The CVBEM for multiply connected domains using a linear trial function

R. R. Harryman III and T. V. Hromadka II*

Williamson and Schmid, Irvine, CA, USA

J. L. Vaughn

The Optical Sciences Company, Placentia, CA, USA

D. P. Watson

Martin Marietta, Baltimore, MD, USA

The objective of this paper is to present a modelling technique that approximates boundary value problems of the Laplace equation over two-dimensional multiply connected regions. By using this method, two-dimensional Laplace equation problems can be solved by use of analytic functions. The flexibility of this technique is demonstrated on problems with multiply connected domains, dissimilar materials, and many types of boundary conditions that have previously been difficult to handle.

Keywords: boundary element methods, cauchy integral equation, boundary value problems, analytic functions

Introduction

The complex variable boundary element method (CVBEM) is a mathematical modelling technique that approximates boundary value problems of the Laplace or Poisson equation. The problems with which the CVBEM deals involve potential problems of the two-dimensional Laplace equation. Specifically, the CVBEM handles problems involving two-dimensional steady-state soil water flow, steady-state heat flow, stress-strain torsion effects, and other similar problems.

The numerical technique follows from the Cauchy integral formula. The produced approximation functions of the CVBEM are analytic in the region enclosed by the problem boundary. Therefore they exactly satisfy the two-dimensional Laplace equation in the entire domain of the problem. The CVBEM integrates the boundary integrals exactly along each boundary element; thus the method does not require numerical integration. The CVBEM can solve problems involving dissimilar materials, flux boundary conditions, and multiply connected domains, all with different types of boundary conditions.

Details regarding the mathematical underpinnings of the CVBEM, as well as a review of the literature, are provided in Ref. 1. A brief development of the CVBEM is presented for the reader's convenience.

* Also at the Department of Applied Mathematics, California State University, Fullerton, CA, USA.

Address reprint requests to Mr. Harryman at Williamson and Schmid, 17782 Sky Park Boulevard, Irvine, CA 92714, USA.

Received 19 June 1989; accepted 27 September 1989

Development of CVBEM approximations

Let Ω be a multiply connected domain enclosed by boundaries C1 and C2. Assume that C_1 and C_2 are polygonal lines composed of V_1 and V_2 straight-line segments and vertices, respectively (*Figure 1*). If $\omega(z) = \phi(z) + i\psi(z)$ is a complex variable function on $R = C_1 \cup C_2 \cup \Omega$, then $\phi(z)$ can be defined to be the state variable and $\psi(z)$ to be the stream function. Consequently, ϕ and ψ are related by the Cauchy-Reimann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{1}$$

where ϕ and ψ are real-valued functions that are harmonic functions for $z \in R$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2}$$

