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Confidence Intervals For Floods

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Abstract:
There is uncertainty in the estimation of the size of the T-year flood due to the uncertain estimation of the parameters in the Log Pearson III distribution which describes the occurrence of the maximum annual discharge. Quantifying this uncertainty by means of computing confidence intervals is discussed.

Introduction:
Flood control agencies generally select a flood protection standard, for example the 100-year flood, to be used in the design of local flood control facilities. To comply with this standard requires estimating the T-year peak flow rate Q_T in the specific example Q_{100} for T = 100.

There are numerous sources of uncertainty in this estimation of Q_T which include:

a) The distribution chosen for Q; the usual choice being the Log Pearson III distribution recommended by the Water Resources Council Bulletins 17A and 17B [1].

b) The estimation of parameters from regional data, such as the skew of the Log Pearson III distribution.

c) Uncertainty in the data due to data measurement or changing catchment conditions.

d) The statistical variation in the estimation of parameters from local data, such as the estimation of the mean and standard deviation of the Log Pearson III distribution.

In spite of these sources of uncertainty, it is a common practice for flood control agencies to adopt a particular flood control goal, e.g., Q_{100} design flows, and then simply utilize a flood frequency curve to estimate Q_{100}. Even supposing that everything is exact except for the estimation of the unknown parameters of the distribution, with the usual estimation by expectorations the true but unknown value of Q_{100} may be just as likely (in the sense of repeated estimations) to be more than the estimate as it is to be less than the estimate. Comparing only the estimated value of Q_{100} does not indicate what confidence one can have that the true value actually is less than the estimated value and so leaves vague a crucial aspect of the protection provided.

In order to quantify the level of protection that the flood control agency receives, confidence intervals for the computed estimate of Q_{100} should be given.

There are too many sources of uncertainty to give definitive confidence intervals, but it is possible, as will be discussed below, to give confidence intervals based on one major source of variability in the estimate.

Discussion:

Two basic assumptions underlie the following analysis:

a) The Q values come from a Log Pearson III distribution.

b) The skew parameter for this distribution is known. (Since the skew is estimated from regional data, following the guidelines of [1], the estimate of the skew is consequently based on much more data than the other parameters in the distribution, and so it is not unreasonable to assume that, relative to the other parameters, the skew is known.)

Thus only that variation in the estimate of Q_T which depends on the statistical variation in the local data is considered, and it is analyzed under the usual assumptions of statistical sampling: that the data form an independent sample from a population with a given distribution whose parameters one wishes to estimate.

It has been known for a long time, and is noted in bulletins 17A and 17B of the Water Resources Council [1], that in the case of a flood distribution whose logarithm is normally distributed, confidence intervals for the T-year flood, under the assumptions above, can be obtained by the use of the non-central student's t-distribution.

In the more general case, following the guidelines of bulletins 17A and 17B, is when the logarithm of the flood distribution has a Pearson III distribution with a non-zero skew parameter. This case of non-zero skew is more complicated than the case of a lognormally distributed flood which is case of zero skew. In an important paper [2] Stedinger shows that the confidence intervals, for quantiles, which are given in the U.S. Water Resources Council guidelines [1] are not satisfactory. He uses a variance formula due to Kite [3] and derives an expression for confidence intervals for the quantiles which he shows is satisfactory in several simulations.

A program described in [4] derives empirical confidence intervals for the T-year flood by means of a simulation. This simulation is similar to that done by Stedinger in [2] and similar to the a simulation done by Hardison [5], who used his results in order to assess the accuracy of an approximate formula for confidence limits for flood-frequency curves given in Bulletin 17A. (This is one of the formulas which Stedinger in [2] found to be unsatisfactory.) The results given in [4] are more accurate and cover a more extensive range of T, sample sizes, skews, and levels of significance than these other simulations.
Mathematical Discussion:

In the case of zero skew, \( X \), the logarithm of the maximum annual discharge, has a normal distribution with mean \( \bar{\mu} \) and standard deviation \( \bar{\sigma} \). For the 1-year flood, take \( p = 1 - 1/T \), and let \( \gamma \) be the \( p \)-th quantile of \( X \); i.e.,

\[
P(X \leq \gamma_p) = p
\]

