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Expansion of the CVBEM into a series using intelligent fractals (IFs)

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ABSTRACT

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the Complex Variable Boundary Element Method (or CVBEM) into a series. The entire approximation effort can be written as a sum of Cauchy integrals of incremental changes in basis functions.

BACKGROUND

The Complex Variable Boundary Element Method, or CVBEM, has been the subject of several papers and books (e.g., Hromadka and Lai, 1987; Hromadka, 1993). The basis of the CVBEM is the use of the Cauchy integal equation to develop approximations of two-dimensional boundary value problems involving the Laplace and Poisson equations. A property of the CVBEM is that the resulting approximation function, $\hat{\omega}(z)$, is analytic in the simply connected domain, Ω, and continuous on the problem boundary, Γ. Thus, $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, where $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are the potential and stream functions, respectively, and both functions satisfy the Laplace equation in Ω.

Let $\omega(z) = \phi(x,y) + i\psi(x,y)$ be analytic on $\Gamma \cup \Omega$, where Ω is a simply connected domain enclosed by the simple closed boundary Γ. Since ω is analytic, ϕ and ψ are related by the Cauchy-Reimann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{1}$$

and thus both functions satisfy the two-dimensional Laplace equation in Ω, namely

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ and } \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2}$$

The Cauchy integral theorem states that

$$\omega(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta)\, d\zeta}{\zeta - z}, \; z \in \Omega, \; z \notin \Gamma \tag{3}$$

Let the boundary Γ be a polygonal line composed of V vertices. Define nodal points with complex coordinates z_j, $j=1,\ldots,m$ on Γ such that $m>V$. Nodal points include all the vertices of Γ and are numbered in a counter-clockwise direction. Let complex boundary element Γ_j be the straight line segment joining z_j and z_{j+1} so that $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$. The CVBEM defines a continuous global trial function, $G(z)$, by

$$G(z) = \sum_{j=1}^{m} N_j(z)\, (\bar{\phi}_j + i\bar{\psi}_j) \tag{4}$$

where,

$$N_j(z) = \begin{cases} \dfrac{z-z_{j-1}}{z_j-z_{j-1}} & z \in \Gamma_{j-1} \\ 0 & z \notin \Gamma_j \cup \Gamma_{j-1} \\ \dfrac{z_{j+1}-z}{z_{j+1}-z_j} & z \in \Gamma_j \end{cases} \tag{5}$$

and where $\overline{\phi}_j$ and $\overline{\psi}_j$ are nodal values of the two conjugate components, evaluated at z_j. An analytic approximation is then determined by

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G(\zeta)\, d\zeta}{\zeta - z}, \; z \in \Omega, z \notin \Gamma \tag{6}$$

If we let $q_{j,j+1}$ be the difference in the polar coordinate angles defined by nodal point coordinates z_{j+1} and z_o, and z_j and z_o, for z_o in Ω, then

$$\hat{\omega}(z_o) = \sum_{j=1}^{m} [\omega_{j+1}(z_o - z_j) - \omega_j(z_o - z_{j+1})] \frac{H_j}{z_{j+1}-z_j} \tag{7}$$

where ω_j and ω_{j+1} are nodal values at coordinates z_j and z_{j+1}, and

$$H_j = \ln \left| \frac{z_{j+1} - z_o}{z_j - z_o} \right| + i q_{j,j+1} \tag{8}$$

Since usually only one of the two specified nodal values $(\overline{\phi}_j, \overline{\psi}_j)$ is known at each z_j, j=1,...,m, values for the unknown nodal values must be estimated as part of the CVBEM approach. It is noted that certain continuity requirements are necessary in the complex logarithm terms.

FRACTAL TRIAL FUNCTION BASIS

The continuous global trial function, $G(\zeta)$, is replaced by a rewriting that is analogous to the triangle fractals used in graphical displays.

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In our case, we use the symbol $\overset{k}{\underset{i\ j}{\Delta}}(z)$ in describing the incremental change between the value of a straight-line interpolation between consecutive nodal point values ϕ_i and ϕ_j, at $x = x_k$, and the true value of ϕ at $x = x_k$, denoted by ϕ_k.

