Expansion of the CVBEM into a series using intelligent fractals (IF's)
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ABSTRACT

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the Complex Variable Boundary Element Method (or CVBEM) into a series. The entire approximation effort can be written as a sum of Cauchy integrals of incremental changes in basis functions.

BACKGROUND

The Complex Variable Boundary Element Method, or CVBEM, has been the subject of several papers and books (e.g., Hromadka and Lai, 1987; Hromadka, 1993). The basis of the CVBEM is the use of the Cauchy integral equation to develop approximations of two-dimensional boundary
value problems involving the Laplace and Poisson equations. A property of the CVBEM is that the resulting approximation function, \( \hat{\omega}(z) \), is analytic in the simply connected domain, \( \Omega \), and continuous on the problem boundary, \( \Gamma \). Thus, \( \hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z) \), where \( \hat{\phi}(z) \) and \( \hat{\psi}(z) \) are the potential and stream functions, respectively, and both functions satisfy the Laplace equation in \( \Omega \).

Let \( \omega(z) = \phi(x,y) + i\psi(x,y) \) be analytic on \( \Gamma \cup \Omega \), where \( \Omega \) is a simply connected domain enclosed by the simple closed boundary \( \Gamma \). Since \( \omega \) is analytic, \( \phi \) and \( \psi \) are related by the Cauchy-Reimann equations

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{1}
\]

and thus both functions satisfy the two-dimensional Laplace equation in \( \Omega \), namely

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2}
\]

The Cauchy integral theorem states that

\[
\omega(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta) \, d\zeta}{\zeta - z}, \quad z \in \Omega, \ z \not\in \Gamma \tag{3}
\]

Let the boundary \( \Gamma \) be a polygonal line composed of \( V \) vertices. Define nodal points with complex coordinates \( z_j, \ j=1,\ldots,m \) on \( \Gamma \) such that \( m>V \). Nodal points include all the vertices of \( \Gamma \) and are numbered in a counter-clockwise direction. Let complex boundary element \( \Gamma_j \) be the straight line segment joining \( z_j \) and \( z_{j+1} \) so that \( \Gamma = \bigcup_{j=1}^{m} \Gamma_j \). The CVBEM defines a continuous global trial function, \( G(z) \), by

\[
G(z) = \sum_{j=1}^{m} N_j(z) \, (\phi_j + i\psi_j) \tag{4}
\]
where,

$$N_j(z) = \begin{cases} \frac{z - z_{j-1}}{z_j - z_{j-1}} & z \in \Gamma_{j-1} \\ 0 & z \not\in \Gamma_j \cup \Gamma_{j-1} \\ \frac{z_{j+1} - z}{z_{j+1} - z_j} & z \in \Gamma_j \end{cases}$$  \hspace{1cm} (5)

and where $\phi_j$ and $\psi_j$ are nodal values of the two conjugate components, evaluated at $z_j$. An analytic approximation is then determined by

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta) \, d\zeta}{\zeta - z} \quad z \in \Omega, \ z \not\in \Gamma$$  \hspace{1cm} (6)

If we let $q_{j,j+1}$ be the difference in the polar coordinate angles defined by nodal point coordinates $z_{j+1}$ and $z_0$, and $z_j$ and $z_0$, for $z_0$ in $\Omega$, then

$$\hat{\omega}(z_0) = \sum_{j=1}^{m} \left[ \omega_j(z_0 - z_j) - \omega_j(z_0 - z_{j+1}) \right] \frac{H_j}{z_{j+1} - z_j}$$  \hspace{1cm} (7)

where $\omega_j$ and $\omega_{j+1}$ are nodal values at coordinates $z_j$ and $z_{j+1}$, and

$$H_j = \ln \left| \frac{z_{j+1} - z_0}{z_j - z_0} \right| + iq_{j,j+1}$$  \hspace{1cm} (8)

Since usually only one of the two specified nodal values $(\phi_j, \psi_j)$ is known at each $z_j$, $j=1, \ldots, m$, values for the unknown nodal values must be estimated as part of the CVBEM approach. It is noted that certain continuity requirements are necessary in the complex logarithm terms.

**FRACTAL TRIAL FUNCTION BASIS**

The continuous global trial function, $G(\zeta)$, is replaced by a rewriting that is analogous to the triangle fractals used in graphical displays.
In our case, we use the symbol $\Delta_{ij}^k (z)$ in describing the incremental change between the value of a straight-line interpolation between consecutive nodal point values $\phi_i$ and $\phi_j$, at $x = x_k$, and the true value of $\phi$ at $x = x_k$, denoted by $\phi_k$.

