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Expanding the CVBEM approximation in a series

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Abstract

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In the past, the CVBEM (complex variable boundary element method) has been approached as a collocation problem or by least squares. In this work, the CVBEM analog is redeveloped as a series expansion of nodal point functions with unknown nodal point values as the coefficients. This series expansion provides further insight into the theoretical and approximation aspects of the CVBEM. Applications demonstrate the utility of the CVBEM as a computational approach to solving two-dimensional potential problems involving Laplace and Poisson equations.

Keywords. CVBEM; series expansion; boundary elements; complex variables; complex variable boundary element method.

1. Introduction

The complex variable boundary element method [3], or CVBEM, has gained increased use in approximating two-dimensional potential problems since its inception nearly ten years ago [2]. The CVBEM has been extended to include collocation techniques, least-squares minimization, and use of singular approximation functions. The CVBEM numerical analog is reformulated into an expansion of nodal point approximation functions such that several techniques described previously can be unified. With this formulation, the CVBEM is now described as a series expansion, developed from the Cauchy integral. Applications to potential flow problems demonstrate the utility of this numerical technique.

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2. Formulation

The CVBEM is developed from numerically approximating the Cauchy contour integral [3]

$$\omega(z)=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z}, \tag{1}$$

where Γ is a simple closed contour enclosing a simply connected domain Ω; $\omega(z)$ is an analytic function on $\Omega\cup\Gamma$; ζ is an integration variable; $\mathrm{i}=\sqrt{-1}$. It is assumed in (1) that Γ is a polygon with υ vertices, and values of $\omega(\zeta)$ are given on Γ such as to provide a well-defined mixed boundary value potential problem.

Contour Γ is discretized into m boundary elements, Γ_j, by m nodal points such that $m>\upsilon$, and a node is located at each vertex of Γ [3]. A piecewise linear spline function $N_j(\zeta)$ is defined for each node j (located at coordinate $z_j\in\Gamma$) by

$$N_j(\zeta)=\begin{cases}0, & \zeta\notin\Gamma_j\cup\Gamma_{j-1},\\ (\zeta-z_j)/(z_j-z_{j-1}), & \zeta\in\Gamma_{j-1},\\ (z_{j+1}-\zeta)/(z_{j-1}-z_j), & \zeta\in\Gamma_j,\end{cases} \tag{2}$$

where $z_{m+1}\equiv z_1$ appropriately.

A global trial function, based upon the sum of nodal contributions, is

$$G(\zeta)=\sum_{j=1}^{m}N_j(\zeta)\omega_j, \tag{3}$$

where $G(\zeta)$ is a continuous piecewise linear function on Γ; ω_j are nodal point values of a potential function $\phi(z_j)$ and its conjugate $\psi(z_j)$, where $\omega_j=\phi_j+\mathrm{i}\psi_j$. Generally, nodal points are located on Γ such that the given boundary conditions are matched on Γ by the global trial function in (3). Higher-order spline functions may be used in (3), or an increase in nodal point density, in order to obtain a match of the known values of $\omega(z)$ on Γ.

The CVBEM approximator, for linear trial functions (higher-order polynomial splines may be used directly in this development) is

$$\hat{\omega}(z)=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{G(\zeta)\,\mathrm{d}\zeta}{\zeta-z},\quad z\in\Omega, \tag{4}$$

where $\hat{\omega}(z)$ is analytic in Ω and continuous on Γ. Note that $\hat{\omega}(z)=\hat{\phi}(z)+\mathrm{i}\hat{\psi}(z)$, $\nabla^2\hat{\phi}(z)=0$, and $\nabla^2\hat{\psi}(z)=0$.

3. Series expansion of the CVBEM analog

The formulation of (4) is expanded by substituting (3) into (4) by

$$\hat{\omega}(z)=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\sum_{j=1}^{m}\frac{\omega_jN_j(\zeta)\,\mathrm{d}\zeta}{\zeta-z} \tag{5}$$

$$=\sum_{j=1}^{m}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{j-1}\cup\Gamma_j}\frac{\omega_jN_j(\zeta)\,\mathrm{d}\zeta}{\zeta-z}, \tag{6}$$

where in (6), $\Gamma_0\equiv\Gamma_m$ appropriately.

