Complex logarithms, Cauchy principal values, and the complex variable boundary element method

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A derivation of the complex variable boundary element method (CVBEM) is presented which explains, in detail, the properties of complex logarithms that are important to the development of and which can cause problems in the numerical implementation of the CVBEM if not fully understood. The derivation uses the Sokhotski–Plemelj formulas and the Cauchy principal value of the Cauchy integral.

Keywords: Complex variable boundary element method, Divichlet problem, Sokhotski-Plemelj, complex variables

1. Introduction

The complex variable boundary element method (CVBEM)[1] uses analytic function theory for the approximate numerical solution of two-dimensional mixed boundary value problems for Laplace's equation, a major application being the steady-state problems of heat conduction. The approximate solution obtained is in terms of a series:

$$\sum_{K=1}^{N} a_k(z - z_k) \log (z - z_k) \tag{1}$$

To use this series it is necessary to define precisely the logarithms appearing in it. To put this in context, an outline of the derivation is given below.

Although there are different ways to use the CVBEM to solve a given boundary value problem, the logarithms of (1) occur in most. For example, in Ref. 2 a series representation for the CVBEM global trial function is developed, which is the function furnishing the approximate solution for a specific boundary value problem, this series having the property that the effect of adding another nodal point is to add a fractal-like term to the series for the global trial function. This derivation is done under the assumptions that the boundary of the domain be approximated by a polygon joining a sequence of nodes and the solution function $w(z)$ be analytic on the boundary. These assumptions, and the series obtained, are not the same as in this paper. However, the crucial fractal-like term in the series of Ref. 2 (equation (19)) is a combination of functions of the form $(z - z_k)\log(z - z_k)$, and the understanding of the properties of these functions is therefore crucial to the results of Ref. 2.

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2. Problem statement

The problem we will consider can be physically interpreted as the steady-state heat distribution of temperature U in an open set Ω in the plane, which we will take to have no holes, i.e., to be simply connected, with boundary Γ. The Dirichlet problem is, given a function g continues on Γ that represents the temperature prescribed on Γ, find a solution U that satisfies Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{2}$$

and equals g on the boundary:

$$U(z) = g(z) \text{ for } z \text{ on } \Gamma \tag{3}$$

In addition to the Dirichlet problem, the CVBEM can be applied to mixed boundary value problems. An example of a mixed problem that often arises in engineering is to have U prescribed on part of the boundary and the heat flux, or normal derivative of U, prescribed on the remaining part of the boundary. Most of what is done here also will apply to mixed boundary value problems, but will be given in terms of the Dirichlet problem for simplicity.

The famous relation of the Dirichlet problem with analytic function theory is furnished by the theorem that U satisfies equation (2) in the simply connected domain Ω, i.e., U is harmonic in Ω, if and only if there is a V harmonic in Ω with $F = U + iV$ analytic in Ω (Refs. 3–5). Consequently the full power of the theory of analytic functions of a complex variable can be used in solving the Dirichlet problem in two dimensions (see, for example, Ref. 4).

The boundary Γ of Ω will have a parameterization γ, a function mapping the interval $[0, 1]$ onto Γ,

$$\gamma:[0, 1] \to \Gamma$$

The curve Γ will be taken to be a simple closed curve, i.e., $\gamma(s) \neq \gamma(t)$ for $s \neq t$ except in the case where one of s and t is 0 and the other is 1, whereupon $\gamma(0) = \gamma(1)$. The curve will also be assumed to be piecewise smooth so that the parameterization of Γ can be taken to be piecewise continuously differentiable with the derivative $\gamma'(t)$ existing except for a finite number of corner points $\{c_1, c_2, \ldots, c_m\}$. At a corner point c_j, the derivative is assumed to exist from the right as $\gamma'(c_j^+)$ and from the left as $\gamma'(c_j^-)$ with $\gamma'(c_j^+) + \gamma'(c_j^-) \neq 0$, i.e., the corner is not a cusp.

