EXPANDING THE CVBEM ANALOG INTO A SERIES

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In the past, the CVBEM (complex variable boundary element method) has been approached as a collocation problem or by least squares. In this work, the CVBEM analog is redeveloped, for the first time, as a series expansion of nodal point functions with unknown nodal points values as the coefficients. This series expansion provides further insight into the theoretical and approximation aspects of the CVBEM.

Key words: CVBEM, series expansion, boundary elements, complex variables, complex variable boundary element method.

Introduction

The complex variable boundary element method[1] or CVBEM, has gained increased use in approximating two-dimensional potential problems since its inception nearly ten years ago[2]. The CVBEM has been extended to include collocation techniques, least-squares minimization, and use of singular approximation functions. In this note, the CVBEM numerical analog is reformulated into a new expansion of nodal point approximation functions such that the several techniques described previously can be unified. With this new formulation, the CVBEM is now described as a series expansion, developed from the Cauchy integral.

Formulation

The CVBEM is developed from numerically approximating the Cauchy contour integral[1]

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\mathrm{d}\zeta}{\zeta - z} \tag{1}$$

where Γ is a simple closed contour enclosing a simply connected domain Ω; $\omega(z)$ is an analytic function on $\Omega \cup \Gamma$; ζ is an integration variable; $i = \sqrt{-1}$. It is assumed in (1) that Γ is a polygon with v vertices, and values of $\omega(\zeta)$ are given on Γ such as to provide a well-defined mixed boundary value potential problem.

Contour Γ is discretized into m boundary elements, Γ_j by m nodal points such that $m > v$, and a node is located at each vertex of Γ.[1] A piecewise linear spline function $N_j(\zeta)$ is defined for each node j (located coordinate $z_j \in \Gamma$) by

$$N_j(\zeta) = \begin{cases} 0, & \zeta \notin \Gamma_j \cup \Gamma_{j-1} \\ (\zeta - z_j)/(z_j - z_{j-1}), & \zeta \in \Gamma_{j-1} \\ (z_{j+1} - \zeta)/(z_{j+1} - z_j), & \zeta \in \Gamma_j \end{cases} \tag{2}$$

where $z_{m+1} \equiv z_1$ appropriately.

A global trial function, based upon the sum of nodal contributions is

$$G(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\omega_j \tag{3}$$

where $G(\zeta)$ is a continuous piecewise linear function on Γ; ω_j are nodal point values of a potential function $\phi(z_j)$ and its conjugate $\psi(z_j)$, where $\omega_j = \phi_j + i\psi_j$. (Generally, nodal points are located on Γ such that the given boundary conditions are matched on Γ by the global trial function in (3). Higher order slpline functions may be used in (3), or an increase in nodal point density, in order to obtain a match of the known values of $\omega(z)$ on Γ).

The CVBEM approximator, for linear trial functions (higher order polynomial splines may be used directly in this development) is

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\mathrm{d}\zeta}{\zeta - z}, z \in \Omega \tag{4}$$

where $\hat{\omega}(z)$ is analytic in Ω and continuous on Γ. Note that $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, and $\nabla^2\hat{\phi}(z) = 0$, $\nabla^2\hat{\psi}(z) = 0$.

Series expansion of CVBEM analog

The formulation of (4) is now expanded by substituting (3) into (4) by

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \sum_{j=1}^{m} \frac{\omega_j N_j(\zeta)\mathrm{d}\zeta}{\zeta - z} \tag{5}$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\Gamma_{j-1}\cup\Gamma_j} \frac{\omega_j N_j(\zeta)\mathrm{d}\zeta}{\zeta - z} \tag{6}$$

where in (6), $\Gamma_0 \equiv \Gamma_m$ appropriately.

Rewriting (6),

$$\hat{\omega}(z) = \frac{1}{2\pi i}\sum_{j=1}^{m}\omega_j \int_{\Gamma_{j-1}\cup\Gamma_j} \frac{N_j(\zeta)\mathrm{d}\zeta}{\zeta - z} \tag{7}$$

Integrating (7), the series expansion is developed by

$$\hat{\omega}(z) = \sum_{j=1}^{m}\omega_j \frac{1}{2\pi i}\left[\left(\frac{z - z_{j-1}}{z_j - z_{j-1}}\right)\right. \tag{8}$$

$$(\ln(z_j - z) - \ln(z_j - z)) + \left(\frac{z_{j+1} - z}{z_{j+1} - z_j}\right)$$

$$\left.(\ln(z_j - z) - \ln(z_{j+1} - z))\right]$$

$$= \sum_{j=1}^{m}\omega_j H_j(z) \tag{9}$$

where in (9), $H_j(z)$ follows immediately from (8).

Note that in (8) and (9), the limit of $\hat{\omega}(z)$, as point $z \in \Omega$ approaches any nodal coordinate $z_j \in \Omega$ is well defined; this limiting value leads to the collocation techniques used in CVBEM.[1] Also, the complex logarithm branch-cut issues are avoided by the expansion of (8).

The least squares techniques used in the CVBEM [3] can be seen in using (9) by first considering the known and the unknown nodal values given respectively by ξ_p^k, ξ_p^u. Letting δ be a characteristic type function assuming the values 1 or i, for ξ being real or imaginary, respectively, then (9) is written as the series expansion of nodal point functions by

$$\hat{\omega}(z) = \sum_{p=1}^{nu} \delta_p^u \xi_p^u H_p(z) + \sum_{q=1}^{nk} \delta_q^k \xi_q^k H_q(z) \tag{10}$$

where δ_p^u and δ_q^k are the complex-valued characteristic type functions associated with the unknown and known nodal values, respectively; nu and nk are the number of unknown and known nodal values, respectively; and $nu + nk = 2m$.

In (10), the ξ_q^k are known by the boundary conditions of the problem, and the unknown nodal values, ξ_p^u, are to be estimated by the $\hat{\xi}_p^u$.

From (10)

$$\hat{\omega}(z) = f^u(z) = f^k(z) \tag{11}$$

where $f^k(z)$ is a known function of z, and

$$f^u(z) = \sum_{p=1}^{nu} \delta_p^u \hat{\xi}_p^u H_p(z) \tag{12}$$

is a function of the nu unknown nodal value estimates $\{\hat{\xi}_p^u; p = 1,2,...,nu\}$ and the complex variable $z \in \Omega$. Equations (10) through (12) present a new series expansion for the CVBEM.

The CVBEM formulation can now be written as a minimization of a norm

$$\| f^u(z) - (\omega(z) - f^k(z)) \| \tag{13}$$

on the problem boundary, Γ, with respect to the given boundary condition values. With a match of the boundary conditions, given on Γ, by the selected spline functions and ξ_q^k values, there is no approximation error associated with the complex function $f^k(z)$.

Thus, the CVBEM approximation analog is to minimize the error in fitting, in a least squares sense (or other norm), the difference in boundary values of $f^u(z)$ and $(\omega(z) - f^k(z))$ on Γ, where boundary condition values are given.

Conclusions

The CVBEM analog is, for the first time, redeveloped into a series expansion of nodal point functions. From the series expansion, the well-known collocation and least-squares formulations currently in use, can be seen to be applications of different norms in minimizing a fitting of the CVBEM approximation function, $f^u(z)$, to the boundary conditions of the problem.

References

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