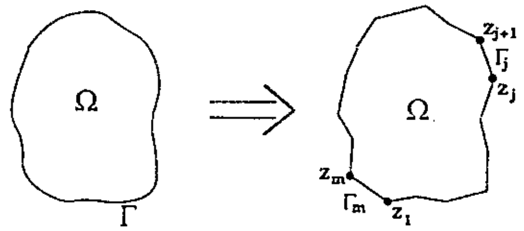

Fig. 2. Discretization.

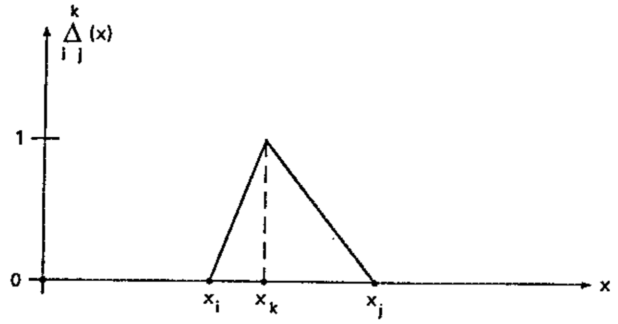

Fig. 3. Graph of $\overset{k}{\underset{i\,j}{\triangle}}(x)$ function, for nodes i, j, and k.

where ω_j and ω_{j+1} are nodal values at coordinates z_j and z_{j+1}, and

$$H_j = \ln\left|\frac{z_{j+1} - z_0}{z_j - z_0}\right| + iq_{j,j+1} \tag{8}$$

Since usually only one of the two specified nodal values $(\bar{\phi}_j, \bar{\psi}_j)$ is known at each $z_j, j = 1, \ldots, m$, values for the unknown nodal values must be estimated as part of the CVBEM approach to developing an analytic approximation function. The CVBEM develops a matrix system for use in solving for these unknown nodal values,[1] solves the resultant matrix system, and uses these nodal value estimates along with the known nodal values in defining $\hat{\omega}(z_0)$ in eqn (7).

FRACTAL TRIAL FUNCTION REPRESENTATION

In this paper, the CVBEM continuous global trial function, $G(\zeta)$, is replaced by a rewriting that is analogous to the triangle fractals used in graphical displays.

In our case, we use the symbol $\overset{k}{\underset{i\,j}{\triangle}}(z)$ in describing the incremental change between the value of a straightline interpolation between consecutive nodal point values ϕ_i and ϕ_j, at $x = x_k$, and the true value of ϕ at $x = x_k$, denoted by ϕ_k.

That is, for nodes i and j being consecutive nodal points on Γ, in the counter-clockwise path of contour integration, the addition of node k inbetween nodes i and j is accomplished by adding to the global trial function, $G(\zeta)$, the incremental contribution from the newly added node j. Given coordinates z_i, z_j, z_k for nodes i, j, k, respectively (see Fig. 3),

$$\overset{k}{\underset{i\,j}{\triangle}}(z) \equiv \begin{cases} 0; & z \notin \text{the boundary element containing nodes } i,j \\ \dfrac{z - z_i}{z_k - z_i}; & z \text{ between nodes } i,k \text{ and } z \in \Gamma \\ \dfrac{z_j - z}{z_j - z_k}; & z \text{ between nodes } k,\ j \text{ and } z \in \Gamma \end{cases} \tag{9}$$

Hereafter, $\overset{k}{\underset{i\,j}{\triangle}}(z)$ will be simply written as $\overset{k}{\underset{i\,j}{\triangle}}(z)$ as it is

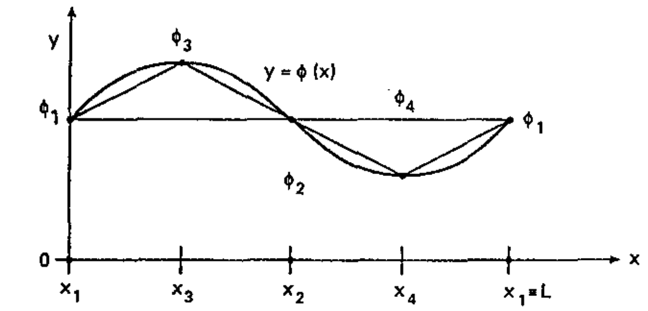

Fig. 4. $G(\zeta) = \phi_1 + \overset{2}{\underset{1\,1}{\Delta}}\phi_2 + \overset{3}{\underset{1\,2}{\Delta}}\phi_3 + \overset{4}{\underset{2\,1}{\Delta}}\phi_4$. L is domain length; x_i are nodal point coordinates, with values $\phi(x_i) = \phi_i$, $i = 1, \ldots, 4$.

understood that a function of z is involved. Also note that $\overset{k}{\underset{i\,j}{\Delta}}$ has nonzero values only between nodes i and j on Γ.