3. Derivation

Choose points $0 = t_1 < t_2 < \ldots t_n < t_{n+1} = 1$ in $[0, 1]$ and apply γ to obtain the points

$$z_1 = \gamma(t_1), \ldots, z_n = \gamma(t_n), z_{n+1} = z_1$$

on Γ. These points divide the curve Γ into arcs Γ_j, which are the image of the interval $[t_j, t_{j+1}]$ under γ

$$\Gamma_j = \gamma[t_j, t_{j+1}] \; j = 1, 2, \ldots, n$$

with endpoints z_j and z_{j+1}.

Consider a set of complex numbers $g_1, \ldots, g_n$ and define the function $\hat{g}$ on Γ by the complex analog of linear interpolation

$$\hat{g}(z) = g_j \frac{(z - z_{j+1})}{(z_j - z_{j+1})} + g_{j+1} \frac{(z - z_j)}{(z_{j+1} - z_j)} \text{ for } z \text{ in } \Gamma_j \tag{4}$$

Use this function and the line integral to define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\hat{g}(\zeta)}{\zeta - z} \, d\zeta \tag{5}$$

It is an elementary property of the integral in (5) that the function $h(z)$ is analytic for z in Ω. The notation $\{g_1, \ldots, g_n\}$ is meant to suggest that the g_js are numbers that are chosen so that the function $h(z)$ has real parts approximating the given real value function g on Γ. The maximum property of harmonic functions[3,4,6] then guarantees that the real part of $h(z)$ is close to the solution U of the Dirichlet problem in all of Ω.

Substituting (4) into (5) gives

$$h(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{\hat{g}(\zeta)}{\zeta - z} \, d\zeta \tag{6}$$

For ζ on Γ_j, write

$$\hat{g}(\zeta) = g_j \left\{ \frac{(\zeta - z + z - z_{j+1})}{(z_j - z_{j+1})} \right\} + g_{j+1} \left\{ \frac{(\zeta - z + z - z_j)}{(z_{j+1} - z_j)} \right\} \tag{7}$$

and substitute into the integral to obtain

$$2\pi i h(z) = \sum_{j=1}^{n} \left[g_j \frac{(z - z_{j+1})}{(z_j - z_{j+1})} + g_{j+1} \frac{(z - z_j)}{(z_{j+1} - z_j)} \right] \times \int_{\Gamma_j} \frac{d\zeta}{\zeta - z} \tag{8}$$

Note that the coefficient multiplying the integral in (8) is $\hat{g}(z)$ only if z belongs to Γ_j; otherwise it is a linear extension of $\hat{g}$ from the arc Γ_j.

Example 1 below shows that care must be taken in evaluating the integrals $\int_{\Gamma_j} \frac{d\zeta}{\zeta - z}$ appearing in equation (8).

Example 1

The domain Ω is the interior of a square

$$\Omega = \{(x, y): -1 < x < 1, \; -1 < y < 1\}$$

Let $z_1 = (1, 1)$, $z_2 = (-1, 1)$, $z_3 = (-1, -1)$, $z_4 = (1, -1)$, $z_5 = z_1$, the vertices of the square, be the points chosen on Γ, so that the arcs Γ_j are the sides of the square. Choose the numbers $g_1 = g_2 = g_3 = g_4 = 1$ so that $\hat{g}(z) = 1$ for all z on Γ.

The integral (5) represents h as the integral over the boundary of the analytic function, which is identically 1 on Ω; thus for z in Ω the Cauchy integral formula gives

$$h(z) = 1 = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}$$

However, from equation (8)

$$h(z) = \sum_{j=1}^{4} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{d\zeta}{\zeta - z} \tag{9}$$

If the integrals in (9) are carelessly evaluated as

$$\int_{\Gamma_j} \frac{d\zeta}{\zeta - z} \overset{(?)}{=} \log(z_{j+1} - z) - \log(z_j - z) \tag{10}$$

for $j = 1, 2, 3, 4$, then substituting (10) into (9) gives the incorrect value:

$$\begin{aligned} 2\pi i h(z) &= \log(z_2 - z) - \log(z_1 - z) \\ &\quad + \log(z_3 - z) - \log(z_2 - z) \\ &\quad + \log(z_4 - z) - \log(z_3 - z) \\ &\quad + \log(z_1 - z) - \log(z_4 - z) \\ &= 0 \end{aligned}$$

To understand why this error occurs, a review of some basic facts about logarithms of complex numbers is necessary.