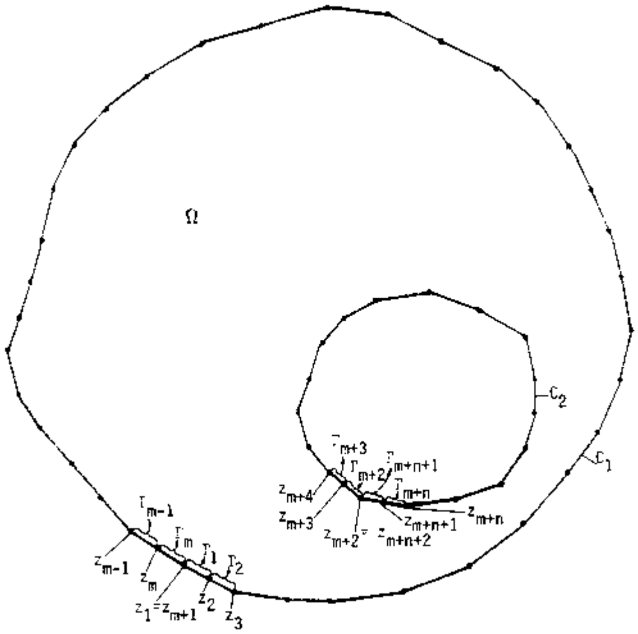

Figure 1. Boundary of multiply connected domain

Define nodal points $[z_j, j = 1, \ldots, m + 1]$ on the outer curve C_1 such that $m \geq V_1$ and a nodal point is located at each boundary vertex, $z_{m+1} = z_1$, and these points are located on C_1 in the counterclockwise direction. Similarly, define nodal points $[z_j, j = m + 2, \ldots, m + n + 2]$ on the inner curve C_2 such that $n \geq V_2$, a nodal point is located at each boundary vertex, $z_{m+2} = z_{m+n+2}$, and these points are located on C_2 in a clockwise direction (*Figure 1*).

At each nodal point $z_j, j = 1, \ldots, m + n + 2$, let $\bar{\phi}_j$ and $\bar{\psi}_j$ be the specified real nodal values. Let Γ_j be the line segment joining nodes z_j and $z_{j+1}, j = 1, \ldots, m, m + 2, \ldots, m + n + 2$. Notice that Γ_m is the line segment joining nodes z_m and z_1 and Γ_{m+n+1} joins nodes z_{m+n+1} and z_{m+2} (*Figure 1*). Therefore $C_1 = \bigcup_{j=1}^{m} \Gamma_j$ and $C_2 = \bigcup_{j=m+2}^{m+n+1} \Gamma_j$.

We can now define a continuous global trial function $G_1(z)$ on $C_1 \cup C_2$ by

$$G_1(z) = N_j(z) \sum_{\substack{j=1 \\ j \neq m+1}}^{m+n+1} (\bar{\phi}_j + i\bar{\psi}_j) \tag{3}$$

An analytic approximation function[1] is determined by

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_{C_1 \cup C_2} \frac{G_1(a)}{a - z} da \qquad z \in \Omega \tag{4}$$

Since $\hat{\omega}(z)$ is analytic in Ω, its real and imaginary parts individually solve the two-dimensional Laplace equation in Ω. Simplifying $\hat{\omega}(z)$, we obtain

$$\begin{aligned}\hat{\omega}(z) &= \frac{1}{2\pi i} \int_{C_1 \cup C_2} \frac{G_1(a)}{a - z} da \qquad z \in \Omega \\ &= \frac{1}{2\pi i} \int_{\substack{m+n+1 \\ \cup \Gamma_j \\ j=1 \\ j \neq m+1}} \frac{G_1(a)}{a - z} da \qquad z \in \Omega \\ &= \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq m+1}}^{m+n+1} \int_{\Gamma_j} \frac{G_1(a)}{a - z} da \qquad z \in \Omega \end{aligned} \tag{5}$$

We define the basis function $N_j(z)$ by linear trial functions,

$$N_j(z) = \begin{Bmatrix} \dfrac{z - z_{j-1}}{z_j - z_{j-1}} & z \in \Gamma_{j-1} \\ 0 & z \notin \Gamma_j \cup \Gamma_{j+1} \\ \dfrac{z_{j+1} - z}{z_{j+1} - z_j} & z \in \Gamma_j \end{Bmatrix} \qquad j = 1, \ldots, m + n + 1 \qquad j \neq m + 1 \tag{6}$$

where $\Gamma_1 = \Gamma_{m+1}, \Gamma_{m+2} = \Gamma_{m+n+2}$. Then on Γ_j,

$$\begin{aligned} G_1(z) &= N_j(z)\bar{\omega}_j + N_{j+1}(z)\bar{\omega}_{j+1} \\ &= (N_j(z)\bar{\phi}_j + N_{j+1}(z)\bar{\phi}_{j+1}) + i(N_j(z)\bar{\psi}_j + N_{j+1}\bar{\psi}_{j+1}) \end{aligned}$$

where $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$. Therefore

$$\begin{aligned} \int_{\Gamma_j} \frac{G_1(a)}{a - z_0} da &= \int_{\Gamma_j} \frac{(z_{j+1} - a)\,\bar{\omega}_j + (a - z_j)\,\bar{\omega}_{j+1}}{(z_{j+1} - z_j)(a - z_0)} da \\ &= \frac{z_{j+1}\bar{\omega}_j - z_j\bar{\omega}_{j+1}}{z_{j+1} - z_j} \int_{\Gamma_j} \frac{da}{a - z_0} + \frac{\bar{\omega}_{j+1} - \bar{\omega}_j}{z_{j+1} - z_j} \int_{\Gamma_j} \frac{a\,da}{a - z_0} \end{aligned}$$

