AN APPROXIMATE ABSOLUTE RELATIVE ERROR BOUND FOR THE COMPLEX VARIABLE BOUNDARY ELEMENT METHOD

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Research in the complex variable boundary element method has recently focused towards development of approximation error analyses and error bound estimates. This paper provides the first development of approximate absolute relative error bounds for the estimation of nodal point unknown boundary values. Because maximum approximation error occurs on the problem boundary due to the analog with the maximum modulus theorem, the developed bounds apply, as an approximation, throughout the problem domain as well. The provided formulae are readily programmable by computer.

Key words: complex variable boundary element method, CVBEM, boundary elements, relative error, error bounds

Introduction

The complex variable boundary element method, or CVBEM, has been the subject of several recent papers in the development of mathematical analyses for evaluating approximation error. Topics include Taylor series expansion (Hromadka [3]), series expansion using trial functions (Harryman *et al.*[2]), approximate boundary analysis (Hromadka *et al.*[6]), among others.

In the current work, approximation error is evaluated using the computational matrix system developed by application of the CVBEM. In this way, nodal point approximation error may be identified, and its influence on other nodal point approximation error may be examined.

The complex variable boundary element method

The details regarding the CVBEM and its application may be found in the references, and will not be repeated here. In brief, the CVBEM develops approximations of two-dimensional boundary value problems involving the Laplace or Poisson equations; in particular, approximations of analytic functions, $\omega(z)$. The resulting approximation function, $\hat{\omega}(z)$, is analytic inside the simply connected problem domain, Ω, and is continuous on the simple closed finite length problem boundary, Γ. Because $\hat{\omega}(z)=\hat{\phi}(z)+i\hat{\psi}(z)$, both the $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are potential functions of Ω, and hence exactly solve the Laplace equation in Ω. Thus, approximation error occurs on the boundary, Γ, and techniques used to better match $\hat{\omega}(z)$ to the boundary conditions improves the accuracy of $\hat{\omega}(z)$ in Ω due to the maximum modulus theorem.

CVBEM matrix system development

Generally, straight-line piecewise continuous spline functions, $\eta_j(\zeta)$ are used as nodal point basis functions, for $\zeta \in \Gamma$. As a boundary value problem, each nodal point, j, has nodal value $\omega_j=\phi_j+i\psi_j$. However, in general, only ϕ_j or ψ_j is known as a boundary condition at node j, not both. So part of the CVBEM is to estimate values for the nodal point unknown component, and then use these estimated nodal values (along with the known values) to define $\hat{\omega}(z)$ in Ω. Notation used for identifying the known and unknown nodal values, at node j, is

$$\omega_j = \phi_j + i\psi_j = \Delta\xi_k + \Delta\xi_u \tag{1}$$

where ξ_k and ξ_u indicate the known and unknown values, respectively, and Δ is a characteristic function that equals 1 or i appropriately. (Note that if both ϕ_j and ψ_j are known, then non nodal equation arises for solving for an unknown nodal value).

Using $\eta_j(\zeta)$ as notation for the nodal point j basis function, for $\zeta \in \Gamma$, then solving the Cauchy integral at each nodal point j defined on Γ at coordinate z_j,

$$\hat{\omega}_j = \hat{\omega}(z_j) = \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{j=1}^{m}\eta_j(\zeta)\bar{\omega}_j\,d\zeta}{\zeta - z_j}, \qquad j = 1, 2, \ldots, m \tag{2}$$

where m nodes are defined on Γ (with a node at each vertex of Γ); Γ is a polygon; $\bar{\omega}_j$ are the nodal values $\Delta\xi_{kj} + \Delta\bar{\xi}_{uj}$; and the limiting value of the integral is used in eqn(2) where z_j is approached from the interior of Ω (see Hromadka & Lai[4]).

