COMPUTATIONAL HYDROLOGY '87

Proceedings of the First International Conference
Anaheim, California, U.S.A., July – 1987

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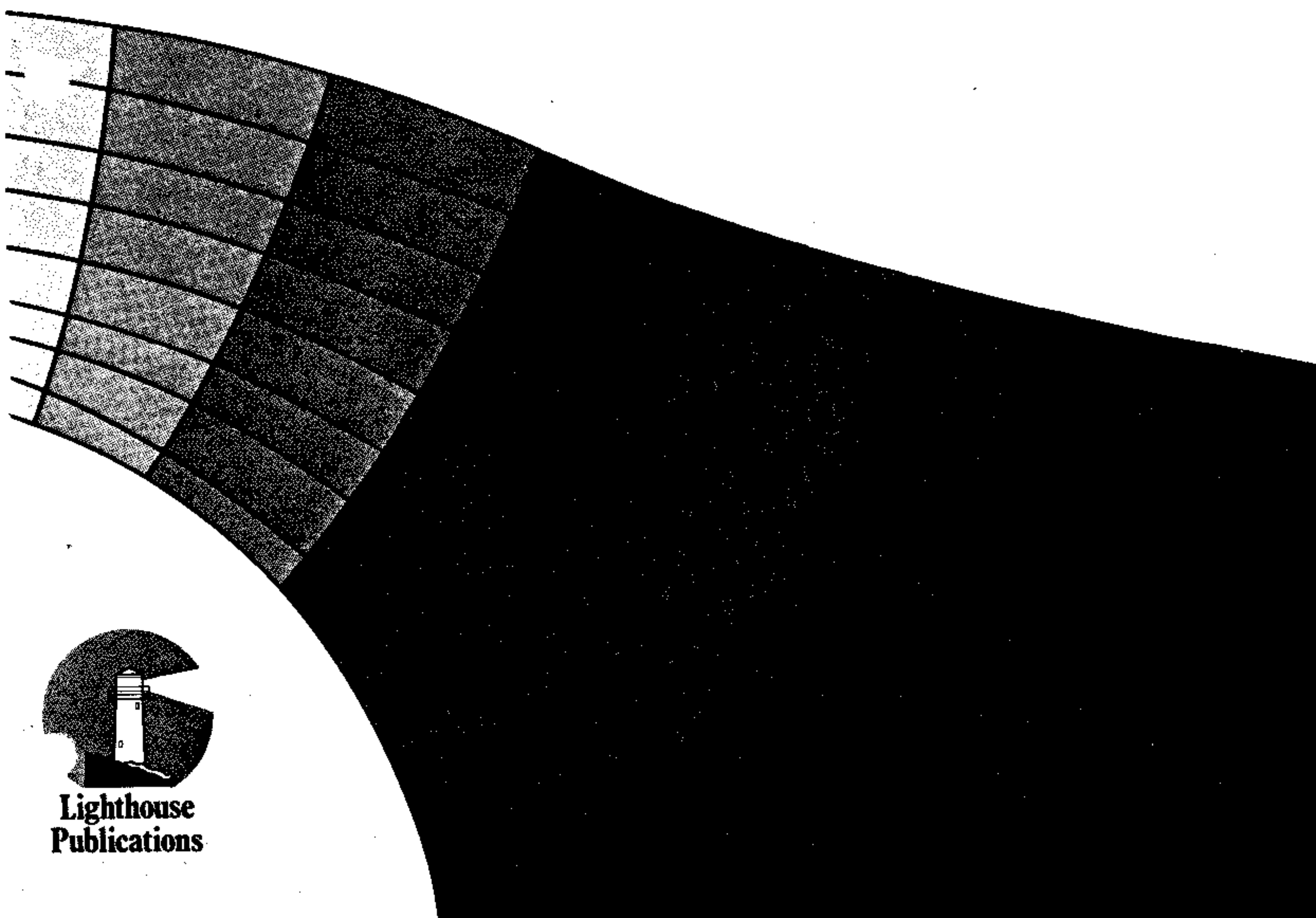

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Publications

CVBEM Analysis in Subsurface Hydraulics

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ABSTRACT

The Complex Variable Boundary Element Method or CVBEM provides solutions to two-dimensional potential problems. Especially unique to this method is the approximate boundary which represents the ture problem boundary transformed to the spatial configuration where the problem's boundary conditions are satisfied. This deforming approximate boundary indicates a ture measure of numerical accuracy which is easy to interpret and understand.
Applications of this technique are demonstrated in studing slow moving subsurface flow problems.

