Error Bounds for Numerical Solution of Partial Differential Equations

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INTRODUCTION AND PROBLEM FORMULATION

The use of numerical methods to approximately solve partial differential equations is widespread, yet the use of error bounds to evaluate the success of the approximation effort appears to be limited. In this paper, an error bound inequality presented by Protter and Weinberger [1] is used to generate useful information regarding the approximation of partial differential equations by numerical methods. The error bound inequality requires only boundary condition error information and approximation error information that can be readily evaluated, and the construction of an error bound function $\omega(x, y)$ as described below.

Let Ω be bounded domain in the plane with boundary $\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$, the union of two disjoint sets (where Γ_2 is not void). It is assumed that for each point z of Γ_1 there is a solid circle C with z on its boundary, and with $C - \{z\} \subset \Omega$.

The equation to be solved is

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= g_1 \text{ on } \Gamma_1, \\ u &= g_2 \text{ on } \Gamma_2, \end{aligned} \tag{1}$$

where $L = \nabla^2$ is the Laplace operator and $\partial u/\partial n$ is the normal derivative of u on Γ.

However, the approximate solution U that is found actually satisfies the equation

$$\begin{aligned} LU &= F \text{ in } \Omega, \\ \frac{\partial U}{\partial n} &= G_1 \text{ on } \Gamma_1, \\ U &= G_2 \text{ on } \Gamma_2. \end{aligned} \tag{2}$$

Numerical Methods for Partial Differential Equations, 7, 339–346 (1991)
 CCC 0749-159X/91/040339-08$04.00

Suppose that a function ω on $\Omega \cup \Gamma$ can be found that satisfies

$$\omega > 0 \text{ on } \Omega \cup \Gamma, \tag{3a}$$

$$L\omega \leq -1 \text{ in } \Omega, \tag{3b}$$

$$\frac{\partial \omega}{\omega n} \geq 1 \text{ on } \Gamma_1, \tag{3c}$$

$$\omega \geq 1 \text{ on } \Gamma_2. \tag{3d}$$

Then, from Ref. [1],

$$|U(x,y) - u(x,y)| \leq \omega(x,y) \max\{\sup_{\Omega} |F - f|, \sup_{\Gamma_1} |G_1 - g_1|, \sup_{\Gamma_2} |G_2 - g_2|\}. \tag{4}$$

So, if F is uniformly close to f on Ω, and if G_1 is uniformly close to g_1 on Γ_1, and if G_2 is uniformly close to g_2 on Γ_2, then U is close to u on Ω.

This result is not an existence theorem for the solution u to Eq. (1); that would require various additional conditions on Γ_1, Γ_2, Ω, g_1, and g_2. The inequality of Eq. (4) gives a measure of how close the approximate solution U is to the true solution u, assuming there is a solution u, which is often clear on physical grounds.

The aforementioned condition on Γ_1 (that for each z on Γ_1, there is a ball B in $\Omega \cup \Gamma$ with z in B and $B - \{z\} \subset \Omega$) prohibits Γ_1 from containing, for example, two straight line segments meeting at an angle. But it is permissible, for example, to have Ω the square $\{(x,y): 0 < x < 1, 0 < y < 1\}$ and Γ_1 the line $\{(x,0): 0 < x < 1\}$ (but not the line $\{(x,0): 0 \leq x \leq 1\}$, a distinction of little physical significance).

The condition that Γ_2 not be void is necessary because a solution to the Neumann problem

$$Lu = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \Gamma \tag{5}$$

is only unique to within an additive constant; and the inequality (4) implies uniqueness.

We remark that the same inequality holds for the more general problem

$$Lu + hu = 0 \text{ in } \Omega,$$

where L is a strongly elliptic operator on R^n and h is a given function, and the first boundary condition is replaced by

$$\frac{\partial u}{\partial n} + \alpha u = g_1 \text{ on } \Gamma_1,$$

where α is a given function.