(1)

It is \( \gamma_p \) that we want to estimate. For the estimate

\[
\hat{\gamma}_p = \hat{\mu} + \hat{\sigma} \hat{z}_p,
\]

where \( \hat{z}_p \) is the \( p \)-th quantile for an \( N(0,1) \) distribution, i.e., a normal distribution with mean \( 0 \) and standard deviation \( 1 \), and \( \hat{\mu} \) and \( \hat{\sigma} \) are the usual estimates for \( \mu \) and \( \sigma \) based on \( m \) data points. It can be shown that

\[
(y_p - \hat{\mu})/\hat{\sigma} \sim (\mu - \bar{\mu})/(\sigma/\sqrt{m})
\]

(2)

has the same distribution as

\[
(1/\sqrt{m})[(y_p - \hat{\mu})/\hat{\sigma}]
\]

(3)

where the random variable in brackets in (3) has a non-central \( t \)-distribution, with non-centrality parameter \( \delta = (\mu - \bar{\mu})/\sigma/\sqrt{m} \); the special case \( \delta = 0 \) is the student's \( t \)-distribution.

Since the distribution of (3) can be written in terms of the known non-central \( t \)-distribution, confidence intervals for \( \gamma_p \) can be given. Because of the way in which the flood design value is used, the confidence interval of greatest interest is a one-sided interval which gives an upper bound. The choice of the value \( T \), for the 1-year flood, and the number \( m \) of data points determine the non-centrality parameter \( \delta \). The other quantity which must be specified is a probability \( p' \) for the one-sided confidence interval. If \( \hat{z}_{p'} \) is the \( p' \)-th quantile of the non-central \( t \)-distribution, then

\[
P(y_p < \hat{\mu} + \mathcal{G}(\hat{z}_{p'}/\sqrt{m})) = p'
\]

(4)

gives the one-sided 100\( p' \)% confidence interval for \( \gamma_p \).

In the case of non-zero skew, the logarithm \( X \) of the yearly peak discharge is assumed to have a Pearson type III distribution with density function

\[
f(x) = (1/\alpha)\Gamma(b)\left[(x-c)/\alpha\right]^{b-1}\exp[-(x-c)/\alpha]
\]

(5)

where, in the case of positive \( \alpha \), the density is given by (5) for \( x > c \) and is zero for \( x < c \), while in the case of negative \( \alpha \), the density is given by (5) for \( x < c \) and is zero for \( x > c \). Computing the mean \( \bar{\mu} \), standard deviation \( \bar{\sigma} \), and skew \( \gamma \) from (5) shows that

\[
\gamma^2 = \bar{\sigma}^2/\bar{\mu},
\]

\[
\mu = \bar{\mu} + \bar{\sigma}^2/\bar{\mu}
\]

(6)

where \( \alpha \) has the same sign as \( \gamma \). As mentioned above, the skew coefficient \( \gamma \) is estimated either from a map of regional skew or from a large pool of data from that region. Consequently, the error in estimating \( \gamma \), which is ignored in this analysis, is of an entirely different kind than the error in estimating \( \mu \) and \( \sigma \) by means of \( m \) data points for the specific area for which the 1-year flood is being estimated.

The form of the density (5) shows that the random variable

\[
Z = (X - \mu)/\sigma
\]

(7)

has the one parameter density

\[
g(x) = (1/\Gamma(b))x^{b-1}e^{-x}
\]

(8)

for \( x > 0 \) and 0 \( x < 0 \); i.e., \( Z \) has a gamma distribution with shape parameter \( b \) and scale parameter \( 1 \).

In [4] it is noted that if \( \alpha \) is the 100\( q \)-th percentile for the gamma distribution \( Z \), \( \hat{\alpha}_Z \) and \( \hat{\alpha}_Z^2 \) are the usual estimators for the mean of \( Z \) and the variance of \( Z \) using \( m \) sample points, then

\[
(y_p - \hat{\mu})/\hat{\sigma} \sim (\hat{\alpha}_Z - \bar{\alpha}_Z)/\hat{\alpha}_Z^2 \quad \text{for} \quad a > 0
\]

(9)

\[
(y_p - \hat{\mu})/\hat{\sigma} \sim (\hat{\alpha}_Z - \bar{\alpha}_Z)/\hat{\alpha}_Z^2 \quad \text{for} \quad a < 0
\]

Confidence intervals for \( y_p \) were obtained in [4], by simulating the distribution of

\[
(\hat{\alpha}_Z - \bar{\alpha}_Z)/\hat{\alpha}_Z^2
\]

(10)

for \( Z \) having the gamma distribution (8).