Simplifying the last integral, we get

$$\begin{aligned}\int_{\Gamma_j} \frac{a\,da}{a - z_0} &= \int_{\Gamma_j} \frac{(a - z_0 + z_0)}{a - z_0} da \\ &= \int_{\Gamma_j} \frac{a - z_0}{a - z_0} da + \int_{\Gamma_j} \frac{z_0\,da}{a - z_0} \\ &= \int_{\Gamma_j} da + z_0 \int_{\Gamma_j} \frac{da}{a - z_0} \\ &= z_{j+1} - z_j + z_0 \ln(a - z_0)\Big|_{z_j}^{z_{j+1}} \\ &= z_{j+1} - z_j + z_0\left[\ln\left|\frac{z_{j+1} - z_0}{z_j - z_0}\right| + i\theta_{j,j+1}\right]\end{aligned} \tag{7}$$

where $\theta_{j,j+1}$, $j = 1, \ldots, m, m + 2, \ldots, m + n + 1$ is the central angle between the straight-line segments joining z_j and z_{j+1} to interior point z_0 (*Figure 2*).

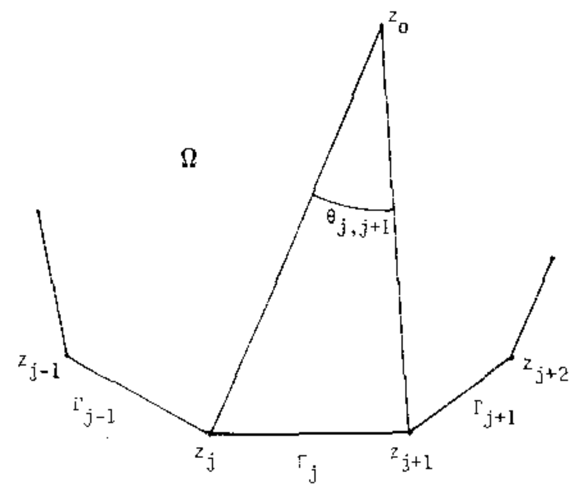

Figure 2. Central angle $\theta_{j,j+1}$

We define the term H_j by

$$H_j = \ln\left|\frac{z_{j+1} - z_0}{z_j - z_0}\right| + i\theta_{j,j+1} \tag{8}$$

Then,

$$\begin{aligned}\int_{\Gamma_j} \frac{G_1(a)}{a - z_0} da &= \frac{z_{j+1}\overline{\omega}_j - z_j\overline{\omega}_{j+1}}{z_{j+1} - z_0} H_j + \frac{\overline{\omega}_{j+1} - \overline{\omega}_j}{z_{j+1} - z_j}(z_{j+1} - z_j + z_0 H_j) \\ &= \frac{z_{j+1}\overline{\omega}_j - z_j\overline{\omega}_{j+1}}{z_{j+1} - z_j} H_j + \overline{\omega}_{j+1} - \overline{\omega}_j + \frac{\overline{\omega}_{j+1}z_0 - \overline{\omega}_j z_0}{z_{j+1} - z_j} H_j \\ &= \overline{\omega}_{j+1} - \overline{\omega}_j + \left[\overline{\omega}_{j+1}\left(\frac{z_0 - z_j}{z_{j+1} - z_j}\right) - \overline{\omega}_j\left(\frac{z_0 - z_{j+1}}{z_{j+1} - z_j}\right)\right] H_j\end{aligned} \tag{9}$$