From the above, the value of $\phi(z_k)$ is ϕ_k. Straightline interpolation between nodes i, j gives an estimate, $\hat{\phi}_k$, at z_k of (see Fig. 4)

$$\hat{\phi}(z_k) = \phi_i\left(\frac{z_j - z_k}{z_j - z_i}\right) + \phi_j\left(\frac{z_k - z_i}{z_j - z_i}\right) \tag{10}$$

which would be the value of the global trial function at z_k, $G(z_k)$, for the case where node k is not part of $G(\zeta)$. $G(\zeta)$ can be extended to include a node k contribution by simply adding the incremental contribution of ϕ_k,

$$G(\zeta) \rightarrow G(\zeta) + \overset{k}{\underset{i\,j}{\Delta}}(\phi_k - \hat{\phi}_k) \tag{11}$$

The global trial function, $G(\zeta)$, can be written as a sum of nodal incremental contributions by, for the case of an eight-node approximation (see Fig. 5),

$$\begin{aligned} G(\zeta) = {} & \overset{1}{\underset{1\,1}{\Delta}}\phi_1 + \overset{2}{\underset{1\,1}{\Delta}}(\phi_2 - \hat{\phi}_2) + \overset{3}{\underset{1\,2}{\Delta}}(\phi_3 - \hat{\phi}_3) \\ & + \overset{4}{\underset{2\,1}{\Delta}}(\phi_4 - \hat{\phi}_4) + \overset{5}{\underset{1\,3}{\Delta}}(\phi_5 - \hat{\phi}_5) + \overset{6}{\underset{3\,2}{\Delta}}(\phi_6 - \hat{\phi}_6) \\ & + \overset{7}{\underset{2\,4}{\Delta}}(\phi_7 - \hat{\phi}_7) + \overset{8}{\underset{4\,1}{\Delta}}(\phi_8 - \hat{\phi}_8) \end{aligned} \tag{12}$$

In the above equation, $\overset{1}{\underset{1\,1}{\Delta}}$ refers to the initial case of having a constant-valued $G(\zeta)$ defined on Γ, where

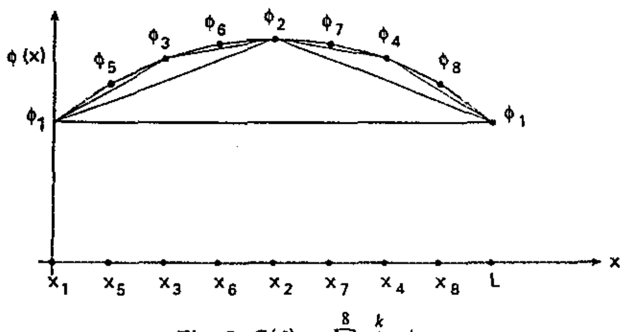

Fig. 5. $G(\zeta) = \sum_{k=1}^{8} \overset{k}{\underset{i\,j}{\Delta}}\phi_k$.

$G(\zeta) = \phi_1$ for all $z \in \Gamma$, due to having only a single node (#1) defined on Γ. Also, note that the order in which the terms involving $\overset{k}{\underset{i\,j}{\Delta}}$ functions appear is important due to the definition of where nodal points occur on Γ. Thus, the above sum of terms cannot be arbitrarily rearranged as the addition is not commutative. Given a specified sequence of nodal point insertion on Γ, such that it is understood where subsequent nodes are to be added on Γ, the index notation of i, j, k can be simplified to simply using i, as it is known that node k is to follow node i (in the counter-clockwise direction) on Γ, and k is known by being the kth index term. The node sequence, S, of Fig. 5 can be written as simply $S = \{1, 1, 1, 2, 1, 3, 2, 4\}$. In the following, it will be assumed that a node installation sequence, S, is defined so that node numbers i, j are understood when given node number k. Consequently, $G(\zeta)$ can be written for m nodal points defined on Γ according to the above sequence, S, by