An analytic logarithm is an analytic function f, defined on an open subset G in the complex plane, with the property that the derivative of f satisfies

$$f'(z) = 1/z \text{ for } z \text{ in } G \quad (11)$$

It is important to note that to completely specify a logarithm, its domain G must be given. It will be seen that there are restrictions on the possible domains; for example, implicit in equation (11) is the restriction that $z = 0$ not belong to G.

To construct a logarithm, begin with the equation

$$e^{f(z)} = z, \quad z \text{ in } G \quad (12)$$

i.e., by considering the logarithm as an analytic inverse to the exponential function. The equation $e^w = z$ has, for $z \neq 0$, the solutions

$$w = \log|z| + i(\arg(z) + 2k\pi) \text{ for } k = 0, \pm 1, \pm 2, \ldots \quad (13)$$

Here, $\arg(z)$ is any one angle in a polar representation of z, and the infinitely many other possible values for $\arg(z)$ are indicated in (13). For a given z, the value of the logarithm $f(z)$ must be one of the w values in the set of values given in (13).

A general theorem (Ref. 3, pp. 39–40, or Ref. 6, pp. 85–96), which is used in defining many inverse analytic functions, applies to the problem of finding an analytic solution to (12). The theorem states that a solution to (12) that is continuous on G is also analytic on G. Because the function $\log|z|$, appearing in (13), is continuous at any nonzero z, the problem of defining a logarithm on G lies entirely in definining a continuous argument function $\arg(z)$ for z in G.

The simplest case, discussed in most texts, is defining the principal branch of the logarithm. Let $R^- = \{(x, 0): x \leq 0\}$ be the nonpositive real axis and let G be the set of all complex numbers not belonging to R^-

$$G = \mathbb{C} - R^- \quad (14)$$

Specifying

$$-\pi < \arg(z) < \pi \quad (15)$$

is easily seen to give a continuous argument in G. The principal branch of the logarithm, often denoted by $\text{Log}(z)$, is the function defined on G, given by (14), by $\text{Log}(z) = \log|z| + \arg(z)$, where $\arg(z)$ satisfies (15). This function $\text{Log}(z)$ is analytic on G and satisfies (11) and (12) for z in G.

The way in which the domain G is specified in (14) is typical of the general situation in which a non-self-intersecting curve C joining 0 to ∞ is given and G is defined to be all the complex numbers not on the curve. The curve C is a *branch cut* for the logarithm, and a logarithm defined using the curve C is a *branch of the logarithm*. On this simply connected open set G, a theorem applies (Ref. 3, p. 202) to show that there is a branch of the logarithm defined on G. To see how the branch cut interacts with the determination of a continuous argument, select one point z_0 in G and choose one of the possible values for $\arg(z_0)$. Any point z in G can be joined to z_0 by a (polygonal) curve C_0 lying entirely in G, because G is connected. The requirement that $\arg(\cdot)$ be continuous means that in moving from z_0 to z along C_0 the values of $\arg(\cdot)$ will be specified in a unique way because at each point all the possible values for $\arg(\cdot)$ differ by some multiple $2k\pi$ of 2π, and a continuous determination of $\arg(\cdot)$ cannot jump from one value to another $2k\pi \neq 0$ distant from the first.