One approach to finding a suitable ω to use in the bound of Eq. (4) is to find a function η that approximately satisfies

$$L\eta = -2 \text{ in } \Omega,$$
$$\frac{\partial \eta}{\partial n} = 2 \text{ on } \Gamma_1, \tag{6}$$
$$\eta = 2 \text{ on } \Gamma_2,$$

with sufficient accuracy so that the last three inequality conditions of Eq. (3) are satisfied (with substitution of η for ω). If $-B \leq 0$ is a lower bound for this solution η, then $\omega = \eta + B + 1$ will satisfy all four conditions of Eq. (3). In a given practical problem, the function η of Eq. (6) can be found by numerical methods, such as the complex variable boundary element method (CVBEM). As an example, the following discussion focuses upon the analysis of potential flow problems, which have many applications.

ERROR BOUNDS FOR POTENTIAL PROBLEMS IN TWO-DIMENSIONS, AND DEVELOPMENT OF NORMS FOR ANALYTIC FUNCTION APPROXIMATIONS

If the two-dimensional problem

$$Lu = 0 \text{ in } \Omega,$$
$$\frac{\partial u}{\partial n} = g_1 \text{ on } \Gamma_1, \tag{7}$$
$$u = g_2 \text{ on } \Gamma_2,$$

has a harmonic solution u, with conjugate harmonic function v,

$$f(x + iy) = u(x, y) + iv(x, y) \tag{8}$$

defines a function analytic in Ω. The relation between v and the normal boundary condition on Γ_1 is as follows:

Suppose that Γ is a simple smooth curve, except for a finite number of corners, with a parameterization

$$\Gamma = \{(x(t), y(t)) : 0 \leq t \leq 1\} \tag{9}$$

with a nonzero tangent $[x'(t), y'(t)]$ at all noncorner points of the curve. Since the vector $[y'(t), -x'(t)]$ is perpendicular to the tangent to the curve,

$$\pm \frac{(y'(t), -x'(t))}{s'(t)} \tag{10}$$

are unit normals to the curve, where

$$s'(t) = (x'(t)^2 + y'(t)^2)^{1/2}. \tag{11}$$

Suppose that in Eq. (10) the plus sign gives the outward pointing unit normal at a point t_0; since by hypothesis the vector in Eq. (1) is never zero,

$$n(t) = \frac{(y'(t), -x'(t))}{s'(t)} \tag{12}$$

is the outward pointing unit normal to Γ for all t, excepting those t for which the curve has corners.

If, in addition, we suppose that f is actually analytic on Γ_1, using the Cauchy-Riemann equations gives

$$\begin{aligned} \frac{\partial u}{\partial n} &= (u_x, u_y) \cdot n = \{u_x y'(t) - u_y x'(t)\}/s'(t) \\ &= \{v_y y'(t) + v_x x'(t)\}/s'(t) \\ &= \left\{\frac{dv}{dt}(x(t), y(t))\right\}/s'(t)\,. \end{aligned} \tag{13}$$

Consequently, the boundary condition on Γ_1

$$\frac{\partial u}{\partial n} = g_1 \tag{14}$$

is equivalent to

$$v(x(t), y(t)) = G_1(t) = \int_0^t g_1(x(\tau), y(\tau))s'(\tau)\,d\tau\,. \tag{15}$$

The hypotheses under which Eqs. (14) and (15) are equivalent can be weakened and will depend on what sense the derivatives in Eqs. (14) and (15) are defined and how limits are taken to obtain the functional boundary values.

Define

$$\|u\|_a = \sup_{\Gamma_1} |v| + \sup_{\Gamma_2} |u|\,. \tag{16}$$

For functions u satisfying the conditions above, this is a norm. The only difficult property of the norm to check is that if $\|u\|_a = 0$, then $u = 0$. This follows because if $\|u\|_a = 0$, then the equivalence of Eqs. (14) and (15) shows that u is the solution to Eq. (7) with $g_1 = g_2 = 0$. Then u is zero if inequality (4) holds, and Eq. (4) will hold if the function ω can be found; this will be so under very mild conditions.