Rewriting (6),

$$\hat{\omega}(z)=\frac{1}{2\pi i}\sum_{j=1}^{m}\omega_j\int_{\Gamma_{j-1}\cup\Gamma_j}\frac{N_j(\zeta)\,d\zeta}{\zeta-z}. \tag{7}$$

Integrating (7), the series expansion is developed by

$$\hat{\omega}(z)=\sum_{j=1}^{m}\omega_j\frac{1}{2\pi i}\left[\left(\frac{z-z_j}{z_j-z_{j-1}}\right)(\ln(z_j-z)-\ln(z_{j-1}-z))\right.$$
$$\left.+\left(\frac{z_{j-1}-z}{z_{j-1}-z_j}\right)(\ln(z_{j+1}-z)-\ln(z_j-z))\right] \tag{8}$$
$$=\sum_{j=1}^{m}\omega_j H_j(z). \tag{9}$$

where in (9), $H_j(z)$ follows immediately from (8).

In evaluating

$$\int_{\Gamma_j}\frac{1}{\zeta-z}d\zeta\equiv\ln(z_{j+1}-z)-\ln(z_j-z), \tag{10}$$

a branch of the logarithm is chosen so that $\ln(\zeta-z)$ is an analytic function of ζ on Γ_j and all z in Ω, resulting in a right-hand side in (10) which is analytic in z for z in Ω. Note that in (8) and (9), the limit of $\hat{\omega}(z)$, as point $z\in\Omega$ approaches any nodal coordinate $z_j\in\Gamma$, is well defined; this limiting value leads to the collocation techniques used in the CVBEM [3].

For any nodal value ω_j, let $k(\omega_j)$ be the given known value. Then $k(\omega_j)$ is generally either the real number $\phi(z_j)$ or the pure imaginary number $i\Omega(z_j)$, but can also be $\phi(z_j)+i\psi(z_j)$ if the values of both $\phi(z_j)$ and $\psi(z_j)$ are known; in the case that neither $\phi(z_j)$ nor $\psi(z_j)$ is known, $k(\omega_j)=0$. Similarly $u(z_j)$ is the unknown value so that

$$k(\omega_j)+u(\omega_j)=\phi(z_j)+i\psi(z_j). \tag{11}$$

From (10)

$$\hat{\omega}(z)=f^u(z)+f^k(z), \tag{12}$$

where $f^k(z)$ is the known function of z,

$$f^k(z)=\sum k(\omega_j)H_j(z), \tag{13}$$

and

$$f^u(Z)=\sum u(\omega_j)H_j(z) \tag{14}$$

is a function of the unknown nodal value estimates and the complex variable $z\in\Omega$. Equations (10)–(14) present a new series expansion for the CVBEM.

The CVBEM formulation can now be written as a minimization of a norm

$$\| f^u(z)-(\omega(z)-f^k(z))\| \tag{15}$$

on the problem boundary, Γ, with respect to the given boundary condition values. With a match of the boundary conditions, given on Γ, by the selected spline functions and $k(\omega_j)$ values, there is no approximation error associated with the complex function $f^k(z)$.

Thus, the CVBEM approximation analog is to minimize the error in fitting, in a least-squares sense (or other norm), the difference in boundary values of $f^u(z)$ and $(\omega(z)-f^k(z))$ on Γ, where boundary condition values are given.