A similar argument using the continuity of the argument can be used to characterize all the logarithms on $G = \mathbb{C} - C$: Suppose that f and g are two logarithms on $G = \mathbb{C} - C$, i.e., two analytic functions satisfying (11), or equivalently (12), on G. From (13), for each z in G there is an integer $k(z)$ with

$$f(z) = g(z) + i2\pi k(z) \quad (16)$$

At this point in the argument it appears that the integer $k = k(z)$ possibly depends on z, as the notation indicates. However, $k(z) = [f(z) - g(z)]/2\pi i$ is continuous on G and integer valued, which is only possible on the connected set G if $k(z)$ is constant. Thus any two logarithms on G differ only by a constant $i2\pi k$ for some fixed integer k.

The domain $G = \mathbb{C} - R^-$ for the principal branch of the logarithm $\text{Log}(z)$ is a largest domain on which a logarithm can be defined. To see this suppose that g is a logarithm defined on G', a domain strictly larger than G. The restriction of g to G is a logarithm on G, and by the result described in the paragraph above, there is an integer k with

$$g(z) = \text{Log}(z) + i2\pi k \quad (17)$$

for all z in G. Because G' is larger than G, there is a point $z_1 \neq 0$ in R^- at which g is continuous, in fact analytic. For ε real and positive, $g(z_1 + i\varepsilon) = \text{Log}(z_1 + i\varepsilon) + 2k\pi i$ converges to $i\pi + i2\pi k$ as ε tends to zero, while $g(z_1 - i\varepsilon) = \text{Log}(z_1 - i\varepsilon) + 2\pi k i$ converges to $-i\pi + i2\pi k$; thus, contrary to assumption, g is not continuous at the point z_1.

Similarly each domain that is the complex plane minus a branch cut is a largest domain on which a logarithm can be defined.

Example 2

Consider the domain G defined by removing the spiral curve C given by

$$h(\theta) = \theta\, e^{i2\pi\theta}, \; 0 \leq \theta < \infty$$

from the complex plane. To determine a logarithm on G begin by taking the argument of 1/2 to be zero. (Therefore all points on the interval $\{(x, 0): 0 < x < 1$ also have argument zero.) From the requirement that $\arg(\cdot)$ be continuous on G, it follows that $\arg(3/2) = 2\pi$, $\arg(5/2) = 4\pi$, $\arg(7/2) = 6\pi$, etc.

The complex variable version of the fundamental theorem of calculus states that if f is analytic on an open set G and a curve Γ lies in G and joins z_1 in G to z_2 in G, then

$$\int_\Gamma f'(z)dz = f(z_2) - f(z_1) \quad (18)$$

The proof for a piecewise smooth curve Γ simply involves passing to the parameterization of the curve $\gamma:[0, 1] \to \Gamma$ and applying the real variable fundamental theorem of calculus to the real and imaginary parts of the derivatives of $f(\gamma(t))$:

$$\int_\Gamma f'(z)dz = \int_0^1 f'(\gamma(t))\gamma'(t)dt = f(\gamma(1)) - f(\gamma(0))$$

Example 3

Example 1 is reconsidered and the correct value for h is obtained.

First consider the computation of $h(z)$ for $z = 0$. In Example 1, three sides of the square lie in the domain of the principal branch of the logarithm Log(z): Γ_1 joining $z_1 = (1, 1)$ to $z_2 = (-1, 1)$, Γ_3 joining $z_3 = (-1, -1)$ to $z_4 = (1, -1)$, and Γ_4 joining $z_4 = (1, -1)$ to $z_1 = (1, 1)$. Therefore (18) can be applied to show that

$$\int_{\Gamma_j} \frac{d\zeta}{\zeta - 0} = \text{Log}(z_{j+1}) - \text{Log}(z_j),\ j = 1, 3, 4 \tag{19}$$

To evaluate the integral (19) for $j = 2$, a different branch of the logarithm must be chosen, one which is analytic in a domain including the curve Γ_2. One choice is to take as a branch cut the non-negative real axis $R^+ = \{(x, 0):0 \le x\}$ and $G = \mathbb{C} - R^+$. A continuous branch of the argument may be specified on this domain by demanding

$$0 < \arg(z) < 2\pi$$

Denoting the logarithm obtained in this way by $\log^+(z)$:

$$\int_{\Gamma_2} \frac{d\zeta}{\zeta - 0} = \log^+(z_3) - \log^+(z_2) \tag{20}$$