Since $\hat{\omega}(z_0)$ is the sum of the contributions of each Γ_j divided by $2\pi i$,

$$\begin{aligned}\hat{\omega}(z_0) &= \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j\neq m+1}}^{m+n+1} \int_{\Gamma_j} \frac{G_1(a)}{a - z_0} da \\ &= \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j\neq m+1}}^{m+n+1} \left\{\overline{\omega}_{j+1} - \overline{\omega}_j + \left[\overline{\omega}_{j+1}\left(\frac{z_0 - z_j}{z_{j+1} - z_j}\right) - \overline{\omega}_j\left(\frac{z_0 - z_{j+1}}{z_{j+1} - z_j}\right)\right] H_j\right\} \\ &= \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j\neq m+1}}^{m+n+1} \left[\overline{\omega}_{j+1}\left(\frac{z_0 - z_j}{z_{j+1} - z_j}\right) - \overline{\omega}_j\left(\frac{z_0 - z_{j+1}}{z_{j+1} - z_j}\right)\right] H_j \\ &= \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j\neq m+1}}^{m+n+1} [\overline{\omega}_{j+1}(z_0 - z_j) - \overline{\omega}_j(z_0 - z_{j+1})] \frac{H_j}{z_{j+1} - z_j}\end{aligned} \tag{10}$$

This can be represented as the complex function

$$\begin{aligned}\hat{\omega}(z_0) &= \hat{\phi}(z_0) + i\hat{\psi}(z_0) \\ &= \hat{\phi}(z_0, \overline{\phi}_1, \ldots, \overline{\phi}_m, \overline{\phi}_{m+2}, \ldots, \overline{\phi}_{m+n+1}, \overline{\psi}_1, \ldots, \overline{\psi}_m, \overline{\psi}_{m+2}, \ldots, \overline{\psi}_{m+n+1}) \\ &\quad + i\hat{\psi}(z_0, \overline{\phi}_1, \ldots, \overline{\phi}_m, \overline{\phi}_{m+2}, \ldots, \overline{\phi}_{m+n+1}, \overline{\psi}_1, \ldots, \overline{\psi}_m, \overline{\psi}_{m+2}, \ldots, \overline{\psi}_{m+n+1})\end{aligned}$$

where z_0 is the interior of Ω, and $\hat{\phi}$ and $\hat{\psi}$ are real-valued functions representing the real and imaginary components of $\hat{\omega}(z)$.

If $\overline{\omega}_j = \overline{\phi}_j + i\overline{\psi}_j$ is known at each z_j, $j = 1, \ldots, m, m+2, \ldots, m+n+1$, then equation (10) is analytic inside Ω, so $\hat{\phi}(x, y)$ and $\hat{\psi}(x, y)$ both satisfy the Laplace equation in Ω. If $\hat{\omega}(z) = \omega(z)$ everywhere on $C_1 \cup C_2$, then $\hat{\omega}(z) = \omega(z)$ in Ω, and $\hat{\omega}(z)$ is the exact solution of the boundary value problem.

Actually, usually only one, and occasionally neither, of the two specified nodal values $(\overline{\phi}_j, \overline{\psi}_j)$ is known at each z_j, $j = 1, \ldots, m, m+2, \ldots, m+n+1$, and we must estimate values for the unknown nodal values. Using an implicit method, we can evaluate $\hat{\omega}(z)$ arbitrarily close to each nodal point and then generate the unknown nodal variable as functions of all the known nodal variables. This results in $m + n$ equations for $m + n$ unknown nodal variables, which can be solved with matrices.

The above values as estimates of the unknown nodal values can be used along with the known nodal values to define $\hat{\omega}(z)$ by equation (10).

Examples

We now apply the CVBEM to two example problems. For each problem a diagram of the boundary conditions and the CVBEM generated flow net will be presented. The problems considered are

1. flow over a bounded step and
2. flow around objects in two regions.

Flow over a bounded step

In cartesian coordinates the problem boundary is contained by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 2$, with the vertex of the step at the point (1, 1). (See the appendixes and *Figure 3*.)