Stedinger derived an approximate expression for these confidence intervals in [2] using a variance formula due to Kite [3]. When restated in terms of the percentiles above Stedinger's formula is:

\[
\bar{\alpha}_Z = \gamma_p + \lambda \zeta(q)(p')^{-1} \zeta(p')
\]

(11)

The index \( p \) is related to the 1-year flood by \( p = 1 - 1/T \), and the constant \( \gamma_p \) is the \( p \)-th percentile for a normal \( N(0,1) \) distribution. The constant \( \zeta_p \) is given by

\[
\zeta_p = (b-1-p)/\alpha
\]

(12)

\[
\zeta_p = (b-1-p)/\alpha
\]

in which \( \zeta_p \) and \( \zeta_{1-p} \) are the 100\( p \)-th and 100\( (1-p) \)-th percentiles of a gamma distribution with scale parameter \( \alpha \) and shape parameter \( b \), with which can be obtained by applying the Wilson-Hilferty transformation [6] to either \( \zeta_p \) or \( \zeta_{1-p} \). The factor \( \lambda \) is a positive number given by Kite's variance formula

\[
\lambda^2 = [1+\gamma_b^2 + \gamma_b^4(1+1.75\gamma_b^2)]^2/(1+1.5\gamma_b^2)
\]

(13)
In using formula (11) it is only necessary to find 
\( z_{q}(p) \), where 
\( z_{q}(p) \) is the \( q \)-th percentile for 
the non-central \( t \)-distribution with non-centrality 
parameter \( \delta = 2\sqrt{p} \). This non-central \( t \)-distribution 
is well-known and its \( q \)-th percentile can be found by 
means of a double numerical integration. The computa-
tion of this approximation is discussed in [7].

Approximate Confidence Intervals:

In [6] it was shown, by means of comparison with 
simulations, that for skew \( \gamma \) in a normal range, 
\(-0.75 \leq \gamma \leq 0.75\), Stedinger's approximate formula is 
satisfactory for computing not only the higher percentiles 
of the distribution but also for computing all 
of the percentiles 55(52)-95%. This approximation is 
less accurate for larger values of skew, as will be seen 
in Table I; note that it was shown in [9] that the 
value \( \gamma = 2 \) corresponds to the common assumption of 
a linear relation between \( \log Q \) and \( \log T \); the return 
period. Table I compares simulation values to 
Stedinger's approximate values for the percentiles of 
the 100 year flood for skew in the range \(-2 \leq \gamma \leq 2\).

Table I:

| Values for \( m = 10 \) data points, \( T = 100 \) year flood
| Tabulated values are relative percent error

\[ 100 \frac{(A - B)}{A} \text{ for the indicated percentiles} \]

\[ A = \text{values from simulation} \]

\( (20,000 \text{ sets of 10 point samples}) \)

\[ B = \text{Stedinger-Kite approximation} \]

\[ \begin{array}{cccccccccc}
    \text{Percentiles} & \text{-2.0} & \text{-1.5} & \text{-1.0} & \text{-0.5} & \text{0.0} & \text{0.5} & \text{1.0} & \text{1.5} & \text{2.0} \\
    \text{A1} & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 \\
    \text{A2} & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 \\
    \text{A3} & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 \\
    \text{A4} & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 & -0.0 \\
    \end{array} \]

More extensive simulation indicates that generally 
this relative error decreases as \( m \) increases 
\((m \geq 30)\) and decreases as \( T \) increases 
\((10 \leq T \leq 100)\).

Conclusions:

Confidence intervals should be provided for design 
values of the \( T \)-year flood and they can be computed by 
simulation or, for the usual range of hydrologic parame-
ters, by the Stedinger-Kite approximation.

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