A frequently occurring boundary condition in Eq. (7) is that $g_1 = 0$ on Γ_1 (i.e., zero flux on Γ_1). In this case the norms of Eqs. (16) and (17) are simplified in that from Eq. (8), $v(x, y)$ is piecewise constant valued on each contour composing Γ_1; if Γ_1 is a single contour, then $v(x, y) = v_0$ on the entire Γ_1.

APPLICATION TO THE LAPLACE EQUATION (POTENTIAL PROBLEMS)

An important class of problems in engineering and mathematical physics involves the solution of the Laplace or Poisson equation in two dimensions.

Such problems occur frequently in the analysis of potential flow. In the following, the error bound will be applied to potential flow problems.

Let

$$\phi_p(x,y) = -\frac{1}{2}(x^2 + y^2).$$

Then

$$L\phi_p = -2 \text{ in } \Omega. \tag{17}$$

Let $W^* = \phi^* + i\psi^*$ be a CVBEM approximate solution of the potential problem

$$\begin{aligned} \nabla^2\phi^* &= 0 \text{ in } \Omega, \\ \frac{\partial\phi^*}{\partial n} &= 2 - \frac{\partial\phi_p}{\partial n} \text{ on } \Gamma_1, \\ \phi^* &= 2 - \phi_p \text{ on } \Gamma_2. \end{aligned} \tag{18}$$

Note that if the bounded domain Ω is contained in a circle of radius R centered at the origin, then on Ω, $|\phi_p| \le 1/2R^2$ and $|\partial\phi_p/\partial n| \le R$.
The above CVBEM approximation W* is to be determined to within sufficient accuracy to have

$$\begin{aligned} G_1 &= \frac{\partial\phi^*}{\partial n} + \frac{\partial\phi_p}{\partial n} \ge 1 \text{ on } \Gamma_1, \\ G_2 &= \phi^* + \phi_p \ge 1 \text{ on } \Gamma_2, \end{aligned} \tag{19}$$

and where $L(\phi + \phi_p) = -2$ [and hence $L(\phi + \phi_p) \le -1$ in Ω]. Letting $-B$ be a lower bound of $(\phi^* + \phi_p)$ in Ω, then, a lower bound function for use in Eq. (4) is

$$\omega(x,y) = B + 1 + \phi^* + \phi_p \tag{20}$$

for all points (x,y) in Ω.

Note that there are an infinite selection of $\omega(x,y)$ functions to choose from. The particular choice above is readily computable.

REAL VARIABLE AND COMPLEX VARIABLE BOUNDARY ELEMENT METHODS IN SOLVING POTENTIAL PROBLEMS

An advantage to using approximation methods that exactly satisfy the operator relationships (i.e., the Laplace equation), such as real or complex variable boundary element methods, is that in Eq. (4) $F = f = 0$ in Ω, and hence

$$|U(x,y) - u(x,y)| \le \omega(x,y) \max\{\sup_{\Gamma_1} |G_1 - g_1|, \sup_{\Gamma_2} |G_2 - g_2|\} \tag{21}$$

for $\omega(x,y)$ developed as above.

For the case of zero flux on Γ_1, $g_1 = 0$ on Γ_1 and

$$|U(x,y) - u(x,y)| \le \omega(x,y) \max\{\sup_{\Gamma_1} |G_1|, \sup_{\Gamma_2} |G_2 - g_2|\}. \tag{22}$$

APPLICATION OF THE ERROR BOUND INEQUALITY

Steady-state heat transport in two dimensions is mathematically modeled by the Laplace equation. In the application now considered, temperatures are required in a roadway embankment with a buried chilled pipeline (-40 °C). The boundary conditions for the exact solution, $u(x, y)$, and geometric information are given in Figure 1.