Solution of eqn(2) results in a matrix system, depending on whether a fully inplicit solution for the vector nodal unknown values, is $\boldsymbol{\xi}_u$, is sought:

$$\boldsymbol{\xi}_u = \mathbf{A}_1\boldsymbol{\xi}_k + \mathbf{A}_2\boldsymbol{\xi}_u \quad \text{(implicit solution)} \tag{3}$$

$$\boldsymbol{\xi}_k = \mathbf{B}_1\boldsymbol{\xi}_k + \mathbf{B}_2\boldsymbol{\xi}_u \quad \text{(explicit solution)} \tag{4}$$

where $\boldsymbol{\xi}_u$ and $\boldsymbol{\xi}_k$ are column vectors of the nodal unknown and known values, respectively; $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$ are fully-populated square matrices representing the integration of basis functions in eqn(2), (see Hromadka & Lai,[4] for construction details and examples).

CVBEM matrix system error analyses

Using the explicit matrix system of eqn(4) provides the convenience that the resulting CVBEM approximation, $\hat{\omega}(z)$, has the property that known nodal values are produced from $\hat{\omega}(z)$. To proceed with the analysis, it is assumed that the nodal trial functions are 'perfect' in describing the (known) boundary conditions on Γ; that is, there is no approximation error in matching the boundary conditions by the basis functions. Obviously, if 'perfect' trial functions happened to be selected for describing the unknown values on Γ, then $\hat{\omega}(z)=\omega(z)$. For our analysis, it is assumed that perfect trial functions are selected for both the known and unknown boundary values on Γ, except for the basis function for the nodal unknown value at node 1. That is, $\eta_1(\xi)$ is in error for the unknown nodal value, but all other basis are perfect.

Applying the CVBEM with the assumed basis functions results in the matrix system (for the explicit solution case),

$$\xi_k = \mathbf{B}_1\xi_k + \hat{\mathbf{B}}_2\hat{\xi}_u \tag{5}$$

where the $\mathbf{B}_1$ matrix system has no associated approximation error; and $\hat{\mathbf{B}}_2$ contains approximation error due to the imperfect $\eta_1(\zeta)$ basis function. In comparison, had $\eta_1(\zeta)$ also been perfect, then no approximation error in integrating $\omega(\zeta)$ on Γ would occur and

$$\xi_k = \mathbf{B}_1\xi_k + \mathbf{B}_2\xi_u \tag{6}$$

where in eqn(6),ξ_u results in the correct unknown values. but in eqn(5) $\hat{\xi}_u$ contains approximation error due to an imperfect $\eta_1(\zeta)$.

Letting $\varepsilon(\zeta)$ be the error in $\eta_1(\zeta)$, such that $(\eta_1(\zeta)-\varepsilon(\zeta))$ is perfect on Γ, we note that $\varepsilon(\zeta)$ is zero everywhere except on the two boundary elements containing node 1. Integrating $\varepsilon(\zeta)$ in the Cauchy integral of eqn(2) develops the matrix contribution $\mathbf{E}$ where from eqns(5) and (6),

$$\mathbf{B}_2 + \mathbf{E} = \hat{\mathbf{B}}_2 \tag{7}$$

Also from eqns(5), (6) and (7),

$$\mathbf{B}_2\hat{\xi}_u = (\mathbf{B}_2 + \mathbf{E})\hat{\xi}_u = \mathbf{B}_2\xi_u \tag{8}$$

or

$$\mathbf{B}_2(\xi_u - \hat{\xi}_u) = \mathbf{E}\hat{\xi}_u \tag{9}$$

In eqn(9), it is seen that the magnitude of error in estimating the unknown nodal values relates to the error in the basis function and also the magnitude of the estimate for the unknown value.