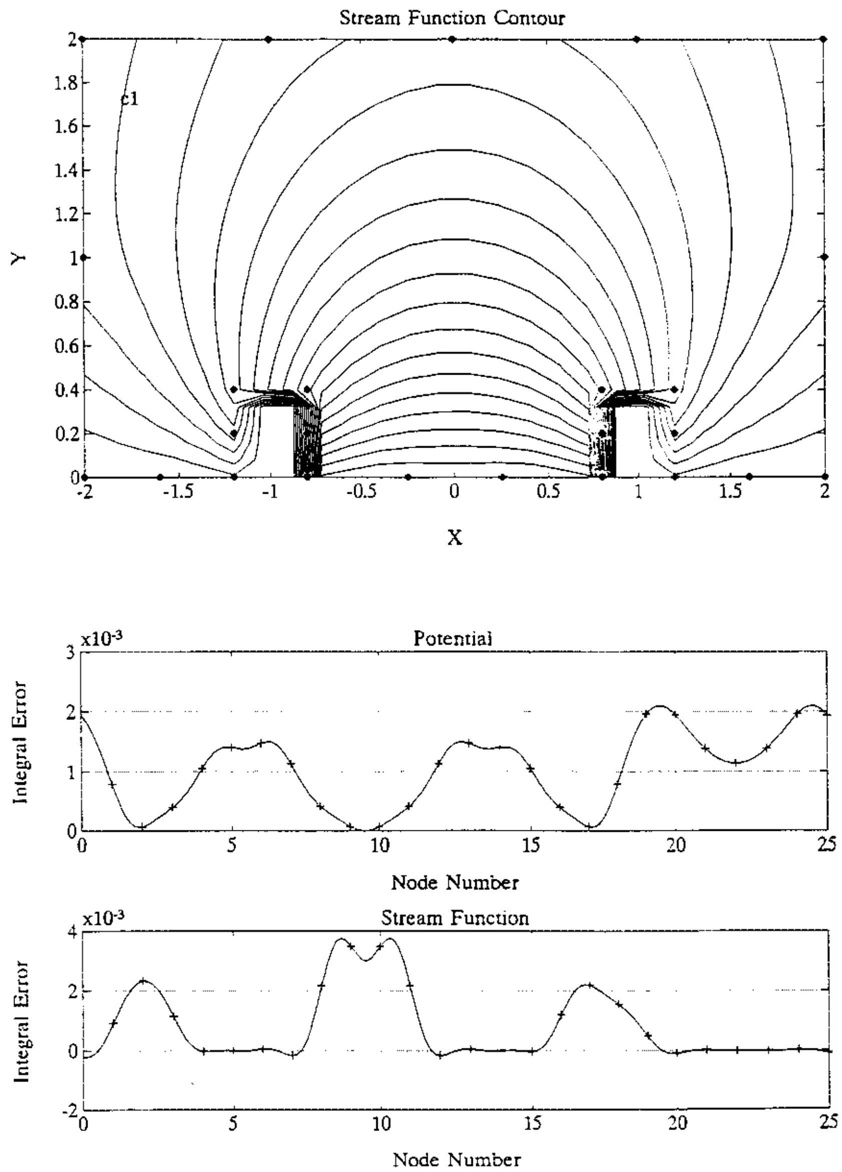

Fig. 1.

4. Application

Two problems, where analytic solutions are readily available, are considered herein to demonstrate the CVBEM accuracy improvement with respect to nodal point placement. Two cases are considered; namely 25-node and 40-node placements on the problem boundary. The problem solution is $\omega(z) = \ln(z+1)/\ln(z-1)$ in the upper half-plane, bounded by $x = -2$, $x = 2$, and $y = 2$, with square insets about the logarithm singularities at $x = -1$ and $x = 1$ as shown in the figures.