Now

$$\text{Log}(z_2) = \log\sqrt{2} + i3\pi/4 = \log^+(z_2)$$

but

$$\text{Log}(z_3) = \log\sqrt{2} - i3\pi/4$$
$$\log^+(z_3) = \log\sqrt{2} + i5\pi/4$$

Substituting into equation (9) gives

$$\begin{aligned} 2\pi i h(0) &= \text{Log}(z_2) - \text{Log}(z_1) + \log^+(z_3) - \log^+(z_2) \\ &\quad + \text{Log}(z_3) - \text{Log}(z_4) + \text{Log}(z_4) - \text{Log}(z_1) \\ &= i5\pi/4 - (-3i\pi/4) = 2\pi i \end{aligned}$$

and $h(0) = 1$ as required.

For $z \ne 0$ inside the square, the choice of logarithms can be thought of in terms of translating the origin to the point z and using branch cuts such as the curves R^- and R^+ used in the case $z = 0$. With the proper choice of branch cut,

$$\int_{\Gamma_j} \frac{d\zeta}{\zeta - z} = \log^j(z_{j+1} - z) - \log^j(z_j - z) \tag{21}$$

where $\log^j(\zeta - z)$ denotes a branch of the logarithm that is analytic as a function of ζ in a domain containing the curve Γ_j.

The discussion has so far focused on evaluating $h(z)$, as given by equation (5), for a point z in Ω. However, to solve the Dirichlet problem using the function h one must compute the limiting values of $h(z)$ as z in Ω approaches a point z' on Γ, for it is the real part of this limiting value that is required to be the boundary value $g(z')$. Denoting the limiting value by $h_+(z')$, it is given by the Sokhotski–Plemelj formula (Ref. 7, p. 32; Ref. 4, p. 94; or Ref. 5).

$$h_+(z') = [1 - (\theta(z')/2\pi)]\hat{g}(z') + \frac{1}{2\pi i}\int_\Gamma \frac{\hat{g}(\zeta)\, d\zeta}{(\zeta - z')} \tag{22}$$

where, as above, $\theta(z')$ is the interior angle the curve makes at the point z', which is π unless z' is a corner point. (The hypotheses under which this formula holds are satisfied by the function $\hat{g}$ given by (4).)

A problem that arises in applying the formula (22) is when the integral

$$\frac{1}{2\pi i}\int_\Gamma \frac{\hat{g}(\zeta)\, d\zeta}{(\zeta - z')}$$

is reduced to the form of equation (8), the integral

$$\int_{\Gamma_j} \frac{d\zeta}{\zeta - z'}$$

in that sum does not exist in the usual sense for z' on Γ_j. To understand how this problem is surmounted, first consider some facts about improper integrals given in Example 4 below.

Example 4

(a) There is a technical aspect relating to how an improper integral is defined in calculus that will be important in what follows. To illustrate this, consider the integral $\int_{-1}^1 x^{-2/3}dx$. This integral is improper because the integrand $x^{-2/3}$ is unbounded. First, the point 0 where the integrand becomes unbounded is singled out and the integral is defined as

$$\int_{-1}^1 x^{-2/3}\, dx = \int_{-1}^0 x^{-2/3}\, dx + \int_0^1 x^{-2/3}\, dx \tag{23}$$

if both integrals on the right-hand side of (23) exist. Second, each integral on the right-hand side of (23) is defined in a similar way, namely:

$$\begin{aligned} \int_{-1}^0 x^{-2/3}\, dx &= \lim_{\varepsilon \to 0^+} \int_{-1}^{-\varepsilon} x^{-2/3}\, dx \\ &= \lim_{\varepsilon \to 0^+} 3((-\varepsilon)^{1/3} - (-1)^{1/3}) = 3 \end{aligned} \tag{24}$$

$$\begin{aligned} \int_0^1 x^{-2/3}\, dx &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-2/3}\, dx \\ &= \lim_{\varepsilon \to 0^+} 3(1^{1/3} - (\varepsilon)^{1/3}) = 3 \end{aligned} \tag{25}$$

so that the integral in (23) exists and has the value 6.