Zero heat flux is assumed on the problem domain's left and right sides, except where the chilled pipeline is located. A 40-node CVBEM model [2] is used to approximate the temperatures by $U(x, y)$ in Ω.

For the error analysis, a $\omega(x, y)$ is needed for the inequality of Eq. (22). The particular solution ϕ_p of Eq. (17) is used. Evaluating ϕ_p and $\partial\phi_p/\partial n$ on the problem boundary Γ, a new set of boundary conditions are obtained for the Laplace problem of Eq. (18): the top and bottom temperatures become $[2 + 1/2(x^2 + y^2]$ °C, and the zero flux values become $[2 - (\partial\phi_p/\partial n]$. The chilled pipeline also changes to a boundary condition of $[2 + 1/2(x^2 + y^2]$ °C.

The CVBEM is again used, but this time the goal is to develop a ϕ^* that satisfies Eq. (18) as discussed above. Using the same 40-node CVBEM configuration as used for the original approximation, a $\phi^*(x, y)$ approximation is obtained, and it is noted that in this case $\phi^*(x, y)$ satisfies

$$\begin{aligned} \nabla^2\phi^* &= 0 \text{ in } \Omega, \\ \frac{\partial\phi^*}{\partial n} + \frac{\partial\phi_p}{\partial n} &> 1.8 \text{ on } \Gamma_1, \\ \phi^* + \phi_p &> 1.9 \text{ on } \Gamma_2, \end{aligned} \tag{23}$$

and hence the necessary inequalities of Eq. (19) are easily satisfied by this choice for ϕ^*. The next step in developing an $\omega(x, y)$ is to find some bound $-B < \phi^* + \phi_p$ in $\Omega \cup \Gamma$. Then, an $\omega(x, y)$ is developed by Eq. (21).

Using $\omega(x, y)$, the magnitude of error in the CVBEM approximation $U(x, y)$ in satisfying the heat transport problem of Fig. 1 can now be evaluated by use of the inequality of Eq. (21) or Eq. (22). Figure 2 shows plots of

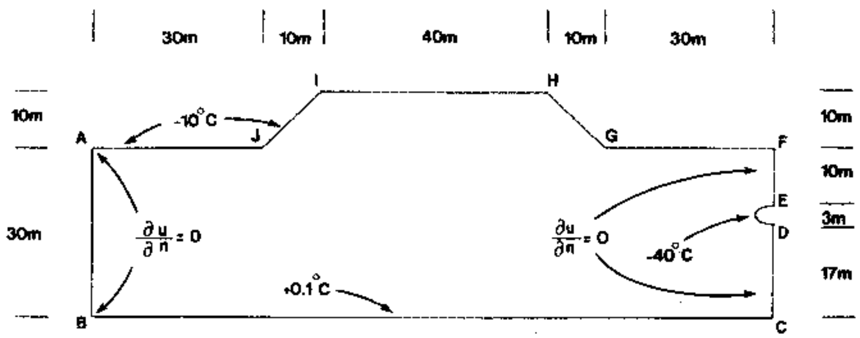

FIG. 1. Application Problem.

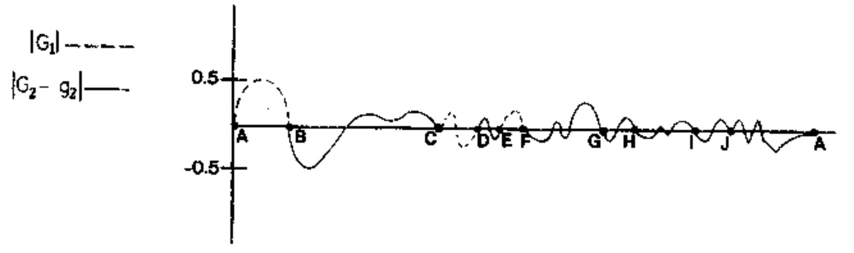

FIG. 2. Plots of $|G_1|$ and $|G_2 - g_2|$.