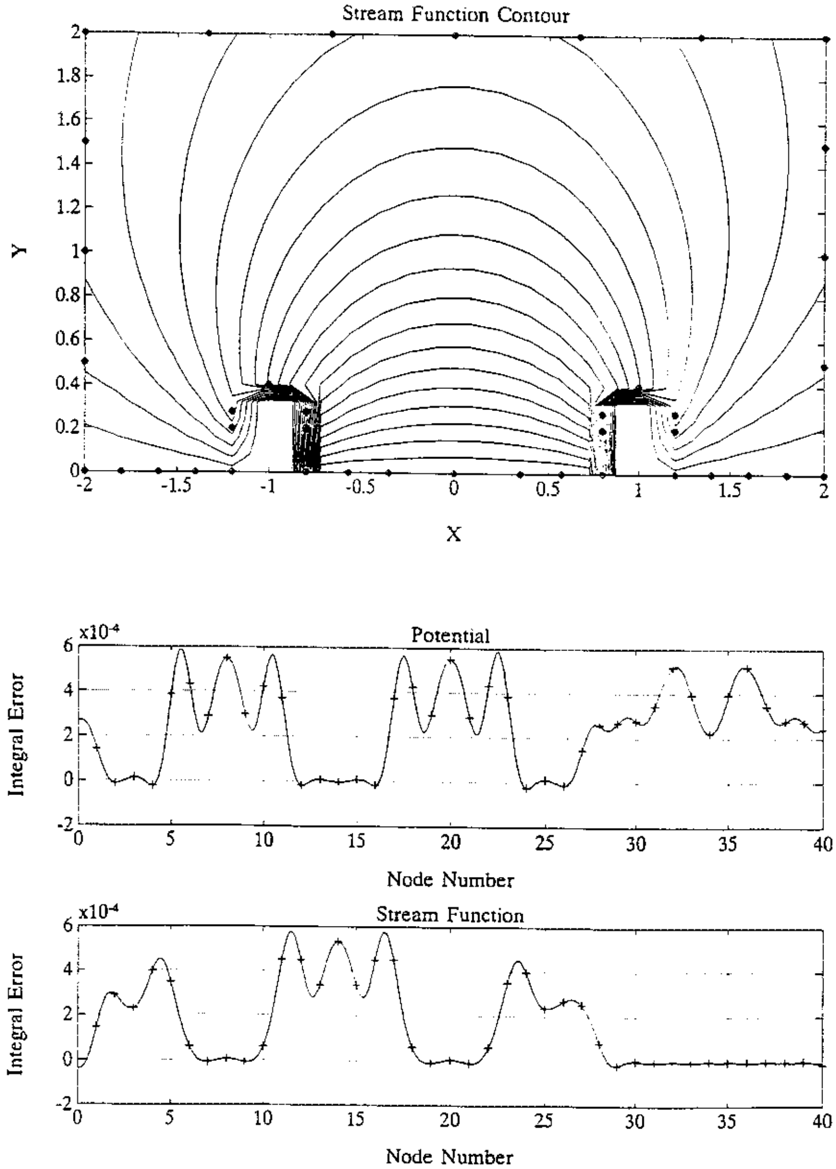

Fig. 2.