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\underset{s_k\,s_{k+1}}{\Delta}}(\phi_k - \hat{\phi}_k) \tag{13}$$

where s_k is the kth term of S, $s_{m+1} = s_1$, and necessarily $\hat{\phi}_1 = 0$ for the initial case of $k = 1$. Equation (13) can be now rewritten into the simpler form,

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\Delta}(\phi_k - \hat{\phi}_k) \tag{14}$$

where it is understood that a nodal point installation sequence, S, is defined, and nodes i and j, as associated with node k in the function, are known given node k.

From the above, the extension of a complex variable function, $\omega(z)$, defined on Γ is given, for m nodes on Γ, by

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\Delta}(\omega_k - \hat{\omega}_k) \tag{15}$$

where $\omega(z_k) = \omega_k$, $k = 1, 2, \ldots m$; and as before, necessarily $\hat{\omega}_1 = 0$.

APPLICATION TO THE CVBEM — NEW SERIES EXPANSION

Using the above expression for the global trial function, $G(\zeta)$, the CVBEM approximation function can be written as, for m nodes on Γ,

$$\begin{aligned} \hat{\omega}(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\,d\zeta}{\xi - z} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{k=1}^{m} \overset{k}{\Delta}(\omega_k - \hat{\omega}_k)\,d\zeta}{\zeta - z}, \quad z \in \Omega \end{aligned} \tag{16}$$

or

$$\hat{\omega}(z) = \frac{1}{2\pi i}\sum_{k=1}^{m}(\omega_k - \hat{\omega}_k)\int_\Gamma \frac{\overset{k}{\triangle}\,d\zeta}{\zeta - z}, \quad z \in \Omega \qquad (17)$$

The above writing provides a new series expansion for the CVBEM approximation function. For the case of $\omega(z)$ being analytic on Γ, then $\omega(z)$ is continuous on Γ where $G(\zeta) \to \omega(\zeta)$ on $m \to \infty$ (and the arclength between successive nodes $\to 0$), and from Schauder's theorem,[3]

$$\omega(z) = \frac{1}{2\pi i}\sum_{k=1}^{\infty}(\omega_k - \hat{\omega}_k)\int_\Gamma \frac{\overset{k}{\triangle}\,d\zeta}{\zeta - z}, \quad z \in \Omega \qquad (18)$$

In the above equation, the integral of

$$\int_\Gamma \frac{\overset{k}{\triangle}\,d\zeta}{\zeta - z}$$

is readily determined as

$$\int_\Gamma \frac{\overset{k}{\triangle}\,d\zeta}{\zeta - z} = \int_\Gamma \frac{\overset{k}{\underset{ij}{\triangle}}\,d\zeta}{\zeta - z} = \int_{z_i}^{z_k} \frac{\overset{k}{\underset{ij}{\triangle}}\,d\zeta}{\zeta - z}\int_{z_k}^{z_j} \frac{\overset{k}{\underset{ij}{\triangle}}\,d\zeta}{\zeta - z}$$

$$= \left(\frac{z - z_i}{z_k - z_i}\right)(\ln(z_k - z) - \ln(z_i - z)) + \left(\frac{z - z_j}{z_k - z_j}\right)(\ln(z_j - z) - \ln(z_k - z)) \qquad (19)$$

where ln is the complex logarithm function.

BINARY TYPE NODE SEQUENCES

If nodal placement on Γ is specified as a partitioning of Γ according to a given proportion, the above series may be simplified. For example, let r be the partition fraction of 1/2, which implies that Γ will be subdivided into boundary element lengths of binary proportions. That is, Γ is subdivided into halves, then quarters, then eighths, and so forth. Then given an initial 'seed' nodal point location on Γ, coordinate z_1, and the parition fraction, r, the previous series expansion is readily determined.

CONCLUSIONS

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the CVBEM into a series. In this paper, a CVBEM series expansion is developed where each additional term of the series is the result of integrating a fractal contribution of the increment between the true nodal value and the previous approximation of the nodal value. Research is currently being conducted in the extension of these results in new applications and theorems.

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