(b) As another example, consider the integral

$$\int_{-1}^{1} x^{-1}\,dx \tag{26}$$

As in (a) above, this integral is defined as the sum of two integrals:

$$\int_{-1}^{1} x^{-1}\,dx = \int_{-1}^{0} x^{-1}\,dx + \int_{0}^{1} x^{-1}\,dx \tag{27}$$

But in this case,

$$\begin{aligned}\int_{0}^{1} x^{-1}\,dx &= \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1} x^{-1}\,dx \\ &= \lim_{\varepsilon\to 0^+}(\log(1) - \log(\varepsilon)) = \infty\end{aligned} \tag{28}$$

and the other integral in the right-hand side of (27) also does not exist. Thus the integral in (26) does not exist in the usual sense. However the integral (26) does exist in another sense, that of the Cauchy principal value. The Cauchy principal value, or principal value, of the integral (26) is defined to be the limit, if it exists, of

$$\begin{aligned}\int_{-1}^{1} x^{-1}\,dx &= \lim_{\varepsilon\to 0^+}\left[\int_{-1}^{-\varepsilon} x^{-1}\,dx + \int_{\varepsilon}^{1} x^{-1}\,dx\right] \\ &= \lim_{\varepsilon\to 0^+}[\log(|-\varepsilon|) - \log(\varepsilon)] = 0\end{aligned} \tag{29}$$

The difference between the usual definition of the improper integral and the principal value is that in the usual definition the limits in (24) and (25) are taken independently, but for the principal value there is only one limit, as in (29), describing the behavior of the integral near the point 0 at which the integrand is unbounded.

The principal value of the line integral[4,5,7]

$$\int_{\Gamma_j} \frac{d\zeta}{(\zeta - z')} \tag{30}$$

for z' on the arc Γ_j is defined, analogous to example 4(b) above, as follows: Suppose that z' lies on Γ_j but is not either of the endpoints. For any $\delta > 0$, let $B(z', \delta)$ be the ball $B(z', \delta) = \{\omega: |\omega - z'| < \delta\}$ and consider

$$\lim_{\delta\to 0^+}\int_{C_j} \frac{d\zeta}{(\zeta - z')} \tag{31}$$

where $C_j = \Gamma_j - B(z', \delta)$. For a smooth curve Γ_j and small $\delta > 0$, the boundary of the ball $B(z', \delta)$ hits Γ_j in two points $\zeta_1 = \gamma(\tau_1)$ and $\zeta_2 = \gamma(\tau_2)$, where $z' = \gamma(t')$ and $0 < t_j < \tau_1 < t' < \tau_2 < t_{j+1} < 1$ (without loss of generality supposing that neither t_j nor t_{j+1} is 0 or 1). Of course ζ_i and τ_i depend on δ for $i = 1, 2$. In terms of the parameterization, equation (31) defines the integral (30) by

$$\lim_{\delta\to 0^+}\left[\int_{t_j}^{\tau_1} \frac{\gamma'(t)\,dt}{\gamma(t) - z'} + \int_{\tau_2}^{t_{j+1}} \frac{\gamma'(t)\,dt}{\gamma(t) - z'}\right] \tag{32}$$

If $\log_{z'}(\zeta - z')$ is a branch of the logarithm that is analytic as a function of ζ in an open set G containing C_j, equation (32) can be evaluated as

$$\log_{z'}(z_{j+1} - z') - \log_{z'}(z_j - z') + i\Theta(z') \tag{33}$$

where $\Theta(z')$ is the interior angle the curve makes at the point z'.