The results of the CVBEM applied to this problem yield a solution for both the boundary and interior points (see the appendixes). Upon evaluating both boundary and interior points, the results can be plotted as in the output diagram. In this example the streamlines are the flow lines over the step, and the state function lines (which are orthogonal to the streamlines) are the lines of equal potential (*Figure 4*).

Flow around objects in two regions

The problem boundary in cartesian coordinates is the area enclosed by the lines $x = -10$, $x = 10$, $y = -5$, and $y = 5$. This area is split by a line at $x = 4$, the area where $x < 4$ having twice the conductivity of the area where $x > 4$. There are two holes in the region $x < 4$; one is a triangle with vertices at $(-8, 0)$, $(-6, 1)$, and $(-6, 1)$, and the other in a circle with radius 1 centered at (0, 0). There is one hole in the region $x > 4$ that is a square. It has vertices at $(6, -1)$, $(8, -1)$, (8, 1), and (6, 1) (see *Figure 5*).

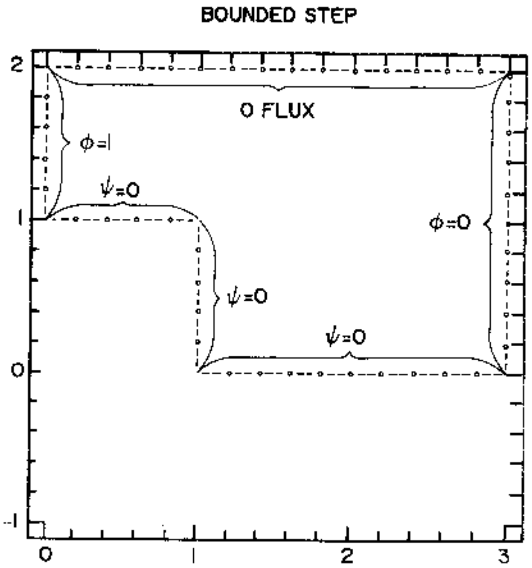

Figure 3. Bounded step boundary conditions

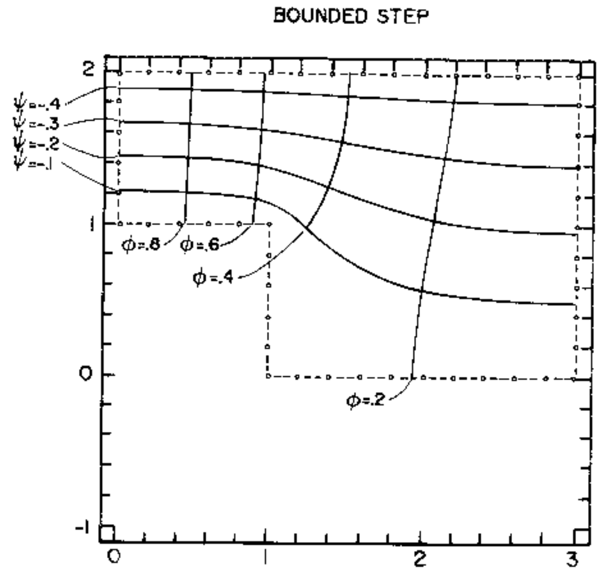

Figure 4. Bounded step flow net

The output results of the CVBEM are evaluated on both the boundary and the interior in order to plot the variables. The streamlines run horizontally through the region, and the state function lines run vertically. The state function is negative for $x < 4$ and positive for $x > 4$. At $x = 4$ the stream variable "jumps." The state function is orthogonal to the stream variable and represents lines of equal potential, while the stream variable represents the flow lines (*Figure 6*).