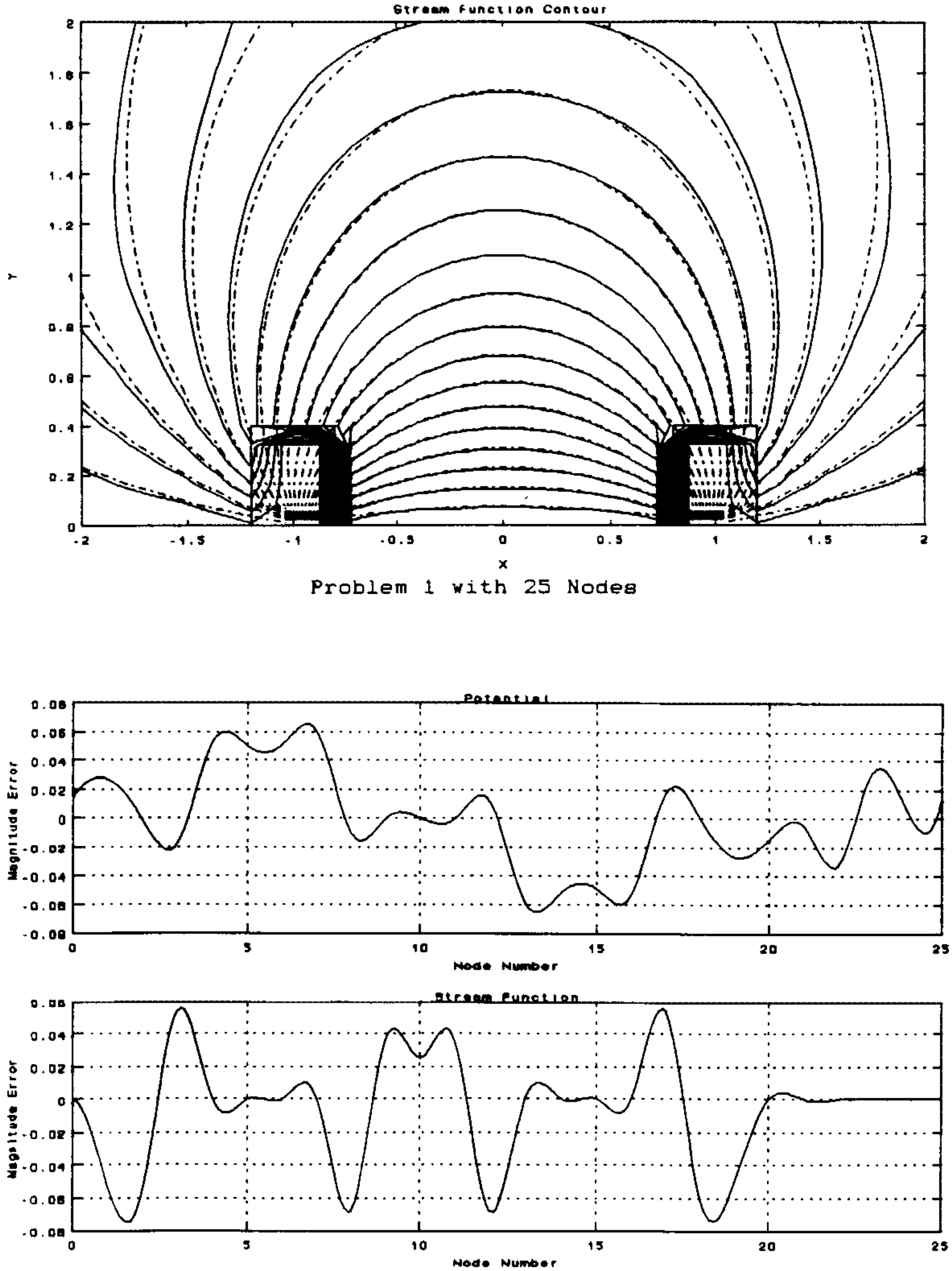
Stream Function Contour
Y
x
Problem 1 with 25 Nodes
Potential
Magnitude Error
Node Number
Stream Function
Magnitude Error
Node Number

Fig. 3.