INTRODUCTION

Potential flow theory may be used to depict streamlines of the groundwater flow analyzing the extent of subsurface flow movement. Especially in the preliminary study, the potential flow theory can be used to determine whether or not a more sophisticated study based on a long period of observation and expensive data collection is required.

For two-dimensioal potential flow problems which governs by the Laplace equation, the Complex Variable Boundary Element Method or CVBEM may be used as an approximate model of the prototype system. Due to the limitation of readily available analytic functions, many flow field problems are not easily solvable. The CVBEM, however, provides an immediate extension.

CVBEM DEVELOPMENT

The CVBEM has been shown to be a powerful numerical technique for the approximation of properly posed boundary-value problems involving the Laplace equation (Hromadka [1]). The keystone of the numerical approach is the Cauchy integral formula

$$\omega(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta)\, d\zeta}{\zeta - z} \qquad (1)$$

where ω is a single-valued analytic function; Γ is a simple closed contour enclosing a simply connected domain Ω; ζ is the variable of integration with $\zeta \in \Gamma$; z is a counterclockwise (positive) sense (see Fig. 1).

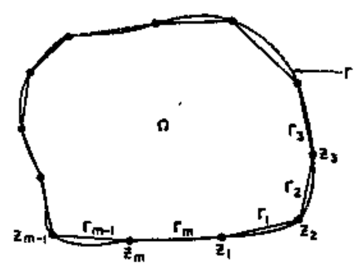

Figure 1. CVBEM Boundary Discretization

In CVBEM approximation, the nodal equations are approximated by taking the limit as the point $z \in \Omega$ approaches a selected nodal point $z_j \in \Gamma$ by

$$\hat{\omega}(z_j) = \lim_{z \to z_j} \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z} \qquad (2)$$

The limiting value is also known as the Cauchy principal value, and the function $G(\zeta)$ is a global trial function which is continuous on Γ. The linear global trial function

$$G(\zeta) = \sum_{j=1}^{m} \delta_j (N_j \bar{\omega}_j + N_{j+1} \bar{\omega}_{j+1}) \qquad (3)$$

where $\delta_j = 1$ if $\zeta \in \Gamma_j$, and $\delta_j = 0$ if $\zeta \notin \Gamma_j$; the functions N_j and N_{j+1} are the usual linear basis functions (see Fig. 2); and $\bar{\omega}_j = \bar{\phi}_j + i\,\bar{\psi}_j$.

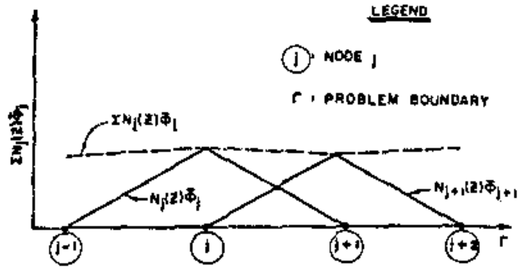

Figure 2. The Linear Basis Function

The above relationship can be written as complex function

$$\begin{aligned}\hat{\omega}(z_j) &= \hat{\phi}(z_j) + i\hat{\psi}(z_j)\\ &= \hat{\phi}(z_j, \bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_m, \bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_m)\\ &\quad + i\hat{\psi}(z_j, \bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_m, \bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_m)\end{aligned} \qquad (4)$$

where $\hat{\phi}$ and $\hat{\psi}$ are real valued functions representing the real and imaginary components of the complex function $\hat{\omega}(z)$. Should values of $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$ be known at each z_j, $j = 1, 2, \ldots, m$, then Eq.(4) defines a complex valued function which is analytic in Ω, and $\hat{\phi}(x,y)$ and $\hat{\psi}(x,y)$ both satisfy the Laplace equation in Ω. If $\hat{\omega}(z) = \omega(z)$ on Γ, then $\hat{\omega}(z) = \omega(z)$ in Ω and $\hat{\omega}(z)$ is the exact solution to the boundary value problem.