the $|G_1|$ and $|G_2 - g_2|$ along Γ. From Fig. 2, the maximum value of either error in absolute value of $|G_1 - g_1|$ or $|G_2 - g_2|$ is easily less than 0.6. Thus, from Eq. (21) or Eq. (22),

$$|u(x,y) - U(x,y)| \leq (0.6)\omega(x,y). \tag{24}$$

The error bound provided in Eq. (24) is useful in examining how the CVBEM approximation $U(x,y)$ approaches the heat transport problem solution $u(x,y)$ as the approximation errors of $|G_1 - g_1|$ and $|G_2 - g_2|$ decrease. By adding another 12 nodal points along Γ where $|G_1 - g_1|$ and $|G_2 - g_2|$ are relatively large, a new CVBEM approximation $U_2(x,y)$ is obtained with the property that

$$\max\{\sup |G_1 - g_1|, \sup_{\Gamma_2} |G_2 - g_2|\} < 0.1.$$

Thus, from Eq. (22) and the newly computed $\omega(x,y)$,

$$|u(x,y) - U_2(x,y)| < (0.1)\omega(x,y). \tag{25}$$

Because $\omega(x,y)$ is strictly positive and continuous on $\Omega \cup \Gamma$ (for the CVBEM numerical technique), there exists a bound M such that $\omega(x,y) \leq M$ over $\Omega \cup \Gamma$. For $U_2(x,y)$, M is found to be approximately 3.3. Hence Eq. (25) can be written, for $(0.1M) \leq 0.33$, that $|u(x,y) - U_2(x,y)| \leq 0.33$.

DISCUSSION OF APPLICATION PROBLEM

The application problem demonstrates use of an error bound inequality for the Laplace equation. In our application, the CVBEM is used to approximately solve the governing operator equation with auxilliary conditions. Additionally, the CVBEM is also used to construct the error bound function $\omega(x,y)$. Other numerical methods can be used, given that the required inequalities can be shown to hold.

It is apparent that the accuracy of the CVBEM approximation function $U(x,y)$, to $u(x,y)$, depends on many factors, including the numerical method's nodal point placement and density. The construction of an $\omega(x,y)$ function depends on similar numerical factors.

The error bound of Eq. (21) depends both on how well the $U(x,y)$ approximates $u(x,y)$ pursuant to Eq. (4), and also on the values of $\omega(x,y)$. It is obvious that from Eq. (4), $G_1 = g_1$ on Γ_1 and $G_2 = g_2$ on Γ_2 (with $F = f$ in Ω) guarantees that $U(x, y) = u(x, y)$ irrespective of $\omega(x, y)$ values. Indeed, $\omega(x, y) > 0$ by Eq. (3). Thus, "better" $\omega(x, y)$ functions can be obtained by trying to obtain smaller $\omega(x, y)$ values over Ω. The CVBEM is useful in developing the error bound functions in conjuction with a particular solution to the Laplace equation.

CONCLUSIONS

An important problem in the use of numerical methods and analysis to approximately solve differential equations is the evaluation of approximation error. In this paper an error bound is obtained for this error. The error bound inequality utilizes (1) the approximation function's maximum error in solving the partial differential equation operator over the problem domain Ω; (2) the maximum error in matching prescribed values of the boundary conditions of either Neumann or Dirichlet type; and (3) a strictly positive function $\omega(x, y)$ constructed such as to solve another, but similar, partial differential equation with auxilliary conditions.

The application problem demonstrates the use of the error bound inequality in a two-dimensional heat transport problem that is approximately solved by the complex variable boundary element method (CVBEM). The error bound inequality can be applied to other classes of similar problems. For two-dimensional Laplace or Poisson type problems, the CVBEM is a good technique for computing the error bound.

References

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