How is the required branch of the logarithm in (33) obtained? What is needed is a non-self-intersecting curve $P_{z'}$, which joints z' to infinity and does not intersect Ω or Γ except at the point z'. That such a curve exists for the simply connected domains with smooth boundaries that we consider here can be shown by an argument such as that given in the proof of lemma 1.2 in Ref. 6 (p. 551), but for any domain arising in an application this will be obvious so we do not give a proof. For such a curve then, the curve $B_{z'}$, which is the translation by z' of $P_{z'}$ to the origin,

$$B_{z'} = P_{z'} - z' \tag{34}$$

will be the branch cut that is used to define the logarithm in (33), there denoted by $\log_{z'}(\zeta - z')$.

There is one more requirement that must be placed on the branch cuts and logarithms that are used to evaluate (31) as (33). This restriction is necessary because the function given in (33) displays a potentially complicated dependence on z' because the branch cut for the logarithm in (33), and therefore the logarithm itself, changes with z'; for example, it is not even clear that (33) represents an analytic function in z'. This problem can be eliminated if the branch cut $P_{z'}$ is chosen so that it also works as a branch cut for any z'' on Γ that is close enough to z'; that is to say that the curve

$$P_{z'} - z' + z''$$

intersects $\Omega \cup \Gamma$ only at the point z''.

Example 5

(a) Consider the unit disk $\Omega = \{(x, y): x^2 + y^2 < 1\}$ and choose the point $z' = (1, 0)$ on Γ. The non-negative real axis is a branch cut $B_{z'}$ for z' that will also work as a branch cut for exactly those $z'' = (x'', y'')$ on Γ, $(x'')^2 + (y'')^2 = 1$, which satisfy $x'' \geq 0$. Similarly, given any z' on Γ, the line from z' to ∞ in the direction of the normal to the circle at z' furnishes a branch cut that works for all z'' in the semicircle whose midpoint is z'.

(b) Consider, as in Example 1, the domain Ω which is the interior of a square.

$$\Omega = \{(x, y): -1 < x < 1, \quad -1 < y < 1\}$$

At a corner on Γ, say $z' = (1, 1)$, the line connecting z' to infinity parallel to the real axis is a branch cut for $\log(\zeta - z')$, but this line will not work as a branch cut for $z'' = (\varepsilon(-1) + (1 - \varepsilon), 1)$, $0 < \varepsilon \leq 1$, which is arbitrarily close to z' for ε small. However, the line connecting z' to infinity making a 45° angle with the x-axis will work as a branch cut for all the points on Γ either of the form $(x, 1)$, $-1 \leq x \leq 1$, or of the form $(1, y)$, $-1 \leq y \leq 1$.

With this final requirement on the branch cuts in place, the integrals in (8) can now be evaluated (for z on Γ) for a mesh of points on Γ that are close enough together. To see this, note that by a compactness argument there is a $\delta > 0$ and branch cuts

$$\{B_{z'} = P_{z'} - z' : z' \text{ in } \Gamma\}$$

so that if z'' is on Γ and $|z' - z''| < \delta$, then $B_{z'}$ also works as a branch cut for $\log(\zeta - z'')$. When the curve is divided into the nodal points $\{z_1, z_2, \ldots z_n\}$, choose these points so close together that B_{z_j} will work as a branch cut for any point on the arc Γ_{j-1} as well as the arc Γ_j. When that has been done, the Cauchy principal value of the integral (30) can be evaluated as

$$\log_j(z_{j+1} - z') - \log_j(z_j - z') + i\Theta(z') \tag{35}$$

where

$$\log_j(\zeta - z') \text{ denotes the log with branch cut } B_{z_j} \tag{36}$$

Use (8) to write the integral in (22)

$$\int_\Gamma \frac{\hat{g}(\zeta)\,d\zeta}{(\zeta - z')} = \sum_{j=1}^{n} \left[g_j \frac{(z - z_{j+1})}{(z_j - z_{j+1})} + g_{j+1} \frac{(z - z_j)}{(z_{j+1} - z_j)} \right] \int_{\Gamma_j} \frac{d\zeta}{\zeta - z} \tag{37}$$