By assumption, only the $\eta_1(\zeta)$ basis function is in error. and the error is the function $\varepsilon(\zeta)$ which is nonzero only in the vicinity of node 1, (i.e. the two boundary elements containing node 1). Thus, for Γ_a and Γ_b being notation for the two boundary elements that contain node 1, (i.e. $\Gamma_a \cap \Gamma_b = z_1$), from eqn(2),

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\varepsilon(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\Gamma_a\cup\Gamma_b}\frac{\varepsilon(\zeta)d\zeta}{\zeta - z} \tag{10}$$

where the only node where $\varepsilon(\zeta)$ is nonzero is at node 1. Then the matrix $\mathbf{E}$ may be written as a square matrix, with nonzero values only in column 1. The entries in column 1 may be seen, from eqn(2), to approximately be proportional to the lengths of Γ_a and Γ_b, and, for (row j) inversely proportional to the distance between node 1 and node j.

Similar to eqn(9), we have from eqn(8),

$$\hat{\mathbf{B}}_2(\xi_u - \hat{\xi}_u) = \mathbf{E}\xi_u \tag{11}$$

which simply swaps $\{\hat{\mathbf{B}}_2,\hat{\xi}_u\}$ for $\{\mathbf{B}_2,\xi_u\}$ in eqn(9).

Sensitivity in estimating relative error is obtained by assuming a trial function for $\varepsilon(\zeta)$ in developing the matrix $\mathbf{E}$, for example, a normal distribution curve or a constant of linear function over Γ_a and Γ_b, among others. After development of a trial $\mathbf{E}$ matrix (again, only column 1 has nonzero entries), the norm of $\mathbf{E}$ is defined, for m nodes on Γ, and using the usual matrix row and column notation,

$$\|\mathbf{E}\| = \overset{\max}{i}\,|\,\mathbf{E}(i,1)\,|;\, i = 1,2,\ldots,m \tag{12}$$

Similarly, norms $\|\hat{\mathbf{B}}_2\|$ and $\|\hat{\xi}_u - \xi_u\|$ are defined as

$$\|\hat{\mathbf{B}}_2\| = \overset{\max}{i}\sum_{j=1}^{m}|\,\hat{\mathbf{B}}_2(i,j)\,|;\, i = 1,2,\ldots,m \tag{13}$$

$$\|\hat{\xi}_u\| = \overset{\max}{i}\,|\,\hat{\xi}_u(i)\,|;\, i = 1,2,\ldots,m \tag{14}$$

$$\|\hat{\xi}_u - \xi_u\| = \overset{\max}{i}\,|\,\hat{\xi}_u(i) - \xi_u(i)\,|;\, i = 1,2,\ldots m \tag{15}$$

From the above

$$\overset{\max}{(\text{Row } i)}|\,\hat{\mathbf{B}}_2(\hat{\xi}_u - \xi_u)\,| \le \|\hat{\mathbf{B}}_2\|\,\|\hat{\xi}_u - \xi_u\| \tag{16}$$

As an estimate, from eqn(11) through (16),

$$\|\hat{\mathbf{B}}_2\|\,\|\hat{\xi}_u - \xi_u\| \approx \|\mathbf{E}\|\,\|\hat{\xi}_u\| \tag{17}$$

where in eqn(17), it is assumed that $\hat{\xi}_u$ values are reasonable approximations of the true (but unknown) ξ_u values, and $\|\mathbf{E}\|$ follows from eqn(12) and the assumed trial function for $\varepsilon(\zeta)$ on Γ.

Then from eqn(17), the relative error is approximated, for the nodal unknown values,

$$\frac{|\,\hat{\xi}_u - \xi_u\,|}{|\,\hat{\xi}_u\,|} \approx \frac{|\,\mathbf{E}\,|}{|\,\hat{\mathbf{B}}_2\,|} \tag{18}$$

The extension to fully-populated $\mathbf{E}$ matrices follows directly from the above development.

Conclusions

An approximate relative error estimation formulation is developed for use with the complex variable boundary element method. The formulation provides an estimate for absolute relative error given an assumed error in the approximations basis functions. The technique is readily programmable, and provides useful error evaluation information to supplement other techniques.

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