Given coordinates z_i, z_j, z_k for nodes i, j, k, respectively,

$$\overset{k}{\underset{i\ j}{\Delta}}(z) \equiv \begin{cases} 0\,; & z \notin \text{the boundary element containing nodes } i,\ j. \\ \dfrac{z - z_i}{z_k - z_i}\,; & z \text{ between nodes } i,\ k \text{ and } z \in \Gamma. \\ \dfrac{z_j - z}{z_j - z_k}\,; & z \text{ between nodes } k,\ j \text{ and } z \in \Gamma. \end{cases} \qquad (9)$$

Hereafter, $\overset{k}{\underset{i\ j}{\Delta}}(z)$ will be simply written as $\overset{k}{\underset{i\ j}{\Delta}}$ as it is understood that a function of z is involved. Straightline interpolation between nodes i, j gives an estimate, $\hat{\phi}_k$, at z_k of (see Fig. 1)

$$\hat{\phi}(z_k) = \phi_i\left(\frac{z_j - z_k}{z_j - z_i}\right) + \phi_j\left(\frac{z_k - z_i}{z_j - z_i}\right) \qquad (10)$$

which would be the value of the global trial function at z_k, $G(z_k)$, for the case where node k is already a node used in $G(\zeta)$. Adding the node z_k to $G(\zeta)$ simply adds the incremental contribution of ϕ_k,

$$G(\zeta) \rightarrow G(\zeta) + \overset{k}{\underset{i\ j}{\Delta}}\,(\phi_k - \hat{\phi}_k) \qquad (11)$$

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For the case of an eight node approximation (see Fig. 2),

$$G(\zeta) = \overset{1}{\underset{1\ 1}{\Delta}} \phi_1 + \overset{2}{\underset{1\ 1}{\Delta}} (\phi_2 - \hat{\phi}_2) + \overset{3}{\underset{1\ 2}{\Delta}} (\phi_3 - \hat{\phi}_3) +$$
$$\overset{4}{\underset{2\ 1}{\Delta}} (\phi_4 - \hat{\phi}_4) + \overset{5}{\underset{1\ 3}{\Delta}} (\phi_5 - \hat{\phi}_5) +$$
$$\overset{6}{\underset{3\ 2}{\Delta}} (\phi_6 - \hat{\phi}_6) + \overset{7}{\underset{2\ 4}{\Delta}} (\phi_7 - \hat{\phi}_7) + \overset{8}{\underset{4\ 1}{\Delta}} (\phi_8 - \hat{\phi}_8) \quad (12)$$

In the above equation, $\overset{1}{\underset{1\ 1}{\Delta}}$ refers to the intial case of having a constant-valued $G(\zeta)$ defined on Γ, where $G(\zeta) = \phi_1$ for all $z \in \Gamma$, due to having only a single node (#1) defined on Γ. Given a specified sequence of nodal point insertion on Γ, the index notation of i,j,k can be simplified to simply using i, as it is known that node k is to follow node i (in the counterclockwise direction) on Γ, and k is known by being the kth index term. In the following, it will be assumed that a node installation sequence, S, is defined so that node numbers i,j are understood when given node number k. Consequently,

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\underset{s_k\ s_{k+1}}{\Delta}} (\phi_k - \hat{\phi}_k) \quad (13)$$

where s_k is the kth term of S, $s_{m+1} = s_1$, and necessarily $\hat{\phi}_1 = 0$ for the initial case of k=1. Equation (13) can be rewritten as,

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\Delta} (\phi_k - \hat{\phi}_k) \quad (14)$$

where a nodal point installation sequence, S, is defined, and nodes i and j are known given node k.

From the above, the extension of a complex variable function $\omega(z)$, defined on Γ is given, for m nodes on Γ, by

$$G(\zeta) = \sum_{k=1}^{m} \overset{k}{\Delta} \; (\omega_k - \hat{\omega}_k) \tag{15}$$

where $\omega(z_k) = \omega_k$, $k = 1,2,...m$; and as before, necessarily $\hat{\omega}_1 = 0$.

EXTENSION TO THE CVBEM

The CVBEM approximation function can be written as, for m nodes on Γ,

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{k=1}^{m} \overset{k}{\Delta} (\omega_k - \hat{\omega}_k)\, d\zeta}{\zeta - z}, z \in \Omega \tag{16}$$

or

$$\hat{\omega}(z) = \frac{1}{2\pi i}\sum_{k=1}^{m} (\omega_k - \hat{\omega}_k)\int_\Gamma \frac{\overset{k}{\Delta}\, d\zeta}{\zeta - z}, z \in \Omega \tag{17}$$

For the case of $\omega(z)$ being analytic on Γ, then $\omega(z)$ is continuous on Γ where $G(\zeta) \rightarrow \omega(\zeta)$ on $m \rightarrow \infty$ (and the arclength between successive nodes $\rightarrow 0$), and from Schauder's theorem (see Cheney, 1966),

$$\omega(z) = \frac{1}{2\pi i}\sum_{k=1}^{\infty} (\omega_k - \hat{\omega}_k)\int_\Gamma \frac{\overset{k}{\Delta}\, d\zeta}{\zeta - z}, z \in \Omega \tag{18}$$