Conclusions

The CVBEM develops approximate solutions to two-dimensional Laplace problems. For problems dealing

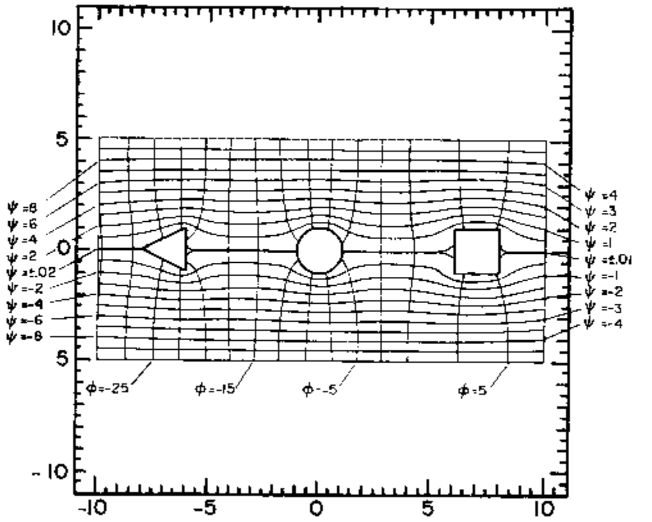

Figure 5. Three-hole, two-region problem boundary conditions

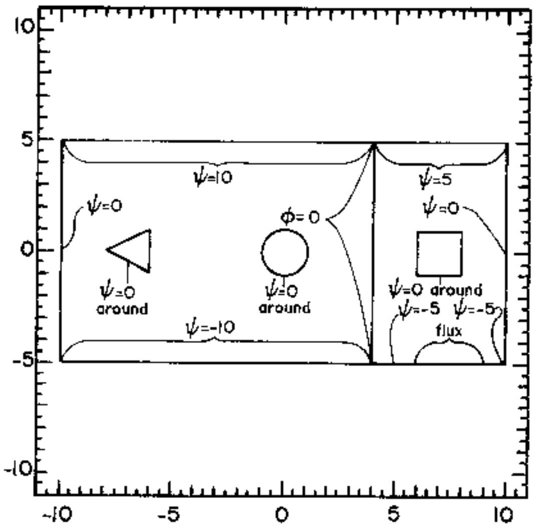

Figure 6. Three-hole, two-region problem flow net

with linear boundary conditions, the results of the CVBEM are exact. For more difficult spaces (that is, nonlinear boundary conditions) the CVBEM exactly solves the Laplace equation over the problem domain but approximates the problem boundary conditions. In this paper the CVBEM is extended to multiply connected regions with applications to domains of dissimilar materials; two example problems demonstrate the utility of the numerical techniques.

References

1 Hromadka, T. V., II and Lai, C. *The Complex Variable Boundary Element Method in Engineering Analysis*. Springer-Verlag, New York, 1987

Appendix 1: Input for "bounded step"

```
1.0.0
1,1
51 'Bounded Step'
0,1,4,1.0
.2,1.2,0.0
.4,1,2,0.0
.6.1,2,0.0
.8,1,2,0.0
1.1,2.0.0
1,.8.2,0.0
1,.6,2.0.0
1..4.2,0.0
1,.2,2,0.0
1.0,2.0.0
1.2.0,2,0.0
1.4,0,2.0.0
1.6.0,2,0.0
1.8,0,2.0.0
2,0,2.0.0
2.2,0,2.0.0
2.4,0.2,0.0
2.6,0.2,0.0
2.8,0,2,0.0
3,0,4,0.0,0
3,.2,1,0.0
3,.4,1,0.0
3,.6,1,0.0
3,.8,1,0.0
3,1,1,0.0
3,1.2,1,0.0
3,1.4,1,0.0
3,1.6,1,0.0
3,1.8,1,0.0
3,2,1.0.0
2.8,2,3,0
2.6,2,3,0
2.4,2,3,0
2.2,2,3,0
2.0,2,3,0
1.8,2.3,0
1.6,2.3,0
1.4,2,3,0
1.2,2,3,0
1.0,2.3,0
.8,2,3,0
.6,2,3.0
.4,2,3,0
.2,2,3,0
.001,2,3.0
0,2,1,1.0
0,1.8,1,1.0
0,1.6,1,1.0
0,1.4,1,1.0
0,1.2,1,1.0
```