Given coordinates $z_i, z_j, z_k$ for nodes $i, j, k$, respectively,

$$
\Delta_{ij}^k (z) \equiv \begin{cases} 
0; & z \notin \text{the boundary element containing nodes } i, j. \\
\frac{z - z_i}{z_k - z_i}; & z \text{ between nodes } i, k \text{ and } z \in \Gamma. \\
\frac{z_j - z}{z_j - z_k}; & z \text{ between nodes } k, j \text{ and } z \in \Gamma.
\end{cases}
$$

Hereafter, $\Delta_{ij}^k (z)$ will be simply written as $\Delta_{ij}$ as it is understood that a function of $z$ is involved. Straightline interpolation between nodes $i, j$ gives an estimate, $\tilde{\phi}_k$, at $z_k$ of (see Fig. 1)

$$
\tilde{\phi}(z_k) = \phi_i \left( \frac{z_k - z_i}{z_j - z_i} \right) + \phi_j \left( \frac{z_k - z_i}{z_j - z_i} \right)
$$

which would be the value of the global trial function at $z_k$, $G(z_k)$, for the case where node $k$ is already a node used in $G(\zeta)$. Adding the node $z_k$ to $G(\zeta)$ simply adds the incremental contribution of $\phi_k$,

$$
G(\zeta) \to G(\zeta) + \Delta_{ij}^k (\phi_k - \tilde{\phi}_k)
$$
For the case of an eight node approximation (see Fig. 2),

\[
G(\zeta) = \Delta_{11} \phi_1 + \Delta_{11} (\phi_2 - \phi_2) + \Delta_{12} (\phi_3 - \phi_3) + \Delta_{21} (\phi_4 - \phi_4) + \Delta_{13} (\phi_5 - \phi_5) + \Delta_{22} (\phi_6 - \phi_6) + \Delta_{24} (\phi_7 - \phi_7) + \Delta_{41} (\phi_8 - \phi_8)
\] (12)

In the above equation, \( \Delta \) refers to the initial case of having a constant-valued \( G(\zeta) \) defined on \( \Gamma \), where \( G(\zeta) = \phi_1 \) for all \( z \in \Gamma \), due to having only a single node (#1) defined on \( \Gamma \). Given a specified sequence of nodal point insertion on \( \Gamma \), the index notation of \( i,j,k \) can be simplified to simply using \( i \), as it is known that node \( k \) is to follow node \( i \) (in the counterclockwise direction) on \( \Gamma \), and \( k \) is known by being the kth index term. In the following, it will be assumed that a node installation sequence, \( S \), is defined so that node numbers \( i,j \) are understood when given node number \( k \). Consequently,

\[
G(\zeta) = \sum_{k=1}^{m} \Delta_{s_k s_{k+1}} (\phi_k - \phi_k)
\] (13)

where \( s_k \) is the kth term of \( S \), \( s_{m+1} = s_1 \), and necessarily \( \phi_1 = 0 \) for the initial case of \( k=1 \). Equation (13) can be rewritten as,

\[
G(\zeta) = \sum_{k=1}^{m} \Delta (\phi_k - \phi_k)
\] (14)

where a nodal point installation sequence, \( S \), is defined, and nodes \( i \) and \( j \) are known given node \( k \).

From the above, the extension of a complex variable function \( \omega(z) \), defined on \( \Gamma \) is given, for \( m \) nodes on \( \Gamma \), by
Boundary Elements

\[ G(\zeta) = \sum_{k=1}^{m} \Delta (\omega_k - \hat{\omega}_k) \]  \hspace{1cm} (15)

where \( \omega(z_k) = \omega_k, \ k = 1,2,\ldots,m; \) and as before, necessarily \( \hat{\omega}_1 = 0. \)

EXTENSION TO THE CVBEM

The CVBEM approximation function can be written as, for \( m \) nodes on \( \Gamma, \)

\[ \hat{\omega}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\Gamma} \sum_{k=1}^{m} \Delta (\omega_k - \hat{\omega}_k) \frac{d\zeta}{\zeta - z}, \ z \in \Omega \]

or

\[ \hat{\omega}(z) = \frac{1}{2\pi i} \sum_{k=1}^{m} (\omega_k - \hat{\omega}_k) \int_{\Gamma} \frac{\Delta d\zeta}{\zeta - z}, \ z \in \Omega \]  \hspace{1cm} (16)

For the case of \( \omega(z) \) being analytic on \( \Gamma, \) then \( \omega(z) \) is continuous on \( \Gamma \) where \( G(\zeta) \to \omega(\zeta) \) on \( m \to \infty \) (and the arclength between successive nodes \( \to 0 \)), and from Schauder's theorem (see Cheney, 1966),

\[ \omega(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} (\omega_k - \hat{\omega}_k) \int_{\Gamma} \frac{\Delta d\zeta}{\zeta - z}, \ z \in \Omega \]  \hspace{1cm} (18)
In the above,

\[
\int_{\Gamma} \Delta \frac{d\zeta}{\zeta - z} = \int_{\Gamma}^{z_k} \Delta \frac{d\zeta}{\zeta - z} = \int_{z_i}^{z_k} \Delta \frac{d\zeta}{\zeta - z} = \int_{z_j}^{z_k} \Delta \frac{d\zeta}{\zeta - z}
\]

\[
= \left( \frac{z_i-z_j}{z_k-z_i} \right) \left( \ln(z_k-z) - \ln(z_i-z) \right) + \left( \frac{z_i-z_k}{z_k-z_j} \right) \left( \ln(z_j-z) - \ln(z_k-z) \right)
\]

(19)

CONCLUSIONS

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the Complex Variable Boundary Element Method (or CVBEM) into a series. Further research is needed in evaluating convergence and uniqueness properties.

REFERENCES


Figure 1. \( G(\zeta) = \phi_1 + \Delta_1^2 \phi_2 + \Delta_1^3 \phi_3 + \Delta_1^4 \phi_4. \)

Figure 2. \( G(\zeta) = \sum_{k=1}^{8} \Delta_{i,j} \phi_k. \)