In the above,

$$\int_{\Gamma} \frac{\overset{k}{\Delta} d\zeta}{\zeta - z} = \int_{\Gamma} \frac{\overset{k}{\underset{ij}{\Delta}} d\zeta}{\zeta - z} = \int_{z_i}^{z_k} \frac{\overset{k}{\underset{ij}{\Delta}} d\zeta}{\zeta - z} \int_{z_k}^{z_j} \frac{\overset{k}{\underset{ij}{\Delta}} d\zeta}{\zeta - z}$$

$$= \left(\frac{z-z_i}{z_k-z_i}\right)\left(\ln(z_k-z) - \ln(z_i-z)\right) + \left(\frac{z-z_j}{z_k-z_j}\right)\left(\ln(z_j-z) - \ln(z_k-z)\right) \tag{19}$$

CONCLUSIONS

In this paper, the development of triangular fractals that geometrically sum into an area whose boundary is a function, of a specific type, is used to expand the Complex Variable Boundary Element Method (or CVBEM) into a series. Further research is needed in evaluating convergence and uniqueness properties.

REFERENCES

1. Cheney, E., *Introduction to Approximation Theory*, McGraw-Hill, 1966.

2. Harryman III, R.R., Hromadka II, T.V., Vaughn, J.L., and Watson, D.P., The CVBEM for Multiply Connected Domains Using a Linear Trial Function, *Applied Mathematical Modeling*, Vol. 14, pp. 104-110, February 1990.

3. Hromadka II, T.V., and Lai, C., *The Complex Variable Boundary Element Method in Engineering Analysis*, Springer-Verlag, New York, 1987.

4. Hromadka II, T.V., *The Best Approximation Method in Computational Mechanics*, Springer-Verlag, London, 1993.

5. Hromadka II, T.V., Wood, M.A., Ciejka, G.J., CVBEM Error Reduction Using the Approximate Boundary Method, *Engineering Analysis*, 1993, in-press.

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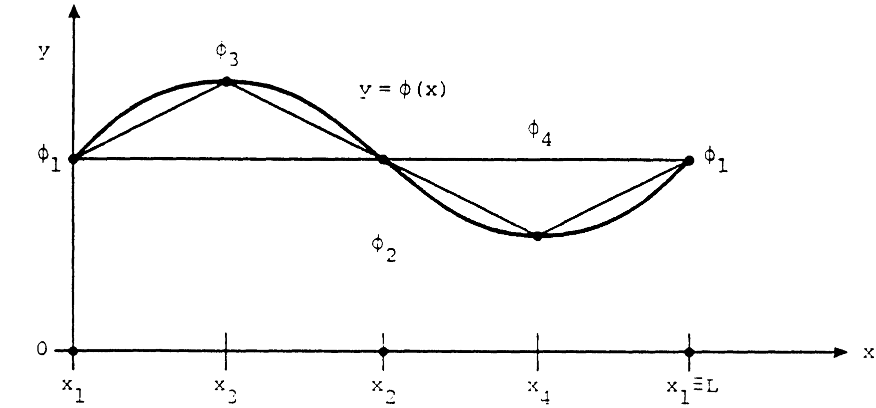

Figure 1. $G(\zeta) = \phi_1 + \underset{1\ 1}{\overset{2}{\Delta}} \phi_2 + \underset{1\ 2}{\overset{3}{\Delta}} \phi_3 + \underset{2\ 1}{\overset{4}{\Delta}} \phi_4.$

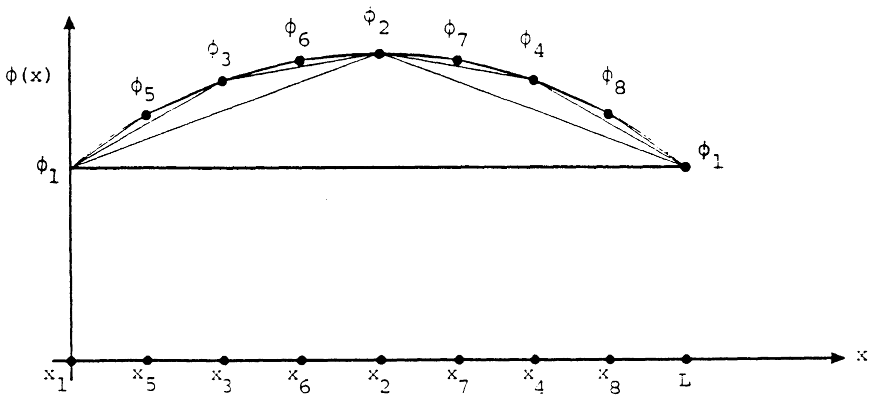

Figure 2. $G(\zeta) = \sum_{k=1}^{8} \underset{i\ j}{\overset{k}{\Delta}} \phi_k.$