NODE NO.	X(i)	Y(i)	KTYPE(i) 1=SV; 2=SF 3=EFFLUX	VALUE(S)		ANGLE(i)
1	0.00000	1.00000	4	1.00000	0.00000	90.00
2	0.20000	1.00000	2	0.00000		180.00
3	0.40000	1.00000	2	0.00000		180.00
4	0.60000	1.00000	2	0.00000		180.00
5	0.80000	1.00000	2	0.00000		180.00
6	1.00000	1.00000	2	0.00000		270.00
7	1.00000	0.80000	2	0.00000		180.00
8	1.00000	0.60000	2	0.00000		180.00
9	1.00000	0.40000	2	0.00000		180.00
10	1.00000	0.20000	2	0.00000		180.00
11	1.00000	0.00000	2	0.00000		90.00
12	1.20000	0.00000	2	0.00000		180.00
13	1.40000	0.00000	2	0.00000		180.00
14	1.60000	0.00000	2	0.00000		180.00
15	1.80000	0.00000	2	0.00000		180.00
16	2.00000	0.00000	2	0.00000		180.00
17	2.20000	0.00000	2	0.00000		180.00
18	2.40000	0.00000	2	0.00000		180.00
19	2.60000	0.00000	2	0.00000		180.00
20	2.80000	0.00000	2	0.00000		180.00
21	3.00000	0.00000	4	0.00000	0.00000	90.00
22	3.00000	0.20000	1	0.00000		180.00
23	3.00000	0.40000	1	0.00000		180.00
24	3.00000	0.60000	1	0.00000		180.00
25	3.00000	0.80000	1	0.00000		180.00
26	3.00000	1.00000	1	0.00000		180.00
27	3.00000	1.20000	1	0.00000		180.00
28	3.00000	1.40000	1	0.00000		180.00
29	3.00000	1.60000	1	0.00000		180.00
30	3.00000	1.80000	1	0.00000		180.00
31	3.00000	2.00000	1	0.00000		90.00
32	2.80000	2.00000	3	0.00000		180.00
33	2.60000	2.00000	3	0.00000		180.00
34	2.40000	2.00000	3	0.00000		180.00
35	2.20000	2.00000	3	0.00000		180.00
36	2.00000	2.00000	3	0.00000		180.00
37	1.80000	2.00000	3	0.00000		180.00
38	1.60000	2.00000	3	0.00000		180.00
39	1.40000	2.00000	3	0.00000		180.00
40	1.20000	2.00000	3	0.00000		180.00
41	1.00000	2.00000	3	0.00000		180.00
42	0.80000	2.00000	3	0.00000		180.00
43	0.60000	2.00000	3	0.00000		180.00
44	0.40000	2.00000	3	0.00000		180.00
45	0.20000	2.00000	3	0.00000		180.00
46	0.00100	2.00000	3	0.00000		180.00
47	0.00000	2.00000	1	1.00000		90.00
48	0.00000	1.80000	1	1.00000		180.00
49	0.00000	1.60000	1	1.00000		180.00
50	0.00000	1.40000	1	1.00000		180.00
51	0.00000	1.20000	1	1.00000		180.00

*** NOTE : KTYPE = 0 INDICATES A COMMON BOUNDARY NODE ***

Appendix 2: Output for "bounded step"

CAUCHY PROGRAM RESULTS

NODE NUMBER	STATE VARIABLE	STREAM FUNCTION
1	1.00000	0.00000
2	0.90881	0.00000
3	0.81636	0.00000
4	0.72130	0.00000
5	0.61924	0.00000
6	0.48245	0.00000
7	0.38140	0.00000
8	0.33562	0.00000
9	0.30842	0.00000
10	0.29352	0.00000
11	0.28901	0.00000
12	0.28463	0.00000
13	0.27106	0.00000
14	0.24991	0.00000
15	0.22274	0.00000
16	0.19104	0.00000
17	0.15605	0.00000
18	0.11874	0.00000
19	0.07986	0.00000
20	0.03999	0.00000
21	0.00000	0.00000
22	0.00000	-0.04072
23	0.00000	-0.08150
24	0.00000	-0.12311
25	0.00000	-0.16585
26	0.00000	-0.20991
27	0.00000	-0.25532
28	0.00000	-0.30202
29	0.00000	-0.34979
30	0.00000	-0.39833
31	0.00000	-0.44727
32	0.04901	-0.44727
33	0.09850	-0.44727
34	0.14887	-0.44727
35	0.20063	-0.44727
36	0.25440	-0.44727
37	0.31091	-0.44727
38	0.37095	-0.44727
39	0.43533	-0.44727
40	0.50464	-0.44727
41	0.57904	-0.44727
42	0.65809	-0.44727
43	0.74081	-0.44727
44	0.82609	-0.44727
45	0.91290	-0.44727
46	0.99940	-0.44727
47	1.00000	-0.44750
48	1.00000	-0.35925
49	1.00000	-0.27107
50	1.00000	-0.18183
51	1.00000	-0.09146