The usual problem in engineering applications is that only one of the specified nodal value pair $(\bar{\phi}_j, \bar{\psi}_j)$ is known at each z_j and, consequently, part of the modeling task is to eva;uate the unknown nodal values. A method of developing such an approximation function is to evaluate $\hat{\omega}(z)$ arbitrarily close to each nodal point location on Γ and, in turn, generate an implicit expression of unknown nodal variavle as a function of all the know variables. The result is m equations for m unknown nodal values which can be solved by the usual matrix techniques such as Gauss-elimination method.

APPROXIMATE BOUNDARY

It is useful to determine an "approximate boundary" $\hat{\Gamma}$ upon which $\hat{\omega}(z)$ satisfies the given boundary conditions for $\omega(z)$ on $\hat{\Gamma}$. That is, given an approximator $\hat{\omega}(z)$, contour curves of constant ϕ or ψ on Γ [where $\omega(z) = \phi + i\psi$ and $\hat{\omega}(z) = \hat{\phi} + i\hat{\psi}$] are compared to contour curves of constant $\hat{\phi}$ or $\hat{\psi}$ on $\hat{\Gamma}$ where $\hat{\Gamma}$ is determined by setting the known $\phi = \hat{\phi}$ and $\psi = \hat{\psi}$.

The resulting boundary $\hat{\Gamma}$ has the property that $\hat{\omega}(z)$ satifies the specified boundary conditions on $\hat{\Gamma}$, and $\hat{\omega}(z)$ satifies the governing Laplace equation in the interior, $\hat{\Omega}$. Consequently, $\hat{\omega}(z)$ is the exact solution to the boundary value problem with the true boundary Γ transformed into the approximate boundary $\hat{\Gamma}$. The approximate boundary provides a direct visual representation of the sensitivity of the approximation $\hat{\omega}(z)$ in accommodating the given boundary condtions.

APPLICATIONS

Groundwater flow in an unconfined aquifer, seepage flow through an earth dam, and two steady-state, advective groundwater contaminant problems are used to demonstrate the CVBEM technique.

Application 1

A long and shallow unconfined aquifer (see Fig.3) is sued to illustrate the CVBEM technique. In a preliminary study, the mean deviation between the exact and the approximate boundary is about 0.001^m and 0.2^m for the water table and the impervious boundary, respectively. The equipotential lines shown on Figure 4 approximate those shown on Figure 8(b) in Frind et al.'s[2] paper. The stream lines are not orthogonal to the equipotential lines because of the different scales in x- and z- directions.

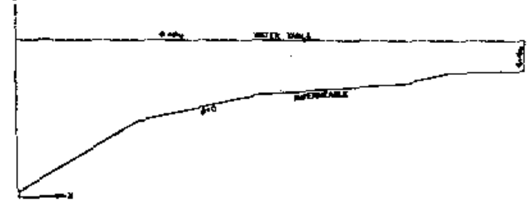

Figure 3. Boundary Conditions for Shallow Unconfined Aquifer

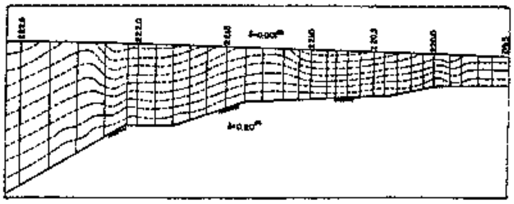

Figure 4. Computed Flow Net for Shallow Unconfined Aquifer

Application 2

Figure 5 shows streamlines and equipotential lines for soil-water flow through a homogeneous earth dam. The locations of the phreatic surface and the seepage face can be determined by the approximate boundary technique.