If z' does not belong to Γ_k, then

$$\int_{\Gamma_k} \frac{d\zeta}{(\zeta - z')} = \log_k(z_{k+1} - z') - \log_k(z_k - z') \tag{38}$$

If z' belongs to Γ_j, but is not an endpoint, then

$$\int_{\Gamma_j} \frac{d\zeta}{(\zeta - z')} = \log_j(z_{j+1} - z') - \log_j(z_j - z') + i\Theta(z') \tag{38}$$

By construction of the branch cuts, for any index m, $\log_m(z_m - z')$ and $\log_{m-1}(z_m - z')$ are analytic logarithms in the variable z' for z' in $\mathbb{C} - [B_{z_1} \cup B_{z_2} \ldots \cup B_{z_n}]$, and so, as noted following equation (16),

$$\log_m(z_m - z') = \log_{m-1}(z_m - z') + i2\pi k_m \tag{40}$$

for some integer k_m holds for z' any point on Γ which is not one of the points $\{z_1, z_2, \ldots, z_n\}$. Combine (37), (38), (39), and (40), and collect terms of the form $(z_m - z')\log_m(z_m - z')$, noting that the factors $i2\pi k_m$ contribute to a constant term and a term in z'. Denote the coefficients so obtained by $a_0, a'_0, a_1, \ldots, a_n$, and write

$$\frac{1}{2\pi i} \int_\Gamma \frac{\hat{g}(\zeta)\,d\zeta}{(\zeta - z')} = \hat{g}(z')[\theta(z')/2\pi] + a_0 + a'_0 z' + \sum_{j=1}^{n} a_j(z_j - z') \log_j(z_j - z') \tag{41}$$

Using the Sokhotski–Plemelj formula (22)

$$h_+(z') - \hat{g}(z') = a_0 + a'_0 z' + \sum_{j=1}^{n} a_j(z_j - z') \log_j(z_j - z') \tag{42}$$

The function $\varphi(z)$ given by

$$\varphi(z) = a_0 + a'_0 z + \sum_{j=1}^{n} a_j(z_j - z) \log_j(z_j - z) \tag{43}$$

is analytic for z in the domain

$$G = \mathbb{C} - [B_{z_1} \cup B_{z_2} \ldots \cup B_{z_n}]$$

and is continuous on all of Ω and Γ, even at the points $z_1, z_2, \ldots, z_n$ because $(z_j - z)\log_j(z_j - z)$ tends to zero as z approaches z_j from points of $\Omega \cup \Gamma$. Consequently the function $\mathrm{Re}(\varphi(z))$ is harmonic in Ω and continuous in $\Omega \cup \Gamma$.

The CVBEM for approximately solving the Dirichlet problem uses the function $\varphi(z)$, determining, in various ways, the coefficients $a_0, a'_0, a_1, \ldots, a_n$ in (43) so as to approximate a given continuous boundary function $g(z)$ by the real part of $\varphi(z)$ for z on Γ. The same function $\varphi(z)$ is also used in the approximate solution of mixed boundary value problems.

Note that the derivation presented does not prove that any continuous real-value boundary function g can be approximated to any given degree of accuracy by the real part of $\varphi(z)$, as in (43), for some choice of the coefficients $a_0, a'_0, a_1, \ldots, a_n$. This has been shown, for the Dirichlet problem, in Ref. 8. It has not been proved that the mixed boundary value problem can be so solved under general conditions, although many such specific problems have been solved in practice; however, *a posteriori* bounds on solutions to mixed problems can be computed.[9]

4. Summary

The expression

$$\varphi(z) = a_0 + a'_0 z + \sum_{j=1}^{n} a_j(z_j - z) \log_j(z_j - z)$$

for the function $\varphi(z)$, on which the CVBEM can be based, has been derived using the Sokhotski–Plemelj formulas and the properties of the singular Cauchy integral. This derivation has shown how the branches $\log_j(z_j - z)$ of the logarithm should be chosen in any numerical implementation of the CVBEM.

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