CVBEM APPROXIMATION FUNCTION NODAL VALUES
Region - 1 Boundary Section - 1 : Bounded Step

NODE NUMBER	STATE VARIABLE	STREAM FUNCTION	STATE ERROR	STREAM ERROR
1	1.00016	-0.00018	-0.00016	0.00018
2	0.90881	-0.00022	0.00000	0.00022
3	0.81636	-0.00021	0.00000	0.00021
4	0.72130	-0.00028	0.00000	0.00028
5	0.61924	-0.00020	0.00000	0.00020
6	0.48245	-0.02429	0.00000	0.02429
7	0.38140	0.00060	0.00000	-0.00060
8	0.33562	0.00006	0.00000	-0.00006
9	0.30842	0.00008	0.00000	-0.00008
10	0.29352	0.00014	0.00000	-0.00014
11	0.28901	0.00080	0.00000	-0.00080
12	0.28463	0.00004	0.00000	-0.00004
13	0.27106	-0.00008	0.00000	0.00008
14	0.24991	-0.00015	0.00000	0.00015
15	0.22274	-0.00019	0.00000	0.00019
16	0.19104	-0.00020	0.00000	0.00020
17	0.15605	-0.00021	0.00000	0.00021
18	0.11874	-0.00021	0.00000	0.00021
19	0.07986	-0.00022	0.00000	0.00022
20	0.03999	-0.00027	0.00000	0.00027
21	-0.00020	-0.00022	0.00020	0.00022
22	-0.00024	-0.04072	0.00024	0.00000
23	-0.00016	-0.08150	0.00016	0.00000
24	-0.00013	-0.12311	0.00013	0.00000
25	-0.00010	-0.16585	0.00010	0.00000
26	-0.00007	-0.20991	0.00007	0.00000
27	-0.00005	-0.25532	0.00005	0.00000
28	-0.00003	-0.30202	0.00003	0.00000
29	-0.00002	-0.34979	0.00002	0.00000
30	-0.00001	-0.39833	0.00001	0.00000
31	-0.00002	-0.44727	0.00002	0.00000
32	0.04901	-0.44729	0.00000	0.00002
33	0.09850	-0.44731	0.00000	0.00004
34	0.14887	-0.44732	0.00000	0.00005
35	0.20063	-0.44733	0.00000	0.00006
36	0.25440	-0.44732	0.00000	0.00005
37	0.31091	-0.44731	0.00000	0.00004
38	0.37095	-0.44728	0.00000	0.00001
39	0.43533	-0.44723	0.00000	-0.00004
40	0.50464	-0.44717	0.00000	-0.00010
41	0.57904	-0.44710	0.00000	-0.00017
42	0.65809	-0.44705	0.00000	-0.00022
43	0.74081	-0.44702	0.00000	-0.00025
44	0.82609	-0.44700	0.00000	-0.00027
45	0.91290	-0.44692	0.00000	-0.00035
46	0.99954	-0.44719	-0.00013	-0.00008
47	1.00006	-0.44731	-0.00006	-0.00019
48	1.00033	-0.35925	-0.00033	0.00000
49	1.00021	-0.27107	-0.00021	0.00000
50	1.00017	-0.18183	-0.00017	0.00000
51	1.00019	-0.09146	-0.00019	0.00000