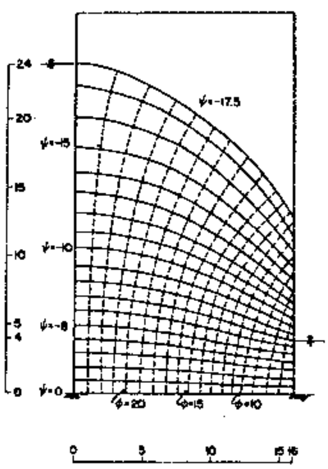

Figure 5. Computed Flow Net for Soil-Water Flow Through a Homogeneous Earth Dam

Application 3

Figure 6 shows a completely penetrating ground-water well (discharge 50 m^3/hr) located at the coordinates (300, 300) in a homogeneous isotropic aquifer of thickness 10 m. Contaminated water is being recharged (recharge 50 m^3/hr) at a second well (injection well) located at the coordinates (300, -300) with a distance of 848.5 m from the supply well (discharge well). Effective porosity is 0.25, saturated hydraulic conductivity is 1 m/hr, and negligible background groundwater flow is assumed.

Depicted in Fig. 6 are the limits of the groundwater contamination corresponding to model times of 0.5, 2, and 4 years. Additionally, the CVBEM model predicts a first arrival of contamination of time 4.33 years for injected water to reach the pumping site which agrees well with the Javendal et al. [3] estimate of 4.3 years.

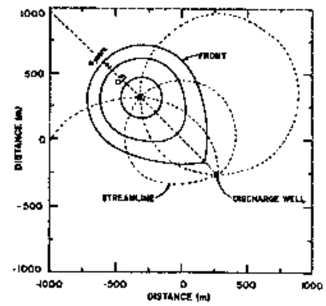

Figure 6. Flowline and Front Positions Between Injection Well and Production Well

Furthermore, if two discharge wells are added at the coordinates (500, 500) and (-500, -500). It takes 4.32 years for the contaminant water to reach the middle discharge well, and about 5.58 years to reach the other two production wells (see Fig. 7).

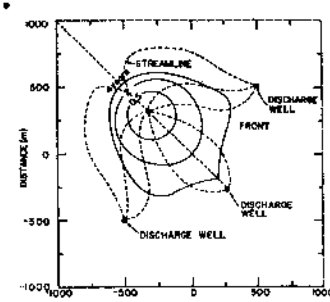

Figure 7. Flowine and Front Positions Between Injection and Three Production Wells

Application 3

Let's consider the steady flow pattern produced by a single pumping well whose strength equals to 50 m^3/hr at (0, 0) near a landfill site with an equipotential boundary $\phi = 2$ m along x = -1000, and a liquid-waste disposal pond with a diameter of 100m fully penetrates the aquifer which is centered at (500, 500) on the Cartesian system shown on Fig. 8. Liquid level in the pond is such that the volume rate of leachate leaving the pond is about 20 m^3/hr. It takes 15.7 years and 7.3 years for the contaminant liquid to reach the discharge well from the left boundary and from the disposal pond, respectively.

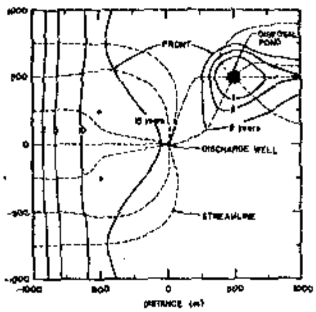

Figure 8. Flowline and Front Positions for Application 4

SUMMARY AND CONCLUSIONS

In this paper, the CVBEM technique is used to develop a two-dimensional, steady-state, subsurface flow model. Because with the CVBEM approach the Laplace equation is solved exactly, all modeling error occurs in matching the prescribed boundary conditions.

Because the CVBEM approach is based upon a boundary integral equation, domain mesh generators or control-volume (finite element) discretizations are not required. Nodal points are required only along the problem boundary rather than in the interior of the domain. Consequently, the computer-coding requirements are small and can be accommodated by many personal computers that support a FORTRAN compiler. Although this study focues upon subsurface flow problem, the numerical analog can be extended to other equivalent problems such as involved in heat and mass transport problems.

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