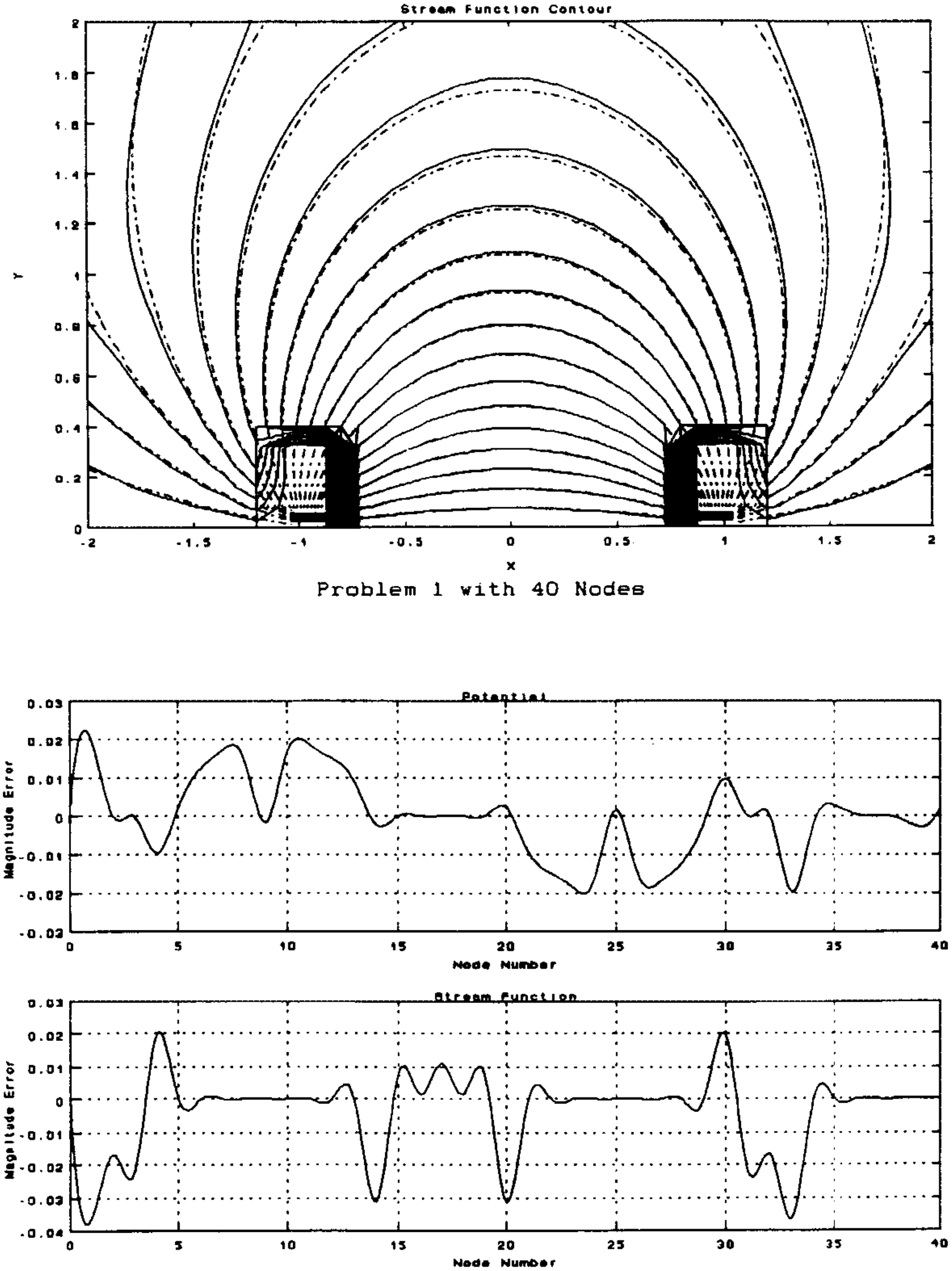
Stream Function Contour
Y
x
Problem 1 with 40 Nodes
Potential
Magnitude Error
Node Number
Stream Function
Magnitude Error
Node Number

Fig. 4.

In the CVBEM application, boundary conditions of the potential (ϕ) are given everywhere on the boundary, except along the two square insets (see figures) where stream function (ψ) values are given. Node placement is shown in the figures, with counterclockwise numbering, beginning with number 1 at the lower left-hand corner. Because the CVBEM develops a continuous solution throughout $\Omega \cup \Gamma$, a maximal error and also a root mean square error is computed along each boundary element and plotted for both the stream function and potential functions. Figure 1 shows the nodal placement and computed stream function contours for a 25-node CVBEM model, and includes the root mean square integral error for both the stream function and potential functions, respectively. Figure 2 considers the 40-node case study. For further comparison, Fig. 3 examines a maximal error for both the stream and potential functions, and also compares computed stream function contours to the solution contours. Similar to Fig. 3, Fig. 4 compares the 40-node model stream function contours to the solution contours.

5. Conclusions

The CVBEM analog is redeveloped into a series expansion of nodal point functions. From the series expansion, the well-known collocation and least-squares formulations, currently in use, can be seen to be applications of different norms in minimizing a fitting of the CVBEM approximation function, $f^u(z)$, to the boundary conditions of the problem. Applications provided demonstrate the utility of the CVBEM in approximating mixed boundary value problems.

References

[1] T.V. Hromadka II, *The Best Approximation Method in Computational Mechanics* (Springer, 1992).
[2] T.V. Hromadka II and G.L. Guymon, A complex variable boundary element method: development, *Internat. J. Numer. Methods Engrg.* 20 (1984) 25–37.
[3] T.V. Hromadka II and Chintu Lai, *The Complex Variable Boundary Element Method in Engineering Analysis* (Springer, New York, 1987).